

APPLICATIONS OF CERTAIN DIFFERENTIAL INEQUALITIES TO THE UNIVALENCE OF AN INTEGRAL OPERATOR

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*Dedicated to Professor Josip Pečarić
 on the occasion of his 60th birthday*

Abstract. In [1] we have introduced the integral operator denoted by $I(f_1, f_2, \dots, f_m)$ given in Definition 2. Also, certain sufficient conditions of univalence were given for this operator. In this paper we take a different approach for proving the univalence of this operator.

1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Let $\mathcal{H}(U)$ denote the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$$

with $A_1 = A$.

Let

$$S = \{f \in A : f \text{ is univalent in } U\}.$$

DEFINITION 1. (St. Ruscheweyh [3]). For $f \in A$, $n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \rightarrow A$

$$\begin{aligned} R^0 f(z) &= f(z) \\ (n+1)R^{n+1} f(z) &= z[R^n f(z)]' + nR^n f(z), \quad z \in U. \end{aligned}$$

REMARK 1. If $f \in A$

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U$$

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then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j, \quad z \in U. \tag{1}$$

In [1] we have defined the following integral operator:

DEFINITION 2. [1] Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, 3, \dots, m\}$, $\alpha_i \in \mathbb{C}$, $\alpha \in \mathbb{C}$, $\text{Re } \alpha > 0$, $A^m = \underbrace{A \times A \times \dots \times A}_m$. We let $I: A^m \rightarrow A$ be the integral operator given by

$$I(f_1, f_2, \dots, f_m)(z) = F(z) \tag{2}$$

$$= \left[\alpha \int_0^z t^{\alpha-1} \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt \right]^{\frac{1}{\alpha}},$$

where $f_i \in A$, $i \in \{1, 2, 3, \dots, n\}$ and R^n is the Ruschewyh differential operator (Definition 1).

In order to prove our main results, we shall use the following lemma:

LEMMA A. [2, Theorem 5] Let $n, m \in \mathbb{N} \cup \{0\}$, α be a complex number with $\text{Re } \alpha > 0$ and c a complex number with $|c| \leq 1$, $c \neq -1$. If $f_k \in A$, $\alpha_k \in \mathbb{C}$, $k \in \{1, 2, \dots, m\}$ and if

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{z}{\alpha} \cdot \frac{f''(z)}{f'(z)} \right| \leq 1,$$

holds for all $z \in U$, where f is given by

$$f(z) = \int_0^z \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt, \quad z \in U,$$

with R^n given by Definition 1, then the function F given by (2), belongs to the class S .

REMARK 2. In [1], the author has shown that this operator is a generalization of other operators.

REMARK 3. In [1], the author has stated and proven the following theorems:

THEOREM 1. Let $n, m \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C}$ with $\text{Re } \alpha > 0$, $f_i \in A$, $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \dots, m\}$ with $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leq 1$.

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \quad i \in \{1, 2, \dots, m\}$$

then $F(z)$ given by (2) belongs to the class S .

THEOREM 2. Let $n, m \in \mathbb{N} \cup \{0\}$, $\alpha, \beta \in \mathbb{C}$, with $\text{Re } \beta \geq \text{Re } \alpha > 0$, let $f_i \in A$ and let $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \dots, m\}$, with $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leq 1$.

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \quad i \in \{1, 2, \dots, m\}$$

where R^n is the Ruschweyh differential operator, then the function given by

$$(2') \quad F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t} \right) dt \right]^{\frac{1}{\beta}}$$

belongs to the class S .

THEOREM 3. Let $n, m \in \mathbb{N} \cup \{0\}, \mu > 0, \alpha \in \mathbb{C}$ with $\text{Re } \alpha > 0, f_i \in A, \alpha_i \in \mathbb{C}, i \in \{1, 2, \dots, m\}$ with $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leq \frac{1}{2\mu + 1}$.

If

(i) $|R^n f_i(z)| \leq \mu,$

(ii) $\left| \frac{z^2 (R^n f_i(z))'}{[R^n f_i(z)]^2} - 1 \right| \leq 1, z \in U, i \in \{1, 2, \dots, m\}$

where R^n is the Ruschweyh differential operator, then the function $F(z)$ given by (2) belongs to the class S .

