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HILBERT INEQUALITY AND GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. By using the integral representation of Gaussian hypergeometric function, we obtain Hilbert type inequalities with some fractional kernels and non-conjugate parameters. Such inequalities include the constant factors expressed in terms of hypergeometric functions. Further, we obtain the best possible constants for some general cases, in conjugate case.

1. Introduction

Let p and q be the real parameters such that

$$p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}\geqslant 1, \tag{1}$$

and let p' and q' respectively be their conjugate exponents, that is, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. It is obvious that $p',q'>1$.

Further, define

$$\lambda := \frac{1}{p'}+\frac{1}{q'} \tag{2}$$

and note that $0<\lambda\leqslant 1$, for all p and q as in (1). Especially, $\lambda=1$ holds if and only if $q=p'$, that is, only when p and q are mutually conjugate. Otherwise, we have $0<\lambda<1$, and in such case p and q will be referred to as non-conjugate exponents.

Considering p, q and λ as in (1) and (2), Hardy, Littlewood and Pólya (see [4]), proved that there exist a constant C dependent only on the parameters p and q, such that the following Hilbert-type inequality holds for all non-negative functions $f\in L^p(\mathbf{R}_+)$ and $g\in L^q(\mathbf{R}_+)$:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy\leqslant C||f||_{L^p(\mathbf{R}_+)}||g||_{L^q(\mathbf{R}_+)}. \tag{3}$$

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However, the original proof did not bring any information about the value of the best possible constant C. That problem was improved by Levin, [8], who obtained an explicit upper bound for C,

$$C \leqslant \left(\pi \operatorname{cosec}\frac{\pi}{\lambda p'}\right)^{\lambda}. \tag{4}$$

This was an interesting result since the right hand side of (4) reduces to the previously known sharp constant $\pi \operatorname{cosec}\frac{\pi}{p'}$ in the conjugate case.

A simpler proof of (3), based on a single application of Hölder's inequality, was given later by Bonsall, [2]. He reduced the case of two non-conjugate parameters to the case of three conjugate parameters. Bonsall has also generalized such conditions to the case of n non-conjugate parameters. Bonsall's idea, used in the proof of (3), has guided us in the research we present here.

During decades, the Hilbert-type inequalities were generalized in many different directions and also the numerous mathematicians reproved them using various technics. Some possibilities of generalizing such inequalities are, for example, various choices of non-negative measures, kernels, sets of integration, extension to multi-dimensional case etc.

Some recent generalizations and extensions of Hilbert's inequality ([5, 6, 7], see also [10]) include the constants which are expressed in terms of beta and gamma functions.

The main objective in this paper are some new Hilbert type inequalities with fractional kernels, which involve the constants expressed in terms of hypergeometric functions. Namely, by using integral representation of Gaussian hypergeometric function we obtain such inequalities. We also consider equivalent form of Hilbert-type inequality, usually called Hardy-Hilbert type inequality, which includes only one measurable function (see [4]).

We also consider the problem of the best possible constant factor in Hilbert-type inequalities, but only in the conjugate case. As we know, the problem of the best possible constant in (3) seems to be very difficult and still remains open. We obtain the best possible constant, in conjugate case, for some general cases.

Before presenting our idea and results, we introduce the notion of Gaussian hypergeometric functions.

2. Hypergeometric functions

We consider hypergeometric series in a power series in z with three parameters, defined as follows in terms of rising factorial powers:

$$F\left(\begin{matrix} a,b \\ c \end{matrix} \,\middle|\, z\right) = \sum_{k\geqslant 0} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}}} \cdot \frac{z^k}{k!},\ a,b,c,z \in \mathbf{R},\ |z| < 1. \tag{5}$$

To avoid division by zero, c is neither zero nor negative integer. The series (5) is often called Gaussian hypergeometric, because many of its subtle properties were first proved by Gauss. In fact, it was the only hypergeometric series until the second half of nineteenth century, when everything was generalized to arbitrary number of upper and lower parameters. Although, hypergeometric series are defined for complex numbers, and many properties hold even for complex numbers, we shall only be interested in the case of real numbers. For more results and properties about general hypergeometric functions reader can consult [3].

Gaussian hypergeometric function can be expressed as an integral, in the following way (see [1]):

$$F\left(\begin{matrix} a,b \\ c \end{matrix} \,\middle|\, z\right) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \tag{6}$$

where $c > b > 0$, $|z| < 1$ and B is standard beta function.

