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OPTIMAL LYAPUNOV INEQUALITIES FOR BOUNDARY VALUE PROBLEMS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. This work is devoted to review some recent results on L_p Lyapunov-type inequalities ($1 \leqslant p \leqslant \infty$) for resonant differential equations. In the case of Ordinary Differential Equations, we consider Neumann boundary conditions and an explicit optimal result is obtained. Moreover, it is also treated the case in which the resonance appears at higher eigenvalues. We also study mixed boundary conditions. From this study, and under some natural restrictions on the linear coefficient, the relation between Neumann boundary conditions and disfocality arises in a natural way. For Partial Differential Equations it is proved that the relation between the quantities p and $N/2$ plays a crucial role in order to obtain L_p Lyapunov-type inequalities, for resonant linear problems with Neumann boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^N$. This fact shows a deep difference with respect to the ordinary case. Combining these linear results with Schauder fixed point theorem, we can obtain some new results about the existence and uniqueness of solutions for resonant nonlinear problems. Finally, we comment some conclusions on systems of equations.

1. Introduction

Let us consider the linear problem

$$u''(x) + a(x)u(x) = 0,\ x \in (0,L),\ u'(0) = u'(L) = 0 \tag{1.1}$$

where $a \in \Lambda_0$ and Λ_0 is defined by

$$\Lambda_0 = \{a \in L^1(0,L) \setminus \{0\} : \int_0^L a(x)\, dx \geqslant 0 \text{ and } (1.1) \text{ has nontrivial solutions }\}. \tag{1.2}$$

The well-known Lyapunov inequality states that if $a \in \Lambda_0$, then $\int_0^L a^+(x)\, dx > 4/L$. Moreover, the constant $4/L$ is optimal (see [2], [3], [13] and [14]). An analogous result is true for Dirichlet boundary conditions. In fact, the original results were proved for this kind of boundary conditions ([13], [16], [18], [23]).

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In this work we review some more general recent results contained in [3], [4], [5], [6] and [7]. We consider for each p with $1 \leqslant p \leqslant \infty$, the quantity

$$\beta_p \equiv \inf_{a \in \Lambda_0} I_p(a) \tag{1.3}$$

where

$$\begin{gathered} I_p(a) = \|a^+\|_p = \left(\int_0^L |a^+(x)|^p \, dx \right)^{1/p}, \ \forall\, a \in \Lambda_0, \ 1 \leqslant p < \infty, \\ I_\infty(a) = \sup \text{ess } a^+, \ \forall\, a \in \Lambda_0 \end{gathered} \tag{1.4}$$

obtaining an explicit expression for β_p as a function of p and L.

In particular, if $p = \infty$, it may be proved (see [3]) that (1.1) has only the trivial solution if function a satisfies

$$0 \prec a \prec \pi^2/L^2 \tag{1.5}$$

where for $c, d \in L^1(0,L)$, we write $c \prec d$ if $c(x) \leqslant d(x)$ for a.e. $x \in [0,L]$ and $c(x) < d(x)$ on a set of positive measure. (1.5) is usually called the nonuniform nonresonance condition with respect to the first two eigenvalues $\lambda_0 = 0$ and $\lambda_1 = \pi^2/L^2$ of the eigenvalue problem

$$u''(x) + \lambda u(x) = 0, \ x \in (0,L), \ u'(0) = u'(L) = 0, \tag{1.6}$$

(see [19], [20] and [22]). From this point of view, it may be affirmed that the nonuniform nonresonance condition (1.5) is in fact the L_∞ Lyapunov inequality at the first eigenvalue λ_0.

On the other hand, the set of eigenvalues of (1.6) is given by $\lambda_n = n^2\pi^2/L^2$, $n \in \mathbb{N} \cup \{0\}$ and by using a general result due to Dolph [10], it can be proved that if for some $n \geqslant 1$ function a satisfies

$$\lambda_n \prec a \prec \lambda_{n+1} \tag{1.7}$$

then (1.1) has only the trivial solution (see [21] for some generalizations of (1.7) to more general boundary value problems). It is clear that if n is sufficiently large, condition (1.7) can not be obtained from L_p Lyapunov inequalities given in [3] and [25] for the set Λ_0.

Previous observations motivated our paper [6] where, for any given natural number $n \geqslant 1$ and function a satisfying $\lambda_n \prec a$, we obtain optimal L_1 Lyapunov inequality.

The case where function a satisfies the condition $A \leqslant a(x) \leqslant B$, a.e. in $(0,L)$ where $\lambda_n < A < \lambda_{n+1} \leqslant B$ for some $n \in \mathbb{N} \cup \{0\}$, has been considered in [24]. In this paper the authors use Optimal Control theory methods, specially Pontryagin's maximum principle.

