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INEQUALITIES OF THE DUNKL–WILLIAMS TYPE FOR ABSOLUTE VALUE OPERATORS

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Abstract. In this paper we give operator-valued versions of some Dunkl-Williams related inequalities for Hilbert space operators. The case of equality is also studied.

1. Introduction

The well-known Dunkl-Williams inequality [7] states that for any two nonzero elements x, y in a normed linear space

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1}$$

This inequality, which estimates the *angular distance* $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$ introduced in [4], has numerous applications (see e.g. [6, 18]). Over the years, many interesting refinements of (1) and their reverse inequalities have been obtained. The refinement

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}, \tag{2}$$

obtained by Massera and Schäffer in [13], is a useful tool to study linear differential equations in functional analysis context. Recently Maligranda [12] has established the following refinement of (1) which seems to be the sharpest one:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{\|x-y\| + |\,\|x\| - \|y\|\,|}{\max\{\|x\|, \|y\|\}}. \tag{3}$$

In the same paper, he also obtained the reverse inequality of (3) by showing that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geqslant \frac{\|x-y\| - |\,\|x\| - \|y\|\,|}{\min\{\|x\|, \|y\|\}} \tag{4}$$

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for any pair of nonzero elements x and y in a normed linear space. (For another proof see [14].)

Further generalizations of (3) and (4) for an arbitrary number of finitely many elements of a normed linear space can be found in [10, 17], where the equality conditions for elements of a strictly convex normed linear space were considered as well. The results from [17] have been recently generalized in the framework of pre-Hilbert C^*-modules (see [16]).

Let us note that for arbitrary elements x and y in a normed linear space it holds

$$\|x-y\|+|\,\|x\|-\|y\|\,|\leqslant\sqrt{2\|x-y\|^2+2(\|x\|-\|y\|)^2}\leqslant 2\|x-y\|.$$

From this we get one more estimate for the angular distance

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leqslant\frac{\sqrt{2\|x-y\|^2+2(\|x\|-\|y\|)^2}}{\max\{\|x\|,\|y\|\}},\tag{5}$$

which is stronger than the Massera-Schäffer inequality (2), but weaker than the Maligranda inequality (3).

In this paper we generalize the inequality (5) to the operator case. More precisely, we estimate $\left|A|A|^{-1}-B|B|^{-1}\right|$ where A and B are Hilbert space operators such that $|A|$ and $|B|$ are invertible. (Here $|T|$ denotes the *absolute value* of a Hilbert space operator T, that is, $|T|=(T^*T)^{\frac{1}{2}}$, where T^* stands for the adjoint operator of T.) The equality conditions are studied as well.

The operator-valued inequalities presented here are derived from the operator-valued inequalities of Bohr's type. The classical Bohr inequality for scalars (see, for instance, [15, p. 312]) was generalized to the operator case by Hirzallah [9] who proved that

$$|A-B|^2\leqslant p|A|^2+q|B|^2$$

for any pair of Hilbert space operators A and B and for scalars $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Thereby, the equality holds if and only if $B=(1-p)A$. Some related operator inequalities of the Bohr type, which we also use here, have been obtained in [5, 21].

Before stating the results we establish the notation and recall some definitions from the literature.

By $B(H)$ we denote the algebra of all bounded linear operators acting on a complex Hilbert space H. The inner product on H will be denoted by $(\cdot,\cdot)$. A self-adjoint operator $A\in B(H)$ is said to be *positive* if $(Ax,x)\geqslant 0$ for all $x\in H$. We write $A\geqslant 0$ if A is positive. If $A,B\in B(H)$ are self-adjoint operators such that $B-A\geqslant 0$ we write $A\leqslant B$.

2. The results

THEOREM 2.1. *Let $A,B\in B(H)$ and $p,q\in\mathbb{R}$ with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$|A|A|^{-1}-B|B|^{-1}|^2\leqslant|A|^{-1}(p|A-B|^2+q(|A|-|B|)^2)|A|^{-1}.\tag{6}$$

The equality in (6) *holds if and only if* $(p-1)(A-B)|A|^{-1}=B(|A|^{-1}-|B|^{-1})$.