We shall use previous integral to obtain some inequalities of Hilbert type with fractional kernels. In the next lemma we give a form of (6), more suitable for our computation.

LEMMA 1. *Suppose $a,b,c,\alpha,\gamma \in \mathbf{R}$ are such that $a+c > b > 0$ and $0 < \alpha < 2\gamma$. Then*

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a(1+\gamma x)^c}dx = \gamma^{-b}B(b,a+c-b)F\left(\begin{matrix} a,b \\ a+c \end{matrix} \,\middle|\, \frac{\gamma-\alpha}{\gamma}\right). \tag{7}$$

Proof. We start with the integral $I = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$. By using the substitutions $1-t = \frac{1}{1+u}$, $u = \gamma x$, $\gamma > 0$, and a label $\alpha = (1-z)\gamma$ we obtain

$$I = \gamma^b \int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a(1+\gamma x)^{c-a}}dx.$$

Now, by using the relation (6) we have

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a(1+\gamma x)^{c-a}}dx = \gamma^{-b}B(b,c-b)F\left(\begin{matrix} a,b \\ c \end{matrix} \,\middle|\, \frac{\gamma-\alpha}{\gamma}\right).$$

Finally, if we replace $c-a$ with c in the previous formula, we obtain (7). $\square$

REMARK 1. By changing the roles of the parameters a and c, and also α and γ, where $0 < \gamma < 2\alpha$, we obtain relation

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a(1+\gamma x)^c}dx = \alpha^{-b}B(b,a+c-b)F\left(\begin{matrix} c,b \\ a+c \end{matrix} \,\middle|\, \frac{\alpha-\gamma}{\alpha}\right). \tag{8}$$

By equalizing the identities (7) and (8) we obtain the relation

$$F\left(\begin{matrix} a,b \\ a+c \end{matrix} \,\middle|\, \frac{\gamma-\alpha}{\gamma}\right) = \left(\frac{\gamma}{\alpha}\right)^b F\left(\begin{matrix} c,b \\ a+c \end{matrix} \,\middle|\, \frac{\alpha-\gamma}{\alpha}\right). \tag{9}$$

Previous relation gives so called Pfaff's reflection law, discovered in 1797. Let's state that transformation in more appropriate form:

$$F\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,\frac{-z}{1-z}\right)=(1-z)^aF\left(\begin{matrix}a,c-b\\c\end{matrix}\,\middle|\,z\right),\text{ where } z\neq 1. \tag{10}$$

This is formal identity in power series. On the other hand, whenever we replace z by a particular numerical value, we have to be sure that the infinite sum is well defined. Since $\left|\frac{\alpha-\gamma}{\alpha}\right|<1$ and $\left|\frac{\gamma-\alpha}{\gamma}\right|<1$, the relation (9) is well defined.

Further, by applying reflection law again to (10), we obtain so called Euler's identity

$$F\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,z\right)=(1-z)^{c-a-b}F\left(\begin{matrix}c-a,c-b\\c\end{matrix}\,\middle|\,z\right). \tag{11}$$

We shall use transformations (10) and (11) in obtaining the best possible constants. □

REMARK 2. Every hypergeometric series always has the value 1 when $z=0$. Gaussian hypergeometric series (5) converges also for $z=1$ if b is non-positive integer or $c>a+b$. Furthermore, the following identity holds (see [3]):

$$F\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,1\right)=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \tag{12}$$

where Γ is gamma function. So, by using (12), the relation (7) also holds for $\alpha=0$ and reduces to the well known formula for beta function:

$$\int_0^\infty \frac{x^{b-1}}{(1+\gamma x)^c}dx=\gamma^{-b}B(b,c-b),\quad \text{where } c>b>0. \tag{13}$$

So, we shall omit such special cases in this paper, because they reduce to many previously known results from the literature. □

3. Main results

Our main goal in this section is to find further generalizations of the inequality (3). Hence, we shall replace the kernel $(x+y)^{-1}$ contained in the left-hand side of (3), with the kernel

$$K(x,y)=(x+\alpha_1 y)^{-s_1}(x+\alpha_2 y)^{-s_2}, \tag{14}$$

where $\alpha_1,\alpha_2>0$, $\frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2$ and $s_1+s_2>0$.

Before we state and prove the main result, we need to define some weighted functions.