On the other hand, under the logical restrictions $a \in L^1(0,L) \setminus \{0\}$ and $\int_0^L a(x)\, dx \geqslant 0$, the relation between Neumann boundary conditions and disfocality arises in a natural way: if $u \in H^1(0,L)$ is any nontrivial solution of (1.1) then u must have a zero c in the interval $(0,L)$ (see [5, Lem. 2.2]). In consequence both problems

$$v''(x) + a(x)v(x) = 0,\ x \in (0,c),\ v'(0) = v(c) = 0 \qquad \mathbf{PM(0,c)}$$

and

$$v''(x) + a(x)v(x) = 0,\ x \in (c,L),\ v(c) = v'(L) = 0 \qquad \mathbf{PM(c,L)}$$

have nontrivial solutions.

This simple observation (which has been previously employed in the case of Dirichlet boundary conditions, [12], [17]) can be used to deduce the following conclusion: if $a \in L^1(0,L) \setminus \{0\}$ with $\int_0^L a \geqslant 0$ is any function such that for any $c \in (0,L)$, either problem **PM(0,c)** or problem **PM(c,L)** has only the trivial solution, then problem (1.1) has only the trivial solution. In [5] we exploit this idea to obtain new results on the existence and uniqueness of solutions for resonant problems with Neumann boundary conditions. For example, by using the L^∞ norm of function a^+, we can prove (see [5, Prop. 3.3]) that if

$$\left.\begin{array}{c} a \in L^\infty(0,L) \setminus \{0\},\ \displaystyle\int_0^L a \geqslant 0 \text{ and } \exists\, x_0 \in (0,L): \\ \max\{x_0^2 \|a^+\|_{L^\infty(0,x_0)},\ (L-x_0)^2 \|a^+\|_{L^\infty(x_0,L)}\} \leqslant \pi^2/4 \end{array}\right. \qquad (\mathbf{H})$$

and, in addition, either a^+ is not the constant $\pi^2/4x_0^2$ in the interval $[0,x_0]$ or a^+ is not the constant $\pi^2/4(L-x_0)^2$ in the interval $[x_0,L]$, then we obtain that (1.1) has only the trivial solution (this kind of functions a are usually named two step potentials).

Hypothesis (**H**) is optimal in the sense that if a^+ is the constant $\pi^2/4x_0^2$ in the interval $[0,x_0]$ and a^+ is the constant $\pi^2/4(L-x_0)^2$ in the interval $[x_0,L]$, then (1.1) has nontrivial solutions (see [5, Rem. 6]).

If $x_0 = L/2$, we have the classical result related to (1.5), but if for instance, $x_0 \in (0,L/2)$ function a can satisfy $\|a^+\|_{L^\infty(0,x_0)} = \pi^2/(4x_0^2)$ (which is a quantity greater than π^2/L^2) as long as $\|a^+\|_{L^\infty(x_0,L)} < \pi^2/4(L-x_0)^2$.

If $\lambda_n \prec a$, condition **(H)** has been generalized by the authors to more general situations ([6]).

In the PDE case, we consider the linear problem

$$\left.\begin{array}{ll} \Delta u(x) + a(x)u(x) = 0, & x \in \Omega \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega \end{array}\right\} \qquad (1.8)$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded and regular domain, $\dfrac{\partial}{\partial n}$ is the outer normal derivative on $\partial\Omega$ and the function $a : \Omega \to \mathbb{R}$ belongs to the set Λ defined as

$$\Lambda = \{a \in L^{\frac{N}{2}}(\Omega) \setminus \{0\} : \int_\Omega a(x)\, dx \geqslant 0 \text{ and (1.8) has nontrivial solutions}\} \qquad (1.9)$$

if $N \geqslant 3$ and

$$\Lambda = \{a : \Omega \to \mathbb{R} \text{ s. t. } \exists q \in (1,\infty] \text{ with } a \in L^q(\Omega) \setminus \{0\},\ \int_\Omega a(x)\,dx \geqslant 0 \text{ and } (1.8) \text{ has nontrivial solutions}\} \text{ if } N = 2.$$

Obviously, the quantity

$$\gamma_p \equiv \inf_{a \in \Lambda} \|a^+\|_p\ ,\ 1 \leqslant p \leqslant \infty \tag{1.10}$$

is well defined and it is a nonnegative real number. The first novelty with respect to the ordinary case is that $\gamma_1 = 0$ for each $N \geqslant 2$. Moreover, if $N = 2$, then $\gamma_p > 0$, $\forall\ p \in (1,\infty]$ and if $N \geqslant 3$, then $\gamma_p > 0$ if and only if $p \geqslant N/2$. Also, for each $N \geqslant 2$, γ_p is attained if $p > N/2$. It seems difficult to obtain explicit expressions for γ_p, as a function of p, Ω and N, at least for general domains.

Combining all the previous linear results with Schauder fixed point theorem, we can obtain some new results about the existence and uniqueness of solutions for resonant nonlinear problems ([3], [4], [5], [11]). We finish this work with some recent results on systems of equations ([7]).