THEOREM 4. Let $n, m \in \mathbb{N} \cup \{0\}, \mu > 0, \alpha, \beta \in \mathbb{C}$, with $\text{Re } \beta \geq \text{Re } \alpha > 0$, let $f_i \in A$ and let $\alpha_i \in \mathbb{C}, i \in \{1, 2, \dots, m\}$, with $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leq \frac{1}{2\mu + 1}$.

If

(i) $|R^n f_i(z)| \leq \mu,$

(ii) $\left| \frac{z^2 (R^n f_i(z))'}{[R^n f_i(z)]^2} - 1 \right| \leq 1, z \in U, i \in \{1, 2, \dots, m\}$

where R^n is the Ruschweyh differential operator, then the function $F(z)$ given by (2') belongs to the class S .

2. Main results

THEOREM 5. Let $n, m \in \mathbb{N} \cup \{0\}, \mu \geq 0, \alpha$ and c be complex numbers with $\text{Re } \alpha > 0, |c| \leq 1, c \neq -1$ and let $f_k \in A, \alpha_k \in \mathbb{C}, k \in \{1, 2, \dots, m\}$.

If

i) $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leq (1 - |c|) \cdot \frac{|\alpha|}{1 + \mu(1 + |\alpha|)}$

ii) $|R^n f_k(z)| \leq \mu$ (3)

iii) $\left| \frac{1 - |z|^{2\alpha}}{\alpha} \left[\frac{z^2 (R^n f_k(z))'}{(R^n f_k(z))^2} - 1 \right] \right| \leq 1, z \in U, k \in \{1, 2, \dots, m\},$

where R^n is the Ruschweyh differential operator, then the function F given by (2) belongs to the class S .

Proof. Let

$$f(z) = \int_0^z \left[\frac{R^n f_1(t)}{t} \right]^{\alpha_1} \dots \left[\frac{R^n f_m(t)}{t} \right]^{\alpha_m} dt, \quad z \in U. \tag{4}$$

By differentiating (4), we obtain

$$f'(z) = \left[\frac{R^n f_1(z)}{z} \right]^{\alpha_1} \dots \left[\frac{R^n f_m(z)}{z} \right]^{\alpha_m} = 1 + p_2 z + p_3 z^2 + \dots, \quad (5)$$

$z \in U$.

By differentiating (5), after a short calculation, we have

$$\frac{z f''(z)}{f'(z)} = \alpha_1 \left[\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right] + \dots + \alpha_m \left[\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right], \quad z \in U. \quad (6)$$

In order to prove the theorem, using (6), we evaluate

$$\begin{aligned} \left| \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{z f''(z)}{f'(z)} \right| &= \left| \frac{1 - |z|^{2\alpha}}{\alpha} \alpha_1 \left[\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right] + \dots \right. \\ &\quad \left. + \frac{1 - |z|^{2\alpha}}{\alpha} \alpha_m \left[\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right] \right| \\ &\leq |\alpha_1| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right| + \dots \\ &\quad + |\alpha_m| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right| \\ &\leq |\alpha_1| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} \right| + 1 \right] + \dots \\ &\quad + |\alpha_m| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} \right| + 1 \right] \\ &= |\alpha_1| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z^2 (R^n f_1(z))'}{(R^n f_1(z))^2} \right| \cdot \left| \frac{R^n f_1(z)}{z} \right| + 1 \right] + \dots \\ &\quad + |\alpha_m| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z^2 (R^n f_m(z))'}{(R^n f_m(z))^2} \right| \cdot \left| \frac{R^n f_m(z)}{z} \right| + 1 \right] \\ &\leq |\alpha_1| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z^2 (R^n f_1(z))'}{(R^n f_1(z))^2} \right| \cdot \mu + 1 \right] + \dots \\ &\quad + |\alpha_m| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z^2 (R^n f_m(z))'}{(R^n f_m(z))^2} \right| \cdot \mu + 1 \right] \\ &= (|\alpha_1| + \dots + |\alpha_m|) \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \\ &\quad + |\alpha_1| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left| \frac{z^2 (R^n f_1(z))'}{(R^n f_1(z))^2} \right| \cdot \mu + \dots \\ &\quad + |\alpha_m| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left| \frac{z^2 (R^n f_m(z))'}{(R^n f_m(z))^2} \right| \cdot \mu \\ &= (|\alpha_1| + |\alpha_2| + \dots + |\alpha_m|) \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \end{aligned} \quad (7)$$