We define $F:(0,\infty)\mapsto \mathbf{R}$ by

$$F(x)=\left[\int_0^\infty \frac{y^{-q'A_2}}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}dy\right]^{\frac{1}{q'}}, \tag{15}$$

where $\alpha_1,\alpha_2>0,\ \frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2,\ s_1+s_2>0$ and $A_2\in\left(\frac{1-s_1-s_2}{q'},\frac{1}{q'}\right)$.

We also define $G:(0,\infty)\mapsto\mathbf{R}$ by

$$G(y)=\left[\int_0^\infty\frac{x^{-p'A_1}}{(x+\alpha_1y)^{s_1}(x+\alpha_2y)^{s_2}}dx\right]^{\frac{1}{p'}}, \tag{16}$$

where $\alpha_1,\alpha_2>0,\ \frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2,\ s_1+s_2>0$ and $A_1\in\left(\frac{1-s_1-s_2}{p'},\frac{1}{p'}\right)$.

Clearly, by using Lemma 1 we can compute the integrals in definitions of weighted functions F and G. So, we easily obtain the following result:

LEMMA 2. *Suppose $\alpha_1,\alpha_2>0,\ \frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2$ and $s_1+s_2>0$. Further, let A_1 and A_2 be real parameters such that $A_1\in(\frac{1-s_1-s_2}{p'},\frac{1}{p'})$ and $A_2\in(\frac{1-s_1-s_2}{q'},\frac{1}{q'})$. If the functions F and G are defined by (15) and (16) respectively, then*

$$F(x)=k(F)\cdot x^{\frac{1-s_1-s_2}{q'}-A_2}, \tag{17}$$

$$G(y)=k(G)\cdot y^{\frac{1-s_1-s_2}{p'}-A_1}, \tag{18}$$

where

$$k(F)=\alpha_2^{A_2-\frac{1}{q'}}B^{\frac{1}{q'}}(1-q'A_2,s_1+s_2+q'A_2-1)F^{\frac{1}{q'}}\left(\begin{matrix}s_1,1-q'A_2\\ s_1+s_2\end{matrix}\,\middle|\,\frac{\alpha_2-\alpha_1}{\alpha_2}\right),$$

$$k(G)=\alpha_1^{\frac{1-s_1}{p'}-A_1}\alpha_2^{-\frac{s_2}{p'}}B^{\frac{1}{p'}}(1-p'A_1,s_1+s_2+p'A_1-1)F^{\frac{1}{p'}}\left(\begin{matrix}s_2,1-p'A_1\\ s_1+s_2\end{matrix}\,\middle|\,\frac{\alpha_2-\alpha_1}{\alpha_2}\right).$$

Now we are able to state and prove our main result. We suppose that all integrals converges and shall omit these types of conditions.

THEOREM 1. *Let parameters $p,\ q,\ p',\ q',\ \lambda$ be as in (1) and (2). Suppose $\alpha_1,\alpha_2>0,\ \frac{1}{2}<\frac{\alpha_1}{\alpha_2}<2$ and $s_1+s_2>0$. If f and g are non-negative measurable functions on $(0,\infty)$, then the following inequalities*

$$\begin{aligned}\int_0^\infty\int_0^\infty&\frac{f(x)g(y)}{(x+\alpha_1y)^{\lambda s_1}(x+\alpha_2y)^{\lambda s_2}}dxdy\\ &\leqslant K\left[\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\\ &\quad\cdot\left[\int_0^\infty y^{\frac{q}{p'}(1-s_1-s_2)+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}},\end{aligned} \tag{19}$$

$$\begin{aligned}&\left[\int_0^\infty y^{\frac{q'}{p'}(s_1+s_2-1)+q'(A_1-A_2)}\left(\int_0^\infty\frac{f(x)}{(x+\alpha_1y)^{\lambda s_1}(x+\alpha_2y)^{\lambda s_2}}dx\right)^{q'}dy\right]^{\frac{1}{q'}}\\ &\qquad\leqslant K\left[\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\end{aligned} \tag{20}$$

hold for any $A_1 \in (\frac{1-s_1-s_2}{p'}, \frac{1}{p'})$ *and* $A_2 \in (\frac{1-s_1-s_2}{q'}, \frac{1}{q'})$, *where*