2. Ordinary differential equations

We begin this section showing an explicit result which generalizes classical Lyapunov inequality:

THEOREM 2.1. ([3]) *Let β_p be defined in (1.3). The following statements hold:*

(1) $\beta_1 = \dfrac{4}{L}$, $\beta_\infty = \dfrac{\pi^2}{L^2}$. *The mapping* $[1,\infty) \to \mathbb{R}$, $p \to \beta_p$, *is continuous and* $\lim_{p\to\infty} \beta_p = \beta_\infty$. *Moreover, the mapping* $\delta : [1,\infty) \to \mathbb{R}$, $p \to L^{-1/p}\beta_p$ *is strictly increasing.*

(2) *If* $1 < p < \infty$,

$$\beta_p = \frac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}}\, p(2p-1)^{1/p}} \left(\int_0^{\pi/2} (\sin x)^{-1/p}\, dx \right)^2. \tag{2.1}$$

(3) β_p *is attained if and only if* $1 < p \leqslant \infty$. *In this case,* β_p *is attained in a unique element* $a_p \in \Lambda_0$ *which is not a constant function if* $1 < p < \infty$ *and* $a_\infty(x) \equiv \frac{\pi^2}{L^2}$.

Main ideas of the proof for the case $1 < p < \infty$. If $a \in \Lambda_0$ (Λ_0 was defined in (1.2)) and $u \in H^1(0,L)$ is a nontrivial solution of

$$u''(x) + a(x)u(x) = 0,\ x \in (0,L),\ u'(0) = u'(L) = 0, \tag{2.2}$$

then

$$\int_0^L u'v' = \int_0^L auv, \ \forall\, v \in H^1(0,L).$$

In particular, choosing $v \equiv u$ and $v \equiv 1,$ we have respectively

$$\int_0^L u'^2 = \int_0^L au^2, \ \ \int_0^L au = 0. \tag{2.3}$$

Therefore, for each $k \in \mathbb{R},$ we have

$$\begin{aligned}\int_0^L (u+k)'^2 = \int_0^L u'^2 = \int_0^L au^2 \leqslant \int_0^L au^2 + k^2 \int_0^L a \\ = \int_0^L au^2 + \int_0^L k^2 a + 2k \int_0^L au = \int_0^L a(u+k)^2.\end{aligned}$$

It follows from Hölder inequality

$$\int_0^L (u+k)'^2 \leqslant \|a\|_p \|(u+k)^2\|_{\frac{p}{p-1}}.$$

Also, since u is a nonconstant solution of (2.2), $u+k$ is a nontrivial function. Consequently

$$\|a\|_p \geqslant \frac{\displaystyle\int_0^L (u+k)'^2}{\|(u+k)^2\|_{\frac{p}{p-1}}}, \ \forall\, a \in \Lambda_0. \tag{2.4}$$

Previous reasoning motivates the study of an special minimization problem given in the following lemma.

LEMMA 2.2. ([3]). *Assume* $1 < p < \infty$ *and let*

$$X_p = \left\{ u \in H^1(0,L) : \int_0^L |u|^{\frac{2}{p-1}} u = 0 \right\}.$$

If $J_p : X_p \setminus \{0\} \to \mathbb{R}$ *is defined by*

$$J_p(u) = \frac{\displaystyle\int_0^L u'^2}{\left(\displaystyle\int_0^L |u|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}}} \tag{2.5}$$

and $m_p \equiv \inf_{X_p \setminus \{0\}} J_p,$ m_p *is attained. Moreover, if* $u_p \in X_p \setminus \{0\}$ *is a minimizer, then* u_p *satisfies the problem*

$$u_p''(x) + A_p(u_p)|u_p(x)|^{\frac{2}{p-1}} u_p(x) = 0, \ \ u_p'(0) = u_p'(L) = 0, \tag{2.6}$$

$$\text{where } A_p(u_p) = m_p \left(\int_0^L |u_p|^{\frac{2p}{p-1}} \right)^{\frac{-1}{p}}. \tag{2.7}$$

Finally, $\beta_p = m_p.$

The proof of this lemma is based on some calculus with Lagrange multipliers and the explicit value of m_p is obtained from a careful analysis of (2.6) (see [3] for the complete proof of Theorem 2.1 and [8], [9] for some similar ideas on the periodic boundary value problem). $\square$

Now we treat with the case where the nonresonance appears at higher eigenvalues. Specifically, if $n \in \mathbb{N}$ is fixed, we introduce the set Λ_n as

$$\Lambda_n = \{a \in L^1(0,L) : \lambda_n \prec a \text{ and } (1.1) \text{ has nontrivial solutions } \}. \tag{2.8}$$

THEOREM 2.3. *([6]).*

$$\beta_{1,n} \equiv \inf_{a \in \Lambda_n} \|a - \lambda_n\|_{L^1(0,L)} = \frac{2\pi n(n+1)}{L} \cot \frac{\pi n}{2(n+1)}.$$

Moreover $\beta_{1,n}$ is not attained.

Proof. It is based on some lemmas. In the first one, we do a careful and optimal analysis about the number and distribution of zeros of nontrivial solutions u of (1.1), and its first derivative u', when $a \in \Lambda_n$. To the best of our knowledge, this result is new.

