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A NOTE ON CERTAIN MAPS BETWEEN ORDERED FIELDS

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Dedicated to J. Pečarić
on the occasion of his 60th birthday

Abstract. We characterize maps f between ordered fields satisfying one of the following five sets of conditions for all x,y:
(A) $f(xy) = f(x)f(y)$, and $f(x+y) \leqslant f(x)+f(y)$,
(B1) $f(xy) = f(x)f(y)$, and $f(x+y) \geqslant f(x)+f(y)$,
(B2) $f(xy) \leqslant f(x)f(y)$, and $f(x+y) = f(x)+f(y)$,
(B3) $f(xy) \geqslant f(x)f(y)$, and $f(x+y) = f(x)+f(y)$.
(C) $f(xy) \geqslant f(x)f(y)$, and $f(x+y) \geqslant f(x)+f(y)$.
Also we pose a problem.

The aim of this note is to comment some obstructions to the definition of field homomorphisms in ordered fields. For generalities on ordered fields see [2], [5] or [9].

Let K,L be two fields. Recall that a nonzero map $f : K \to L$ is a homomorphism if

$$f(x+y) = f(x)+f(y), \text{ and } f(xy) = f(x)f(y),$$

for all $x,y \in K$. Assume that L is an ordered field. Then f is an absolute value (on K with values in L) if

(N1) $f(0) = 0$ and $f(x) > 0$ for $x \neq 0$.
(N2) $f(xy) = f(x)f(y)$ for all x,y.
(N3) $f(x+y) \leqslant f(x)+f(y)$, for all $x,y \in K$.

THEOREM 1. *Let K be a field, let L be an ordered field and let $f : K \to L$ be a nonzero map.*

(A) Assume that $f(xy) = f(x)f(y)$, and $f(x+y) \leqslant f(x)+f(y)$, for all x,y in K. Then f is an injective homomorphism of fields (embedding), a constant map $f(x) := 1$ for all x, or an absolute value.

(B) Assume that f satisfies one of the following three conditions for all x,y in K:
(B1) $f(xy) = f(x)f(y)$, and $f(x+y) \geqslant f(x)+f(y)$,
(B2) $f(xy) \leqslant f(x)f(y)$, and $f(x+y) = f(x)+f(y)$,
(B3) $f(xy) \geqslant f(x)f(y)$, and $f(x+y) = f(x)+f(y)$.
Then f is an injective homomorphism of fields.

Proof. (A) By $f(xy) = f(x)f(y)$ and $f \neq 0$ we get $f(1) = 1$ and $f(-1) = \pm 1$. Also, from $f(0) = f(0)f(x)$ we get that $f(x) = 1$ for all x (which satisfies the conditions) or $f(0) = 0$. From now we have $f(0) = 0$.

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If $f(-1)=-1$ then $f(-x)=-f(x)$ for all x, hence

$$f(x)=f(x+y-y)\leqslant f(x+y)-f(y), \text{ and so } f(x+y)\geqslant f(x)+f(y), \text{ for all } x,y\in K.$$

Now f is a homomorphism of fields. It is injective since it is nonzero. Namely, if $f(t)=0$ for some $t\neq 0$, then $f(x)=f(xtt^{-1})=f(x)f(t)f(t^{-1})=0$ for all x.

If $f(-1)=1$ then $f(-x)=f(x)$ for all x. Therefore $0=f(0)\leqslant f(-x)+f(x)=2f(x)$, which implies that $f(x)\geqslant 0$ for all x, and so f is an absolute value.

(B1) Similarly as in (A) if $f(-1)=-1$ then f is an injective homomorphism of fields. Namely $f(x)=f(x+y-y)\geqslant f(x+y)-f(y)$, and so $f(x+y)\leqslant f(x)+f(y)$, for all $x,y\in K$.

The case $f(-1)=1$ is impossible. Namely, it implies $f(-x)=f(x)$ for all x, from which we get $0=f(0)=f(-x+x)\geqslant 2f(x)$, i.e. $f(x)\leqslant 0$ for all $x\in K$ (a contradiction with $f(1)=1$).

(B2) By $f(x+y)=f(x)+f(y)$ we get $f(0)=0$, from which it follows $f(-x)=-f(x)$ for all $x\in K$. Now, from $f(-xy)\leqslant f(-x)f(y)$, we get $f(xy)\geqslant f(x)f(y)$, hence f is a homomorphism of fields.