$$K = \alpha_1^{\frac{1-s_1}{p'}-A_1}\alpha_2^{A_2-\frac{1}{q'}-\frac{s_2}{p'}} B^{\frac{1}{q'}}(1-q'A_2, s_1+s_2+q'A_2-1) B^{\frac{1}{p'}}(1-p'A_1, s_1+s_2+p'A_1-1)$$

$$\cdot F^{\frac{1}{q'}}\left(\begin{matrix} s_1, 1-q'A_2 \\ s_1+s_2 \end{matrix} \,\Big|\, \frac{\alpha_2-\alpha_1}{\alpha_2}\right) F^{\frac{1}{p'}}\left(\begin{matrix} s_2, 1-p'A_1 \\ s_1+s_2 \end{matrix} \,\Big|\, \frac{\alpha_2-\alpha_1}{\alpha_2}\right). \tag{21}$$

Moreover, inequalities (19) *and* (20) *are equivalent.*

Proof. The left-hand side of (19) can be written as

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dxdy = \int_0^\infty\int_0^\infty I_1^{\frac{1}{q'}}(x,y) I_2^{\frac{1}{p'}}(x,y) I_3^{1-\lambda}(x,y)dxdy$$

where

$$\begin{aligned} I_1(x,y) &= \frac{F^{p-q'}(x)f^p(x)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}} \cdot \frac{x^{pA_1}}{y^{q'A_2}}, \\ I_2(x,y) &= \frac{G^{q-p'}(y)g^q(y)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}} \cdot \frac{y^{qA_2}}{x^{p'A_1}}, \\ I_3(x,y) &= x^{pA_1}F^p(x)y^{qA_2}G^q(y)f^p(x)g^q(y) \end{aligned}$$

and the functions F and G are defined by (15) and (16). Now, since the exponents satisfy identity $\frac{1}{p'}+\frac{1}{q'}+(1-\lambda)=1$, we can use Hölder's inequality with the exponents p', q' and $\frac{1}{1-\lambda}$ on the above transformation. So by using that inequality, Fubini's theorem and the definitions (15) and (16) of the functions F and G respectively, we obtain the inequality:

$$\begin{aligned} &\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dxdy \\ &\quad\leqslant \left[\int_0^\infty x^{pA_1}F^p(x)f^p(x)dx\right]^{\frac{1}{q'}} \left[\int_0^\infty y^{qA_2}G^q(y)g^q(y)dy\right]^{\frac{1}{p'}} \\ &\quad\cdot \left[\left(\int_0^\infty x^{pA_1}F^p(x)f^p(x)dx\right)\left(\int_0^\infty y^{qA_2}G^q(y)g^q(y)dy\right)\right]^{1-\lambda}. \end{aligned}$$

Now, by using Lemma 2 and since $\frac{1}{q'}+1-\lambda=\frac{1}{p}$, $\frac{1}{p'}+1-\lambda=\frac{1}{q}$, the inequality (19) holds.

Let us show that the inequalities (19) and (20) are equivalent. Suppose that the inequality (19) is valid. If we put the function $\widetilde{g}:(0,\infty)\mapsto \mathbf{R}$, defined by

$$\widetilde{g}(y) = y^{\frac{q'}{p'}(s_1+s_2-1)+q'(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dx\right)^{\frac{q'}{q}}$$

into the inequality (19), we obtain

$$\int_0^\infty y^{\frac{q'}{p'}(s_1+s_2-1)+q'(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dx\right)^{q'}dy$$
$$=\int_0^\infty\int_0^\infty \frac{f(x)\widetilde{g}(y)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dxdy$$
$$\leqslant K\left[\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}$$
$$\cdot\left[\int_0^\infty y^{\frac{q'}{p'}(s_1+s_2-1)+q'(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dx\right)^{q'}dy\right]^{\frac{1}{q}},$$

what gives inequality (20).