LEMMA 2.4. ([6]). *Let $a \in \Lambda_n$ be given and u any nontrivial solution of (1.1). If the zeros of u' in $[0,L]$ are denoted by $0 = x_0 < x_2 < \ldots < x_{2m} = L$ and the zeros of u in $(0,L)$ are denoted by $x_1 < x_3 < \ldots < x_{2m-1}$, then:*

(1) $x_{i+1} - x_i \leqslant \frac{L}{2n}$, $\forall\, i:\ 0 \leqslant i \leqslant 2m-1$. *Moreover, at least one of these inequalities is strict.*

(2) $m \geqslant n+1$. *Moreover, any value $m \geqslant n+1$ is possible.*

(3) *Let i, $0 \leqslant i \leqslant 2m-1$, be given. Then, functions a and u satisfy either*

$$\|a - \lambda_n\|_{L^1(x_i,x_{i+1})} \geqslant \frac{\int_{x_i}^{x_{i+1}} u'^2 - \lambda_n \int_{x_i}^{x_{i+1}} u^2}{u^2(x_{i+1})}, \quad \text{if } u(x_i) = 0 \tag{2.9}$$

or

$$\|a - \lambda_n\|_{L^1(x_i,x_{i+1})} \geqslant \frac{\int_{x_i}^{x_{i+1}} u'^2 - \lambda_n \int_{x_i}^{x_{i+1}} u^2}{u^2(x_i)}, \quad \text{if } u(x_{i+1}) = 0. \tag{2.10}$$

Proof. Let i, $0 \leqslant i \leqslant 2m-1$, be given. Then, function u satisfies either the problem

$$u''(x) + a(x)u(x) = 0,\ x \in (x_i, x_{i+1}),\ \ u(x_i) = 0,\ u'(x_{i+1}) = 0, \tag{2.11}$$

or the problem

$$u''(x) + a(x)u(x) = 0,\ x \in (x_i, x_{i+1}),\ \ u'(x_i) = 0,\ u(x_{i+1}) = 0. \tag{2.12}$$

Let us assume the first case. The reasoning in the second case is similar. Note that u may be chosen such that $u(x) > 0,\ \forall\, x \in (x_i, x_{i+1})$. Let us denote by μ_1^i and φ_1^i, respectively, the principal eigenvalue and eigenfunction of the eigenvalue problem

$$v''(x) + \mu v(x) = 0,\ x \in (x_i, x_{i+1}),\ v(x_i) = 0,\ v'(x_{i+1}) = 0. \tag{2.13}$$

It is known that

$$\mu_1^i = \frac{\pi^2}{4(x_{i+1} - x_i)^2},\ \varphi_1^i(x) = \sin \frac{\pi(x - x_i)}{2(x_{i+1} - x_i)}. \tag{2.14}$$

Choosing φ_1^i as test function in the weak formulation of (2.11) and u as test function in the weak formulation of (2.13) for $\mu = \mu_1^i$ and $v = \varphi_1^i$, we obtain

$$\int_{x_i}^{x_{i+1}} (a(x) - \mu_1^i) u \varphi_1^i(x)\, dx = 0. \tag{2.15}$$

Then, if $x_{i+1} - x_i > \frac{L}{2n}$, we have

$$\mu_1^i = \frac{\pi^2 L^2}{4(x_{i+1} - x_i)^2 L^2} < \frac{n^2 \pi^2}{L^2} = \lambda_n \leqslant a(x),\ \text{a.e. in } (x_i, x_{i+1})$$

which is a contradiction with (2.15). Consequently, $x_{i+1} - x_i \leqslant \frac{L}{2n}, \forall\, i:\ 0 \leqslant i \leqslant 2m-1$. Also, since $a \prec \lambda_n$ in the interval $(0, L)$, we must have $a \prec \lambda_n$ in some subinterval (x_j, x_{j+1}). If $x_{j+1} - x_j = \frac{L}{2n}$, it follows $\mu_1^j \prec a$ in (x_j, x_{j+1}) and this is again a contradiction with (2.15). These reasonings complete the first part of the lemma. For the second one, let us observe that

$$L = \sum_{i=0}^{2m-1} (x_{i+1} - x_i) < 2m \frac{L}{2n}.$$

In consequence, $m > n$. Also, note that for any given natural number $q \geqslant n+1$, function $a(x) \equiv \lambda_q$ belongs to Λ_n and for function $u(x) = \cos \frac{q\pi x}{L}$, we have $m = q$.