$$\begin{aligned}
 & + |\alpha_1| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z^2 (R^n f_1(z))'}{(R^n f_1(z))^2} \right| - 1 + 1 \right] \cdot \mu + \dots \\
 & + |\alpha_m| \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[\left| \frac{z^2 (R^n f_m(z))'}{(R^n f_m(z))^2} \right| - 1 + 1 \right] \cdot \mu \\
 & \leq (|\alpha_1| + |\alpha_2| + \dots + |\alpha_m|) \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \\
 & + \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \left[|\alpha_1| \left| \frac{z^2 (R^n f_1(z))'}{(R^n f_1(z))^2} - 1 \right| \cdot \mu + \dots \right. \\
 & \left. + |\alpha_m| \left| \frac{z^2 (R^n f_m(z))'}{(R^n f_m(z))^2} - 1 \right| \right] \cdot \mu \\
 & + \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \cdot \mu [|\alpha_1| + |\alpha_2| + \dots + |\alpha_m|] \\
 & = (|\alpha_1| + |\alpha_2| + \dots + |\alpha_m|) \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| (\mu + 1) + \mu (|\alpha_1| + \dots + |\alpha_m|) \\
 & \leq (|\alpha_1| + |\alpha_2| + \dots + |\alpha_m|) \frac{1}{|\alpha|} (\mu + 1) + \mu (|\alpha_1| + \dots + |\alpha_m|) \\
 & = (|\alpha_1| + \dots + |\alpha_m|) \left[\frac{\mu(1 + |\alpha|) + 1}{|\alpha|} \right] \leq 1 - |c|.
 \end{aligned}$$

Using (7), we evaluate

$$\begin{aligned}
 \left| c|z|^{2\alpha} + \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| & \leq |c| \cdot |z|^{2\alpha} + \left| \frac{1 - |z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| \\
 & \leq |c| + 1 - |c| = 1.
 \end{aligned}$$

By applying Lemma A, we obtain that function F given by (2) belongs to the class S . \square

EXAMPLE 1. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\mu = 2$, $\alpha = 3 - 4i$, $|\alpha| = 5$, $c = \frac{1}{3} + i\frac{\sqrt{3}}{3}$, $|c| = \frac{2}{3}$, $\alpha_1 = \frac{1}{4} + i\frac{\sqrt{3}}{4}$, $\alpha_2 = \frac{\sqrt{3}}{3} - i\frac{1}{3}$, $f_1(z) = z + az^2$, $R^n f_1(z) = z + (n + 1)az^2$, $f_2(z) = z + bz^2$, $R^n f_2(z) = z + (n + 1)bz^2$, where $|a| < \frac{1}{2(n + 1)}$, $|b| < \frac{1}{2(n + 1)}$, $z \in U$.

The conditions given by the hypothesis of Theorem 5 become

- i) $|\alpha_1| + |\alpha_2| \leq \frac{7}{6} < (1 - |c|) \frac{|\alpha|}{1 + \mu(|\alpha| + 1)} = \frac{5}{39}$
- ii) $|z + (n + 1)az^2| \leq |z| [1 + (n + 1)|a||z|] \leq 1 + (n + 1)|a||z| \leq 2$,
 $|z + (n + 1)bz^2| \leq |z| [1 + (n + 1)|b||z|] \leq 1 + (n + 1)|b||z| \leq 2$,
- iii) $\left| \frac{1 - |z|^{2\alpha}}{\alpha} \left[\frac{z^2(1 + 2(n + 1)az^2)}{z^2(1 + (n + 1)az)} - 1 \right] \right| = \left| \frac{1 - |z|^{2\alpha}}{\alpha} \right| \cdot \left| \frac{(n + 1)az}{1 + (n + 1)az} \right|$

$$\leq \frac{1 - |z|^{2\alpha}}{|\alpha|} \cdot \frac{(n + 1)|a||z|}{1 - (n + 1)|a||z|} \leq \frac{1 - |z|^{2\alpha}}{|\alpha|} \leq \frac{1}{|\alpha|} < 1.$$

$$\begin{aligned} \left| \frac{1 - |z|^{2\alpha}}{\alpha} \left[\frac{z^2(1 + 2(n + 1)az^2)}{z^2(1 + (n + 1)az)} - 1 \right] \right| &\leq \frac{1 - |z|^{2\alpha}}{|\alpha|} \cdot \left| \frac{(n + 1)bz}{1 + (n + 1)bz} \right| \leq \\ &\leq \frac{1 - |z|^{2\alpha}}{|\alpha|} \cdot \frac{(n + 1)|b||z|}{1 - (n + 1)|b||z|} \leq \frac{1 - |z|^{2\alpha}}{|\alpha|} \leq 1. \end{aligned}$$

From Theorem 5, we have

$$F(z) = \{(3 - 4i) \int_0^z t^{2-4i} [1 + (n + 1)at]^{\frac{1}{4} + i\frac{\sqrt{3}}{4}} [1 + (n + 1)bt]^{\frac{\sqrt{3}}{3} - i\frac{1}{3}} dt\}^{\frac{1}{3-4i}} \in S,$$

for all $z \in U$.