It remains to prove that the inequality (19) is a consequence of the inequality (20). Then the left hand side of the inequality (19) can be transformed in the following way:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dxdy$$
$$=\int_0^\infty y^{\frac{1-s_1-s_2}{p'}+A_2-A_1}g(y)\left(y^{\frac{s_1+s_2-1}{p'}+A_1-A_2}\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dx\right)dy.$$

Now, by applying Hölder's inequality with conjugate exponents q and q' on that transformation, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dxdy$$
$$\leqslant\left[\int_0^\infty y^{\frac{q}{p'}(1-s_1-s_2)+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}$$
$$\cdot\left[\int_0^\infty y^{\frac{q'}{p'}(s_1+s_2-1)+q'(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}}dx\right)^{q'}dy\right]^{\frac{1}{q'}},$$

and the result follows from (20). Hence, we have showed that the inequalities (19) and (20) are equivalent, what completes the proof. □

Clearly, if $\alpha_1=\alpha_2$ in Theorem 1, then the hypergeometric part of the constant K takes value 1 (see Remark 2) and the inequalities (19) and (20) reduces to some previously known from the literature. Such results can be found in the papers [5, 6, 7] for appropriate choices of the real parameters A_1 and A_2. For example, by putting $\alpha_1=\alpha_2=1$, $s_1=s_2=\frac{1}{2}$, $A_1=A_2=\frac{1}{\lambda p'q'}$ in Theorem 1, the constant K reduces to $(\pi\operatorname{cosec}\frac{\pi}{\lambda p'})^\lambda$, so we obtain the inequality (3).

On the other hand, if $\alpha_1=0$ or $\alpha_2=0$, then by using the relation (13), we can obtain the inequalities involving the constant expressed only in terms of beta function. Here, they are omitted.

It is very interesting to investigate under which assumptions holds the equality in (19) and (20). The answer on that question is given in the following result:

THEOREM 2. *The equality in (*19*) and (*20*) holds if and only if at least one of the functions f or g is equal to zero.*

Proof. The equality in (19) holds if and only if the functions $I_1(x,y)$, $I_2(x,y)$ and $I_3(x,y)$, defined in proof of Theorem 1, are effectively proportional. Hence, there exist non-negative real numbers A and B not both equal to zero, such that

$$\frac{F^{p-q'}(x)f^p(x)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}\cdot\frac{x^{pA_1}}{y^{q'A_2}}=A\frac{G^{q-p'}(y)g^q(y)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}\cdot\frac{y^{qA_2}}{x^{p'A_1}} \tag{22}$$

and

$$\frac{F^{p-q'}(x)f^p(x)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}\cdot\frac{x^{pA_1}}{y^{q'A_2}}=Bx^{pA_1}F^p(x)y^{qA_2}G^q(y)f^p(x)g^q(y). \tag{23}$$

Now, if we suppose that the functions f and g are not equal to zero, from the condition (22), we have

$$x^{(p+p')A_1}F^{p-q'}(x)f^p(x)=Ay^{(q+q')A_2}G^{q-p'}(y)g^q(y). \tag{24}$$

Since the left-hand side of the previous equality is dependent only on the variable x, while the right-hand side is dependent only on y, it follows that the both sides of the equality (24) are constant. By applying that fact to the condition (23), we obtain that there exist constant C such that

$$\frac{1}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}}=CF^{q'}(x)G^{p'}(y).$$

That is a contradiction, since on the right-hand side we have a function with a separated variables. □

4. The best possible constants in the conjugate case

In this section we shall take an attention to the case of conjugate exponent, to obtain the best possible constants in Theorem 1, for some general cases. As we know, in the conjugate case holds $p'=q$, $q'=p$ and $\lambda=1$.

However, we shall deal with an appropriate forms of the inequalities obtained in the previous section, in conjugate case. The main idea is to simplify the constant K defined by (21), i.e. to obtain the constant without exponents. For that sake, it is natural to consider the real parameters A_1 and A_2 satisfying

$$pA_2+qA_1=2-s_1-s_2. \tag{25}$$

So, if the parameters A_1 and A_2 satisfy constraint (25) then the constant K becomes

$$K=\alpha_1^{\frac{1-s_1}{q}-A_1}\alpha_2^{A_2-\frac{1}{p}-\frac{s_2}{q}}B(1-pA_2,1-qA_1)$$
$$\cdot F^{\frac{1}{p}}\left(\begin{matrix}s_1,1-pA_2\\ s_1+s_2\end{matrix}\,\middle|\,\frac{\alpha_2-\alpha_1}{\alpha_2}\right)F^{\frac{1}{q}}\left(\begin{matrix}s_2,1-qA_1\\ s_1+s_2\end{matrix}\,\middle|\,\frac{\alpha_2-\alpha_1}{\alpha_2}\right).$$