Lastly, if i, with $0 \leqslant i \leqslant 2m-1$ is given and u satisfies (2.11), then

$$\int_{x_i}^{x_{i+1}} u'^2(x) = \int_{x_i}^{x_{i+1}} a(x) u^2(x) =$$

$$\int_{x_i}^{x_{i+1}} (a(x) - \lambda_n) u^2(x) + \int_{x_i}^{x_{i+1}} \lambda_n u^2(x).$$

Therefore,

$$\int_{x_i}^{x_{i+1}} u'^2(x) - \lambda_n \int_{x_i}^{x_{i+1}} u^2(x) \leqslant \|a - \lambda_n\|_{L^1(x_i, x_{i+1})} \|u^2\|_{L^\infty(x_i, x_{i+1})}.$$

Since u' has no zeros in the interval (x_i, x_{i+1}) and $u(x_i) = 0$, we have $\|u^2\|_{L^\infty(x_i, x_{i+1})} = u^2(x_{i+1})$. This proves the third part of the lemma when u satisfies (2.11). The reasoning is similar if u satisfies (2.12). $\square$

From the previous lemma and by using Lagrange multipliers Theorem and some minimization problems, it is possible to obtain a lower bound for $\beta_{1,n}$. More precisely, we prove that

$$\begin{aligned}\beta_{1,n} &\geqslant \frac{n\pi}{L}\sum_{i=0}^{2m-1}\cot\Big(\frac{n\pi}{L}(x_{i+1}-x_i)\Big)\\ &\geqslant \frac{n\pi}{L}2m\cot\frac{n\pi}{2m} \geqslant \frac{n\pi}{L}2(n+1)\cot\frac{n\pi}{2(n+1)}.\end{aligned}$$

To conclude the proof of Theorem 2.3, we show in [6, Lem. 2.7] an explicit minimizing sequence of functions $a\in\Lambda_n$. $\square$

As we have mentioned in the introduction, if $a\in\Lambda_0$ the relation between Neumann boundary conditions and disfocality arises in a natural way: if $u\in H^1(0,L)$ is any nontrivial solution of (1.1) then u must have a zero c in the interval $(0,L)$ (see [5, Lem. 2.2]). In consequence both problems **PM(0,c)** and **PM(c,L)** (both defined in the Introduction) have nontrivial solutions. Hence, it is natural to consider

$$\Lambda^* = \{a\in L^1(0,L):\ u''+a(x)u=0; u'(0)=u(L)=0 \text{ has nontrivial solutions }\} \quad (2.16)$$

and

$$\beta_p^* \equiv \inf_{a\in\Lambda^*\bigcap L^p(0,L)} \|a^+\|_{L^p(0,L)},\ 1\leqslant p\leqslant\infty. \quad (2.17)$$

THEOREM 2.5. ([5]). *If* $1\leqslant p\leqslant\infty$, *we have* $\beta_p^*=\beta_p/4$.

From this result and if we establish Hypothesis **(Hp)*** as:

(1) $\|a^+\|_{L^1(c,d)}\leqslant\beta_1^*(c,d)$ if $p=1$.

(2) $a\in L^p(c,d)$, $\|a^+\|_{L^p(c,d)}<\beta_p^*(c,d)$, if $p\in(1,\infty]$.

we obtain, as an application to Neumann problem (1.1), the next result.

THEOREM 2.6. ([5]). *Let* $a\in L^1(0,L)\setminus\{0\}$ *with* $\int_0^L a(x)\,dx\geqslant 0$, *satisfying:*

For each $c\in(0,L)$ *either hypothesis* **(Hp)*** *in the interval* $(0,c)$ *or hypothesis* **(Hq)*** *in the interval* (c,L) *(here,* $p,q\in[1,\infty]$ *may depend on* c*).*

Then the problem (1.1) *has only the trivial solution.*

By using an slight modification of the previous Theorem for the case $p=q=\infty$, we can obtain the result mentioned in the Introduction, related to hypothesis **(H)** ([5]).

3. Partial Differential Equations

This section will be concerned with the linear boundary value problem (1.8) where $a\in\Lambda$ and Λ was defined in the Introduction.

As previously, Ω is a bounded and regular domain of $\mathbb{R}^N$. As in the ordinary case, the positive eigenvalues of the eigenvalue problem

$$\left.\begin{array}{ll} \Delta u(x) + \lambda u(x) = 0 & x \in \Omega \\ \frac{\partial u(x)}{\partial n} = 0 & x \in \partial\Omega \end{array}\right\} \tag{3.1}$$

belong to Λ. Therefore, the quantity

$$\gamma_p \equiv \inf_{a \in \Lambda} \|a\|_p \,, \; 1 \leqslant p \leqslant \infty$$

is well defined. Our main result on linear PDE is the next theorem.