(B3) Analogously as $B2$. □

REMARK 1. The characterization of the maps f from the theorem is far away from an explicit description. For example, if $K=\mathbf{Q}$, the field of rational numbers, then the identity is the unique embedding of K in $\mathbf{R}$. Further, the absolute values (with values in the field $\mathbf{R}$ of real numbers) are described by the Ostrowski theorem (see [3] or [5]):

Each absolute value f on $\mathbf{Q}$ is either the trivial absolute value defined by $f(x)=1$ for $x\neq 0$ and $f(0)=0$, or equivalent to the ordinary absolute value, i.e. of the form $f(x)=|x|^{\alpha}$ with α real and $0<\alpha\leqslant 1$, and $|\ |$ the ordinary absolute value (the Archimedean case) or equivalent to a p-adic absolute value, i.e. of the form $f(x)=c^{v_p(x)}$, for some prime number p, where v_p is the discrete valuation at p and c is real number with $0<c<1$ (non Archimedean or p-adic cases).

Here if $x\neq 0$ and $x=p^r\frac{m}{n}$ with m,n relatively prime and not divisible by p we define $v_p(x):=r$. Also, we define $v_p(0)=+\infty$. The p-adic absolute value $|\ |_p$ is defined by $|x|_p:=p^{-v_p(x)}$.
We have an analogous description for all algebraic number fields K (of a finite degree over $\mathbf{Q}$). If n is the degree of K over $\mathbf{Q}$, then there are exactly n embeddings of K in $\mathbf{C}$ (and at most n real embeddings). Also, there is an extension of the Ostrowski theorem (here prime ideals of the ring of integers stay instead of prime numbers).

REMARK 2. The field K should not be necessarily ordered. However, in case (B) of Theorem 1., K has a natural ordering if we regard it as a subfield of L (under the embedding f). In case (A) we have a new moment. For example, if $K:=\mathbf{Q}(i)$ the field of Gaussian numbers, then K can not be ordered, and so there is no nontrivial homomorphism of K into an ordered field. Nevertheless, K has a lot of absolute values (which are explicitly described by an extension of the Ostrowski theorem).

REMARK 3. The situation becomes more complicated if we allow to add transcendental numbers. Assume, for example, that $K=\mathbf{Q}(T)$ is the field of rational functions over $\mathbf{Q}$, and that $f : \mathbf{Q}(T) \to \mathbf{R}$ is an absolute value. Then, by Remark 1, the restriction of f on $\mathbf{Q}$ is

(i) the trivial absolute value,

(ii) equivalent to the ordinary absolute value,

(iii) or equivalent to a p-adic absolute value.

For (i) there is an analogue of the Ostrowski theorem, which gives an explicit description of all such f.

For (ii) note that for every complex transcendental number α there is a field isomorphism $\mathbf{Q}(T) \cong \mathbf{Q}(\alpha)$. Composing by the ordinary absolute value on $\mathbf{C}$ we get different absolute values on $\mathbf{Q}(T)$ for different complex-conjugate pairs $\{\alpha, \bar{\alpha}\}$.

The following construction provides new examples. Let $\alpha = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be a real Hamilton's quaternion such that the set $\{1, \alpha, \alpha^2, ...\}$ is linearly independent over $\mathbf{Q}$. We define an absolute value $|\ |$ on the ring $\mathbf{Q}[T]$ by $|g(T)| := ||g(\alpha)||$, where $||\ ||$ is the norm on the algebra of real quaternions, defined by

$$||d_0 + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}|| := \sqrt{d_0^2 + d_1^2 + d_2^2 + d_3^2}.$$

It is easy to see that this absolute value has a unique extension from the ring $\mathbf{Q}[T]$ to the field $\mathbf{Q}(T)$. However, we may use suitable generalized quaternions, too (see, for example, [6]). Note also that the construction works on the algebra of real octonions (Cayley algebra), which is a normed division algebra (and provides new examples). Namely, although this algebra is not associative, the ring $\mathbf{Q}[\alpha]$ is well-defined for each octonion α.

For (iii) note that there are infinitely many embeddings of K in the field $\mathbf{Q}_p$ of p-adic numbers. Similarly as in (ii) we may compose by p-adic absolute values. However, neither this list is complete. To see it we may use the fact that the fields $\mathbf{Q}_p$ are not algebraically closed. To be more precise, the p-adic absolute value has a unique extension to the algebraic closure $\bar{\mathbf{Q}}_p$. However, $\bar{\mathbf{Q}}_p$ is not complete, but its completion C_p is both complete and algebraic closed. We may carry out the above construction with transcendental elements of C_p.

Note that in this setting there are essentially different examples. For instance, we may extend a p-adic absolute value $|\ |_p$ on $\mathbf{Q}$ to the ring of polynomials $\mathbf{Q}[T]$ as follows. Let $g(T) := b_0 + b_1T + ... + b_nT^n$ be a polynomial over $\mathbf{Q}$. We define

$$|g(T)|_p := \max\{|b_0|_p, ..., |b_n|_p\}$$

(here we note that p-adic absolute values satisfy the ultrametric inequality: $|x+y|_p \leqslant \max\{|x|_p, |y|_p\}$). This absolute value has a unique extension from the ring $\mathbf{Q}[T]$ to the field $\mathbf{Q}(T)$.