Further, if we apply Pfaff's reflection law (10) twice, i.e. Euler identity (11), the constant K reduces to

$$K^* = {\alpha_2}^{pA_2-1}B(1-pA_2, 1-qA_1)F\left(\begin{matrix} s_1, 1-pA_2 \\ s_1+s_2 \end{matrix} \,\middle|\, \frac{\alpha_2-\alpha_1}{\alpha_2}\right), \tag{26}$$

and we shall see that K^* is the best possible constant. In that case inequalities (19) and (20) become

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}} dxdy \\ &\qquad \leqslant K^* \left[\int_0^\infty x^{pqA_1-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{pqA_2-1} g^q(y)dy\right]^{\frac{1}{q}} \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\int_0^\infty y^{p-1-p^2A_2} \left(\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}} dx\right)^p dy \\ &\qquad \leqslant K^* \int_0^\infty x^{pqA_1-1} f^p(x)dx. \end{aligned} \tag{28}$$

In what follows, we shall see that the constant K^* in (27) and (28) is the best possible in the sense that one can't replace that constant K^* in inequalities (27) and (28) with the smaller constant, so that inequalities are fulfilled for all non-negative measurable functions on $(0,\infty)$.

THEOREM 3. *If conjugate parameters satisfy constraint* (25)*, then the constant* K^* *is the best possible in both inequalities* (27) *and* (28)*.*

Proof. Let's suppose that the constant factor K^* given by (26) is not the best possible in the inequality (27). Then, there exist a positive constant $K_1 < K^*$, such that (27) is still valid when we replace K^* by K_1.

For this purpose, with $0<\varepsilon<1$, set $f(x) = x^{-qA_1}$ in $(\varepsilon, \frac{1}{\varepsilon})$, $f(x)=0$ elsewhere, and $g(y) = y^{-pA_2}$ in $(\varepsilon, \frac{1}{\varepsilon})$, $g(y)=0$ elsewhere.

Now, we shall put these functions in the inequality (27) with the constant K_1. Since the parameters A_1 and A_2 satisfy (25) and by using substitution $y = xt$, the left-hand side of the inequality (27) becomes

$$I = \int_\varepsilon^{\frac{1}{\varepsilon}} \int_\varepsilon^{\frac{1}{\varepsilon}} \frac{x^{-qA_1}y^{-pA_2}}{(x+\alpha_1 y)^{s_1}(x+\alpha_2 y)^{s_2}} dxdy = \int_\varepsilon^{\frac{1}{\varepsilon}} \frac{dx}{x} \int_{\frac{\varepsilon}{x}}^{\frac{1}{\varepsilon x}} \frac{t^{-pA_2}}{(1+\alpha_1 t)^{s_1}(1+\alpha_2 t)^{s_2}} dt.$$

Further, the left-hand side of (27) can be written as

$$I = \int_\varepsilon^{\frac{1}{\varepsilon}} \frac{dx}{x} \int_0^\infty \frac{t^{-pA_2}}{(1+\alpha_1 t)^{s_1}(1+\alpha_2 t)^{s_2}} dt - R_1 - R_2,$$

where

$$R_1 = \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x} \int_0^{\frac{\varepsilon}{x}} \frac{t^{-pA_2}}{(1+\alpha_1 t)^{s_1}(1+\alpha_2 t)^{s_2}} dt$$

$$R_2 = \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x} \int_{\frac{1}{\varepsilon x}}^{\infty} \frac{t^{-pA_2}}{(1+\alpha_1 t)^{s_1}(1+\alpha_2 t)^{s_2}} dt.$$

Now, by using Lemma 1, straightforward computations shows that

$$\int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x} \int_0^{\infty} \frac{t^{-pA_2}}{(1+\alpha_1 t)^{s_1}(1+\alpha_2 t)^{s_2}} dt = 2\ln\left(\frac{1}{\varepsilon}\right) K^*,$$

so, the left-hand side of (27) is

$$I = 2\ln\left(\frac{1}{\varepsilon}\right) K^* - R_1 - R_2. \tag{29}$$

Since the function $f(t) = (1+\alpha_1 t)^{-s_1}(1+\alpha_2 t)^{-s_2}$ is continuous on interval $[0,\infty)$ and since $f(0) = 1$, $\lim_{t\to\infty} f(t) = 0$, it follows that there exist the positive constant M such that $f(t) < M$, $t \in [0,\infty)$. Hence,

$$R_1 < M \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x} \int_0^{\frac{\varepsilon}{x}} t^{-pA_2} dt = M \frac{1-\varepsilon^{s_1+s_2+qA_1-pA_2}}{(1-pA_2)^2} = M \frac{1-\varepsilon^{2(1-pA_2)}}{(1-pA_2)^2}. \tag{30}$$