THEOREM 3.1. ([4]) *The following statements hold:*

(1) $\gamma_1 = 0$, *and* $\gamma_\infty = \lambda_1$, $\forall\, N \geqslant 2$. *Here* λ_1 *is the first positive eigenvalue of the eigenvalue problem (*3.1*).*

(2) *If* $N = 2$, $\gamma_p > 0$, $\forall p \in (1, \infty]$.
If $N \geqslant 3$, $\gamma_p > 0 \Leftrightarrow p \in [\frac{N}{2}, \infty]$.
If $N \geqslant 2$ *and* $\frac{N}{2} < p \leqslant \infty$ *then* γ_p *is attained.*

(3) *The mapping* $(\frac{N}{2}, \infty) \to \mathbb{R}$, $p \mapsto \gamma_p$, *is continuous and the mapping* $[\frac{N}{2}, \infty] \to \mathbb{R}$, $p \mapsto |\Omega|^{-1/p}\gamma_p$, *is strictly increasing.*

Main ideas of the proof:

(1) If $N \geqslant 3$ and $\frac{N}{2} < p < \infty$, the ideas are the same as in the ordinary case since the imbedding of the Sobolev space $H^1(\Omega)$ into $L^{2p/p-1}(\Omega)$ is compact.

(2) If $N = 2$, the imbedding $H^1(\Omega) \subset L^q(\Omega)$ is compact $\forall\, q \in [1, \infty)$ and therefore, if $1 < p < \infty$, the ideas are the same as in the ordinary case.

(3) If $N \geqslant 3$ and $1 \leqslant p < \frac{N}{2}$, we prove that $\gamma_p = 0$ by finding appropriate minimizing sequences. For instance, if $\Omega = B(0,1)$ we can take radial functions $u(x) = f(|x|)$ of the form $f(r) = \alpha r^{-a} - \beta r^{-b}$, $(a > 0,\; b > 0, 0 < r < 1)$.
If $N = 2$ and $p = 1$, we use the fundamental solution $\ln|x|$ to find appropriate minimizing sequences.

(4) If $N \geqslant 3$ and $p = \frac{N}{2}$, then $\frac{2p}{p-1} = \frac{2N}{N-2}$ and the imbedding $H^1(\Omega) \subset L^{2N/N-2}(\Omega)$ is continuous but not compact. This implies that the infimum $\gamma_p > 0$.

By using the Schauder fixed point theorem, we can apply the previous results (on linear problems) to nonlinear boundary value problems. In fact, from each obtained linear result we may deduce the corresponding nonlinear one. For instance, we can consider

$$\left.\begin{array}{ll} -\Delta u(x) = f(x, u(x)) & x \in \Omega \\ \frac{\partial u}{\partial n}(x) = 0 & x \in \partial\Omega \end{array}\right\} \tag{3.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded and regular domain and the function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R},\ (x,u) \mapsto f(x,u),$ satisfies the condition

(C) f, f_u are Caratheodory functions and $0 \leqslant f_u(x,u)$ in $\overline{\Omega} \times \mathbb{R}$.

The existence of a solution of (3.2) implies

$$\int_\Omega f(x,s_0)\, dx = 0 \tag{3.3}$$

for some $s_0 \in \mathbb{R}$ (see [22]). Trivially, conditions **(C)** and (3.3) are not sufficient for the existence of solutions of (3.2). Indeed, consider the problem

$$\left.\begin{array}{ll} -\Delta u(x) = \lambda_1 u(x) + \varphi_1(x) & x \in \Omega \\ \frac{\partial u}{\partial n}(x) = 0 & x \in \partial\Omega \end{array}\right\} \tag{3.4}$$

where φ_1 is a nontrivial eigenfunction associated to λ_1. Here λ_1 is the first positive eigenvalue of the eigenvalue problem (3.1). The function $f(x,u) = \lambda_1 u + \varphi_1(x)$ satisfies **(C)** and (3.3), but the Fredholm alternative theorem shows that there is no solution of (3.4).

If, moreover of **(C)** and (3.3), f satisfies a non-uniform non-resonance condition of the type

(c1) $f_u(x,u) \leqslant \gamma(x)$ in $\overline{\Omega} \times \mathbb{R}$ with $\gamma(x) \leqslant \lambda_1$ in Ω and $\gamma(x) < \lambda_1$ in a subset of Ω of positive measure,

then it has been proved in [22] that (3.2) has solution. Let us observe that supplementary condition **(c1)** is given in terms of $\|\gamma\|_\infty$. In the next result, we provide new supplementary conditions in terms of $\|\gamma\|_p$, where $N/2 < p \leqslant \infty$, obtaining a generalization of Theorem 2 in [22]. In the case of Dirichlet conditions, it is possible to obtain analogous results in an easier way (see [4]). Moreover, it is possible to do a similar analysis related to the operator q-laplacian, where $1 < q < \infty$ (see [5]).

THEOREM 3.2. *([4]) Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded and regular domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R},\ (x,u) \mapsto f(x,u)$, satisfying:*

(1) *f, f_u are Caratheodory functions and $f(\cdot,0) \in L^\infty(\Omega)$.*

(2) *There exists a function $\gamma \in L^\infty(\Omega)$, satisfying*

$$0 \leqslant f_u(x,u) \leqslant \gamma(x) \text{ in } \overline{\Omega} \times \mathbb{R} \tag{3.5}$$

and such that for some p, $N/2 < p \leqslant \infty$, we have $\|\gamma\|_p < \gamma_p$ (or $\|\gamma\|_p = \gamma_p$ and $\gamma(x)$ is not a minimizer of the L_p-norm in Λ), where γ_p is given by Theorem 3.1.