Proof. Applying Corollary 1 of [9] on the operators $(A-B)|A|^{-1}$ and $B(|B|^{-1}-|A|^{-1})$ we have

$$\begin{aligned} |A|A|^{-1}-B|B|^{-1}|^2 &= |(A-B)|A|^{-1}-B(|B|^{-1}-|A|^{-1})|^2 \\ &\leqslant p|(A-B)|A|^{-1}|^2+q|B(|B|^{-1}-|A|^{-1})|^2. \end{aligned} \tag{7}$$

Thereby, the equality in (7) holds if and only if $(p-1)(A-B)|A|^{-1}=B(|A|^{-1}-|B|^{-1})$. Observe that

$$\begin{aligned} |(A-B)|A|^{-1}|^2 &= ((A-B)|A|^{-1})^*(A-B)|A|^{-1} \\ &= |A|^{-1}(A-B)^*(A-B)|A|^{-1} \\ &= |A|^{-1}|A-B|^2|A|^{-1} \end{aligned}$$

and

$$\begin{aligned} |B(|B|^{-1}-|A|^{-1})|^2 &= (B(|B|^{-1}-|A|^{-1}))^*B(|B|^{-1}-|A|^{-1}) \\ &= (|B|^{-1}-|A|^{-1})B^*B(|B|^{-1}-|A|^{-1}) \\ &= (|B|^{-1}-|A|^{-1})|B|^2(|B|^{-1}-|A|^{-1}) \\ &= I-|B||A|^{-1}-|A|^{-1}|B|+|A|^{-1}|B|^2|A|^{-1} \\ &= |A|^{-1}(|A|-|B|)^2|A|^{-1}. \end{aligned} \tag{8}$$

This proves the theorem.

REMARK 2.2. Interchanging the operators A and B in Theorem 2.1 note that it also holds

$$|A|A|^{-1}-B|B|^{-1}|^2 \leqslant |B|^{-1}(p|A-B|^2+q(|A|-|B|)^2)|B|^{-1},$$

with equality if and only if $(p-1)(A-B)|B|^{-1}=A(|A|^{-1}-|B|^{-1})$.

As a consequence of Theorem 2.1 we get the following operator-valued version of the inequality (5).

COROLLARY 2.3. *Let* $A,B\in B(H)$. *Then we have*

$$|A|A|^{-1}-B|B|^{-1}| \leqslant (|A|^{-1}(2|A-B|^2+2(|A|-|B|)^2)|A|^{-1})^{\frac{1}{2}}, \tag{9}$$

$$|A|A|^{-1}-B|B|^{-1}| \leqslant (|B|^{-1}(2|A-B|^2+2(|A|-|B|)^2)|B|^{-1})^{\frac{1}{2}}. \tag{10}$$

Proof. To obtain (9) put $p=q=2$ in (6) and take the positive square root of each side of the inequality (6). Interchanging A and B in (9) we get (10).

REMARK 2.4. Considering the Maligranda inequality (3) one could expect the following extension of this result to the operator valued case:

$$||A|A|^{-1} - B|B|^{-1}| \leqslant |A|^{-\frac{1}{2}}(|A-B| + |A| - |B|)|A|^{-\frac{1}{2}}, \tag{11}$$

where $A, B \in B(H)$ are such that $|B| \leqslant |A|$.

However, the following example shows that (11) need not hold. Indeed, if H is a two-dimensional Hilbert space, and if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.5 \\ -1 & 0 \end{bmatrix}$$

are matrix representations of two operators A and B with respect to some fixed orthonormal basis of H, then it can be easily verified that $|B| \leqslant |A|$,

$$||A|A|^{-1} - B|B|^{-1}| = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

and

$$|A|^{-\frac{1}{2}}(|A-B| + |A| - |B|)|A|^{-\frac{1}{2}} = \begin{bmatrix} 1.4 & 0.2 \\ 0.2 & 1.6 \end{bmatrix}.$$

Since $\begin{bmatrix} 1.4-\sqrt{2} & 0.2 \\ 0.2 & 1.6-\sqrt{2} \end{bmatrix}$ is not a positive matrix, (11) does not hold.

In what follows we give different characterizations of the case of equality in (6). To do this, let us first prove the following technical lemmas.

LEMMA 2.5. *Let $A, B \in B(H)$ and $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements are mutually equivalent.*

(*i*) $(p-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1})$.

(*ii*) $A|A|^{-1} - B|B|^{-1} = qB(|A|^{-1} - |B|^{-1})$.

Proof.

$$\begin{gathered}
(p-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1}) \Leftrightarrow \\
pA|A|^{-1} - pB|A|^{-1} - A|A|^{-1} = -B|B|^{-1} \Leftrightarrow \\
(p-1)(A|A|^{-1} - B|B|^{-1}) = pB(|A|^{-1} - |B|^{-1}) \Leftrightarrow \\
A|A|^{-1} - B|B|^{-1} = qB(|A|^{-1} - |B|^{-1}).
\end{gathered}$$

LEMMA 2.6. *Let $A, B \in B(H)$ and $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let us suppose that $(p-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1})$. Then*

$$(p-1)|A-B|^2 = |A|^2 - |B|^2.$$

In particular, if $p > 1$, then $|B| \leqslant |A|$.