Finally, let us note that the situation becomes simple if the field of constants is finite. For example if $K := \mathbf{F}_p(T)$, for a prime number p, then there is no homomorphism from K to $\mathbf{R}$. Further, each absulute value should be trivial on $\mathbf{F}_p$, and one may prove that there is a full analogue of the Ostrowski theorem which describes completely all absolute values on K.

Recall that the map f was given by one equality and one inequality. It is a question what happens if we replace the equality with an inequality. The following theorem is a stronger version of Theorem 1 (B3).

THEOREM 2. *Assume that a nonzero map* $f : K \to L$ *satisfies the conditions* $f(xy) \geqslant f(x)f(y)$, and $f(x+y) \geqslant f(x)+f(y)$.
Then f *is an injective homomorphism.*

Proof. We see that $f(0) \leqslant 0$, hence

$$f(x)+f(-x) \leqslant 0 \tag{1}$$

for all x. It means that at least one of $f(x)$ and $f(-x)$ is negative (or zero). Especially, there exists $t \in K$ such that $f(t) < 0$. Now, from $f(t) \geqslant f(t)f(1)$ we get $f(1) \geqslant 1$, and so $f(-1) < 0$. By (1) we see:

$$f(xy) \leqslant -f(-xy) \leqslant -f(-x)f(y) \leqslant f(x)f(y) \tag{2}$$

provided $f(y) \leqslant 0$.

Similarly, $f(xy) \leqslant f(x)f(y)$, provided $f(x) \leqslant 0$.

Note at this moment that $f(1) > 1$ implies $f(x) \leqslant 0$ for all x. Therefore, in that case, by (2), we have $f(xy) \leqslant f(x)f(y)$ for all $x,y \in K$, and so $f(xy) = f(x)f(y)$ for all x,y. By theorem 1 (B1), f is a field homomorphism.

It remains the case $f(1) = 1$. Since $f(-1) < 0$, we have, by (2), $1 = f(-1\cdot -1) \leqslant f(-1)^2$, hence $f(-1)^2 = 1$, and so $f(-1) = -1$. It implies $f(-x) = -f(x)$ for all $x \in K$. Now, from $f(-xy) \geqslant f(x)f(-y)$ we get $f(xy) \leqslant f(x)f(y)$ for all x,y. Again, by theorem 1 (B1), we see that f is a field homomorphism. $\square$

REMARK 4. Theorem 2. was proved in [7] for $K = L = \mathbf{R}$ as a special case and extended in [4] to the functions from a ring K to the ordered ring L with property: if $z \in B$ and $z \neq 0$ then $z^2 > 0$. Implicitly, this condition is also in the core of the proof from [7] as well as from [8] where L is the ring of real valued functions on a set (see also [1], Exercises 14, 16, pp. 70–71). Our proof works over rings, too and eliminates that condition (which, for example, is not satisfied for ultra-metric ordered fields or rings).

The following example shows that in other cases appear new functions.

EXAMPLE. (I) Let $f : \mathbf{R} \to \mathbf{R}$ be a map given by $f(x) = 1+|x|$. Then f satisfies the conditions

$$f(xy) \leqslant f(x)f(y), \text{ and } f(x+y) \leqslant f(x)+f(y).$$

(II) Let $f : \mathbf{R} \to \mathbf{R}$ be a map given by $f(x) = -|\sin x|$. Then f satisfy the conditions

$$f(xy) \leqslant f(x)f(y), \text{ and } f(x+y) \geqslant f(x)+f(y).$$

(III) Let $f : \mathbf{R} \to \mathbf{R}$ be a map given by $f(x) = c$, for a real constant c with $0 < c < 1$. Then f satisfy the conditions

$$f(xy) \geqslant f(x)f(y), \text{ and } f(x+y) \leqslant f(x)+f(y).$$

However these maps are not field homomorphisms nor absolute values.

PROBLEM. Give a characterization (at least for subfields of $\mathbf{R}$) of maps f from a field K to an ordered field L satisfying one of the following sets of conditions:
(I) $f(xy) \leqslant f(x)f(y)$, and $f(x+y) \leqslant f(x)+f(y)$.
(II) $f(xy) \leqslant f(x)f(y)$, and $f(x+y) \geqslant f(x)+f(y)$.
(III) $f(xy) \geqslant f(x)f(y)$, and $f(x+y) \leqslant f(x)+f(y)$.

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