(3) *There exists $s_0 \in \mathbb{R}$ such that*

$$\int_\Omega f(x,s_0)\, dx = 0, \text{ and } f_u(x,u(x)) \not\equiv 0 \text{ for all } u \in C(\overline{\Omega}). \tag{3.6}$$

Then problem (3.2) has a unique solution.

Proof. For the uniqueness, if u_1 and u_2 are two solutions of (3.2), then the function $u = u_1 - u_2$ is a solution of the problem

$$-\Delta u(x) = a(x)u(x),\ x \in \Omega,\ \frac{\partial u}{\partial n} = 0,\ x \in \partial\Omega \tag{3.7}$$

where $a(x) = \int_0^1 f_u(x, u_2(x) + \theta u(x))\, d\theta$. From the definition of γ_p, we obtain $u \equiv 0$. For the existence, we write (3.2) in the equivalent form

$$\left.\begin{aligned} -\Delta u(x) &= b(x,u(x))u(x) + f(x,0), && \text{in } \Omega,\\ \tfrac{\partial u}{\partial n} &= 0, && \text{on } \partial\Omega \end{aligned}\right\} \tag{3.8}$$

where the function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by $b(x,z) = \int_0^1 f_u(x, \theta z)\, d\theta$. If $X = C(\overline{\Omega})$, our hypotheses allow to define an operator $T : X \to X$, by $Ty = u_y$, being u_y the unique solution of the linear problem

$$\left.\begin{aligned} -\Delta u(x) &= b(x,y(x))u(x) + f(x,0), && \text{in } \Omega,\\ \tfrac{\partial u}{\partial n} &= 0, && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.9}$$

where $X = C(\overline{\Omega})$ with the uniform norm. Now T is completely continuous and $T(X)$ is bounded. The Schauder's fixed point theorem provides a fixed point for T which is a solution of (3.2). $\square$

As in the ordinary case, the constant γ_p may be obtained as a minimum of some special minimization problems ([4]). This important fact makes possible the application to linear or nonlinear systems of equations. To this respect, we show the next Theorem on systems with variational structure. It is a generalization of the main result given in [22] for the Neumann problem. Moreover, it is a generalization of some results given in [1] and [15] where the authors take all the constants $p_i = \infty$, $1 \leqslant i \leqslant n$. In the Theorem, the relation $C \leqslant D$ between $n \times n$ matrices means that $D - C$ is positive semi-definite and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$.

THEOREM 3.3. ([7]) *Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) be a bounded and regular domain and $G : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$, $(x,u) \to G(x,u)$ satisfying:*

(1) *(a) $u \to G(x,u)$ is of class $C^2(\mathbb{R}^n, \mathbb{R})$ for every $x \in \overline{\Omega}$.*
    (b) $x \to G(x,u)$ is continuous on $\overline{\Omega}$ for every $u \in \mathbb{R}^n$.

(2) *There exist continuous matrix functions $A(\cdot)$, $B(\cdot)$, with $B(x)$ diagonal and with entries $b_{ii}(x)$, and $p_i \in (N/2, \infty]$ $1 \leqslant i \leqslant n$, such that*

$$\left.\begin{aligned} A(x) \leqslant G_{uu}(x,u) \leqslant B(x) \ \text{in } \overline{\Omega} \times \mathbb{R}^n,\\ \|b_{ii}^+\|_{p_i} < \gamma_{p_i},\ 1 \leqslant i \leqslant n,\\ \int_\Omega \langle A(x)k, k\rangle dx > 0,\ \forall\, k \in \mathbb{R}^n \setminus \{0\}. \end{aligned}\right\} \tag{3.10}$$

Then system

$$\left.\begin{aligned} -\Delta u(x) = G_u(x,u(x)),\ x\in\Omega, \\ \frac{\partial u(x)}{\partial n} = 0,\ \ x\in\partial\Omega, \end{aligned}\right\} \tag{3.11}$$

has a unique solution.

Previous Theorem is optimal in the following sense: for any given positive numbers $\delta_i,\ 1\leqslant i\leqslant n,$ such that at least one of them, say $\delta_j,$ satisfies

$$\delta_j > \gamma_{p_j},\ \text{for some}\ p_j\in(N/2,\infty], \tag{3.12}$$

there exists a function $G(x,u)$ satisfying all the hypotheses of the previous Theorem except (3.10), which is replaced with

$$\|b_{ii}^{+}\|_{p_i} < \delta_{p_i},\ 1\leqslant i\leqslant n,$$

and such that (3.11) has more than one solution. To see this, if δ_j satisfies (3.12), then from the definition of $\gamma_{p_j},$ there exists some continuous function $a(x),$ not identically zero, with $\int_\Omega a(x)\,dx \geqslant 0,$ and $\|a^{+}\|_{p_j} < \delta_j,$ such that the scalar problem (1.8) has nontrivial solutions. Then, to get our purpose, it is sufficient to take $G(x,u)=\frac{1}{2}\langle M(x)u,u\rangle,$ where $M(x)$ is a diagonal matrix with entries $m_{jj}(x)=a(x)$ and $m_{ii}(x)=\delta\in\mathbb{R}^{+},$ if $i\neq j,$ with δ sufficiently small.