THEOREM 6. Let $n, m \in \mathbb{N} \cup \{0\}$, α be a complex number, with $\operatorname{Re} \alpha > 0$, $|\alpha| \geq \frac{1}{1 - |c|}$ and c a complex number with $|c| \leq 1$, $c \neq -1$, $\alpha_k \in \mathbb{C}$, $f_k \in A$, $k \in \{1, 2, \dots, m\}$.
If

$$\left| (1 - |z|^{2\alpha}) \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{8}$$

for all $z \in U$, where $\frac{zf''(z)}{f'(z)}$ is given by (6), then the function F given by (2) belongs to the class S .

Proof. If $z \in U$, $|z| \leq 1$, we obtain

$$|z|^{2\alpha} \leq 1.$$

In order to prove the theorem, using (6) and (8), we evaluate

$$\begin{aligned} \left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| &\leq |cz^{2\alpha}| + \left| (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \\ &= |c||z|^{2\alpha} + \left| (1 - |z|^{2\alpha}) \frac{zf''(z)}{f'(z)} \right| \frac{1}{|\alpha|} \leq |c| + \frac{1}{\frac{1}{1 - |c|}} \leq 1. \end{aligned}$$

By applying Lemma A, we obtain that function F given by (2) belongs to the class S . \square

EXAMPLE 2. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha = 5 - 2i$, $|\alpha| = \sqrt{29}$, $c = \frac{1}{5} - i\frac{\sqrt{3}}{5}$, $|c| = \frac{2}{5}$, $\alpha_1 = 1 - i$, $\alpha_2 = 1 + i$, $f_1(z) = z + az^2$, $R^n f_1(z) = z + (n + 1)az^2$, $f_2(z) = z + bz^2$, $R^n f_2(z) = z + (n + 1)bz^2$, where

$$|a| < \frac{1}{(n + 1)(2\sqrt{2} + 1)}, \quad |b| < \frac{1}{(n + 1)(2\sqrt{2} + 1)}$$

and

$$\begin{aligned}\frac{zf''(z)}{f'(z)} &= (1-i) \left[\frac{1+2(n+1)az^2}{1+(n+1)az} - 1 \right] + (1+i) \left[\frac{1+2(n+1)bz}{1+(n+1)bz} - 1 \right] \\ &= (1-i) \frac{(n+1)az}{1+(n+1)az} + (1+i) \frac{(n+1)bz}{1+(n+1)bz}.\end{aligned}$$

We evaluate

$$\begin{aligned}& \left| (1-|z|^{2\alpha}) \cdot \left[(1-i) \frac{(n+1)az}{1+(n+1)az} + (1+i) \frac{(n+1)bz}{1+(n+1)bz} \right] \right| \\ & \leq |1-|z|^{2\alpha}| \cdot \left| (1-i) \frac{(n+1)az}{1+(n+1)az} + (1+i) \frac{(n+1)bz}{1+(n+1)bz} \right| \\ & \leq |1-i| \frac{(n+1)|a||z|}{1-(n+1)|a||z|} + |1+i| \frac{(n+1)|b||z|}{1-(n+1)|b||z|} \\ & = \sqrt{2} \cdot \left[\frac{(n+1)|a||z|}{1-(n+1)|a||z|} + \frac{(n+1)|b||z|}{1-(n+1)|b||z|} \right] \leq \frac{\sqrt{2}}{2\sqrt{2}} + \frac{\sqrt{2}}{2\sqrt{2}} = 1.\end{aligned}$$

From Theorem 6 we deduce

$$F(z) = \left\{ (5-2i) \int_0^z t^{4-2i} [1+(n+1)at]^{1-i} [1+(n+1)bt]^{1+i} dt \right\}^{\frac{1}{5-2i}} \in S,$$

for all $z \in U$.

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