On the other hand, since $g(t) = (1+\alpha_1 t)^{s_1}(1+\alpha_2 t)^{s_2} = O(t^{s_1+s_2})$, there exist the positive constant N such that

$$R_2 < N \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x} \int_{\frac{1}{\varepsilon x}}^{\infty} t^{-pA_2-s_1-s_2} dt = N \frac{1-\varepsilon^{s_1+s_2-qA_1+pA_2}}{(1-qA_1)^2} = N \frac{1-\varepsilon^{2(1-qA_1)}}{(1-qA_1)^2}. \tag{31}$$

Now, by using (29), (30) and (31) we have inequality

$$I > 2\ln\left(\frac{1}{\varepsilon}\right) K^* - M \frac{1-\varepsilon^{2(1-pA_2)}}{(1-pA_2)^2} - N \frac{1-\varepsilon^{2(1-qA_1)}}{(1-qA_1)^2}. \tag{32}$$

For above choice of functions f and g, the right-hand side of the inequality (27), with the constant K_1, become $2\ln\left(\frac{1}{\varepsilon}\right) K_1$. So, by using (32) we have

$$-M \frac{1-\varepsilon^{2(1-pA_2)}}{(1-pA_2)^2} - N \frac{1-\varepsilon^{2(1-qA_1)}}{(1-qA_1)^2} < 2(K_1 - K^*) 2\ln\left(\frac{1}{\varepsilon}\right), \tag{33}$$

and we obtain contradiction by letting $\varepsilon \to 0$. Hence, the constant factor in the inequality (27) is the best possible.

Finally, equivalence of the inequalities (27) and (28) means that the constant K^* is also the best possible in the inequality (28). That completes the proof. □

5. Some examples

If we put $A_1 = \frac{1-s_1}{p'}$, $A_2 = \frac{1-s_2}{q'}$, $s_1, s_2 > 0$ in Theorem 1 we obtain:

COROLLARY 1. *Let parameters p, q, p', q', λ be as in (1) and (2). Suppose $\alpha_1, \alpha_2 > 0$, $\frac{1}{2} < \frac{\alpha_1}{\alpha_2} < 2$ and $s_1, s_2 > 0$. If f and g are non-negative measurable functions on $(0,\infty)$, then the following inequalities hold and are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}} dxdy \leqslant L\left[\int_0^\infty x^{-\lambda s_1 p+p-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{-\lambda s_2 q+q-1} g^q(y)dy\right]^{\frac{1}{q}}, \tag{34}$$

$$\left[\int_0^\infty y^{\lambda s_2 q'-1}\left(\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}} dx\right)^{q'} dy\right]^{\frac{1}{q'}} \leqslant L\left[\int_0^\infty x^{-\lambda s_1 p+p-1} f^p(x)dx\right]^{\frac{1}{p}}, \tag{35}$$

where the constant L is defined by

$$L = \alpha_2{}^{-\lambda s_2} B^\lambda(s_1,s_2) F^\lambda\left(\begin{array}{c} s_1, s_2 \\ s_1+s_2 \end{array}\middle| \frac{\alpha_2-\alpha_1}{\alpha_2}\right). \tag{36}$$

□

Further, if we put $A_1 = \frac{2-s_1-s_2}{2p'}$, $A_2 = \frac{2-s_1-s_2}{2q'}$ in Theorem 1 we obtain the inequalities similar as those in Corollary 1, with the constant M instead L, defined by

$$M = \alpha_2{}^{-\frac{\lambda}{2}(s_1+s_2)} B^\lambda\left(\frac{s_1+s_2}{2}, \frac{s_1+s_2}{2}\right) F^\lambda\left(\begin{array}{c} s_1, \frac{s_1+s_2}{2} \\ s_1+s_2 \end{array}\middle| \frac{\alpha_2-\alpha_1}{\alpha_2}\right). \tag{37}$$

REMARK 3. Both pairs of the parameters A_1 and A_2, defined in this section, satisfy constraint $p'A_1 + q'A_2 = 2 - s_1 - s_2$, which reduces to (25) in the conjugate case. Hence, in the conjugate case, the constants L and M are the best possible.

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