REFERENCES

[1] P. W. BATES, *Solutions of nonlinear elliptic systems with meshed spectra,* Nonlinear Anal., 4 (1979), 1023–1030.

[2] R.C. BROWN AND D.B. HINTON, *Lyapunov inequalities and their applications, Survey on Classical Inequalities,* T.M. Rassias, ed. Kluwer, Dordrecht, 2000, 1–25.

[3] A. CAÑADA, J.A. MONTERO AND S. VILLEGAS, *Liapunov-type inequalities and Neumann boundary value problems at resonance,* Mathematical Inequalities and Applications, 8 (2005), 459–476.

[4] A. CAÑADA, J.A. MONTERO AND S. VILLEGAS, *Lyapunov inequalities for partial differential equations*, J. Funct. Anal., 237, (2006), 176–193.

[5] A. CAÑADA AND S. VILLEGAS, *Optimal Lyapunov inequalities for disfocality and Neumann boundary conditions using L^p norms,* Discrete Contin. Dyn. Syst. 20 (2008), 877–888.

[6] A. CAÑADA AND S. VILLEGAS, *Lyapunov inequalities for Neumann boundary conditions at higher eigenvalues,* to appear in J. Eur. Math. Soc. (JEMS).

[7] A. CAÑADA AND S. VILLEGAS, *Matrix Lyapunov inequalities for ordinary and elliptic partial differential equations,* preprint.

[8] G. CROCE AND B. DACOROGNA, *On a generalized Wirtinger inequality,* Discrete and Continuous Dynamical Systems, 9, (2003), 1329–1341.

[9] B. DACAROGNA, W. GANGBO AND N. SUBÍA, *Sur une généralisation de l'inégalité de Wirtinger,* Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 9, (1992), 29–50.

[10] C.L. DOLPH, *Nonlinear integral equations of Hammerstein type,* Trans. Amer. Math. Soc., 66, (1949), 289–307.

[11] D. G. DE FIGUEIREDO, *The Dirichlet problem for nonlinear elliptic equations: a Hilbert space approach,* Partial differential equations and related topics, Lecture Notes in Math. 446, Springer Verlag, (Berlin 1975), 144–165.

[12] B.J. HARRIS, *On an inequality of Lyapunov for disfocality,* J. Math. Anal. Appl., 146, (1990), 495–500.

[13] P. HARTMAN, *Ordinary Differential Equations,* John Wiley and Sons Inc., New York-London-Sydney, 1964.

[14] W. HUAIZHONG AND L. YONG, *Neumann boundary value problems for second-order ordinary differential equations across resonance,* SIAM J. Control and Optimization, 33, (1995), 1312–1325.

[15] J. L. KAZDAN AND F. W. WARNER, *Remarks on some queasilinear elliptic equations,* Comm. Pure Appl. Math. 28, (1975), 567–597.

[16] L. KOTIN, *A generalization of Liapunov's inequality,* J. Math. Anal. Appl., 102, (1984), 585–598.

[17] M.K. KWONG, *On Lyapunov's inequality for disfocality,* J. Math. Anal. Appl., 83, (1981), 486–494.

[18] A.M. LYAPUNOV, *Sur une série relative a la théorie des équations différentielles linéaires avec coeficient périodiques,* C.R. Acad. Sci. Paris, 123 (1896), 1248–1252.

[19] J. MAWHIN AND J.R. WARD,*Nonuniform nonresonance conditions at the two first eigenvalues for periodic solutions of forced Liénard and Duffing equations*, Rocky Mountain J. Math., 12, (1982), 643–654.

[20] J. MAWHIN AND J.R. WARD, *Periodic solutions of some forced Liénard differential equations at resonance*, Arch. Math (Basel), 41, (1983), 337–351.

[21] J. MAWHIN AND J.R. WARD, *Nonresonance and existence for nonlinear elliptic boundary value problems,* Nonlinear Anal., 5 (1981), 677–684.

[22] J. MAWHIN, J.R. WARD AND M. WILLEM, *Variational methods and semilinear elliptic equations,* Arch. Rational Mech. Anal., 95, (1986), 269–277.

[23] E.R. VAN KAMPEN AND A. WINTNER, *On an absolute constant in the theory of variational stability,* Amer. J. Math., 59 (1937), 270–274.

[24] L. YONG AND W. HUAIZHONG, Neumann problems for second order ordinary differential equations across resonance, Z. Angew Math. Phys. 46, (1995), 393–406.

[25] M. ZHANG, *Certain classes of potentials for p-Laplacian to be non-degenerate*, Math. Nachr., 278, (2005), 1823–1836. SI.

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