Proof. First, notice that

$$(p-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1}) \Leftrightarrow$$
$$(p-1)(A-B)|A|^{-1} - B|A|^{-1} = -B|B|^{-1} \Leftrightarrow$$
$$((p-1)A - pB)|A|^{-1} = -B|B|^{-1} \Leftrightarrow$$
$$(p-1)A - pB = -B|B|^{-1}|A|,$$

from which it follows that $|(p-1)A - pB|^2 = |B|B|^{-1}|A||^2$. Now we have

$$|(p-1)A - pB|^2 = |B|B|^{-1}|A||^2 \Leftrightarrow$$
$$((p-1)A^* - pB^*)((p-1)A - pB) = |A||B|^{-1}B^*B|B|^{-1}|A| \Leftrightarrow$$
$$(p-1)^2|A|^2 - (p-1)p(A^*B + B^*A) + p^2|B|^2 = |A|^2 \Leftrightarrow$$
$$(p^2 - 2p)|A|^2 - (p-1)p(A^*B + B^*A) + p^2|B|^2 = 0 \Leftrightarrow$$
$$(p-2)|A|^2 - (p-1)(A^*B + B^*A) + p|B|^2 = 0 \Leftrightarrow$$
$$(p-1)(|A|^2 - A^*B - B^*A + |B|^2) = |A|^2 - |B|^2 \Leftrightarrow$$
$$(p-1)|A-B|^2 = |A|^2 - |B|^2.$$

Thus, if $p > 1$, it follows that $|A|^2 - |B|^2 \geqslant 0$, i.e., $|B|^2 \leqslant |A|^2$; hence $|B| \leqslant |A|$.

PROPOSITION 2.7. *Let $A, B \in B(H)$ and $p, q \in \mathbb{R}$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements are mutually equivalent.*

- (i) $(p-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1})$.
- (ii) $|A| = |B| + \frac{p}{q}|A-B|$ *and* $A - B = -B|B|^{-1}|A-B|$.
- (iii) $|A| = |B| + \frac{p}{q}|A-B|$ *and there exists an isometry* $U \in B(H)$ *such that* $B = -U|B|$ *and* $A - B = U|A-B|$.

Proof. $(i) \Rightarrow (ii)$: First, note that Lemma 2.6 implies that $|B| \leqslant |A|$. By Theorem 2.1 and Lemma 2.5 we have

$$|qB(|A|^{-1} - |B|^{-1})|^2 = |A|^{-1}(p|A-B|^2 + q(|A| - |B|)^2)|A|^{-1},$$

i.e.,

$$|A||qB(|A|^{-1} - |B|^{-1})|^2|A| = p|A-B|^2 + q(|A| - |B|)^2.$$

From this, by using (8), we get

$$q^2(|A| - |B|)^2 = p|A-B|^2 + q(|A| - |B|)^2,$$

so

$$\frac{q^2}{p}(|A| - |B|)^2 = (q^2 - q)(|A| - |B|)^2 = p|A-B|^2.$$

Therefore, $q^2(|A|-|B|)^2 = p^2|A-B|^2$ wherefrom, by taking the positive square root of each side of this equality, we get $q(|A|-|B|) = p|A-B|$, as $|A|-|B| \geqslant 0$. Hence,

$$|A| = |B| + \frac{p}{q}|A-B|.$$

From this it follows that

$$\begin{aligned}(p-1)(A-B) &= B(|A|^{-1}-|B|^{-1})|A| \\ &= -B(|B|^{-1}|A|-I) \\ &= -B(|B|^{-1}(|B|+\tfrac{p}{q}|A-B|)-I) \\ &= -B(I+\tfrac{p}{q}|B|^{-1}|A-B|-I) \\ &= -\tfrac{p}{q}B|B|^{-1}|A-B|.\end{aligned}$$

Thus,

$$A-B = -\frac{p}{q(p-1)}B|B|^{-1}|A-B| = -B|B|^{-1}|A-B|.$$

$(ii) \Rightarrow (i)$: Since

$$\begin{aligned}(A|A|^{-1}-B|B|^{-1})|A| &= A-B|B|^{-1}|A| \\ &= A-B|B|^{-1}(|B|+\tfrac{p}{q}|A-B|) \\ &= A-B-\tfrac{p}{q}B|B|^{-1}|A-B| \\ &= A-B+\tfrac{p}{q}(A-B) \\ &= p(A-B),\end{aligned}$$

we have

$$A|A|^{-1}-B|B|^{-1} = p(A-B)|A|^{-1}. \tag{12}$$

Also,

$$\begin{aligned}qB(|A|^{-1}-|B|^{-1})|A| &= qB-qB|B|^{-1}|A| \\ &= qB-qB|B|^{-1}(|B|+\tfrac{p}{q}|A-B|) \\ &= qB-qB-pB|B|^{-1}|A-B| \\ &= p(A-B),\end{aligned}$$

so

$$qB(|A|^{-1}-|B|^{-1}) = p(A-B)|A|^{-1}. \tag{13}$$

From (12) and (13) we get $A|A|^{-1}-B|B|^{-1} = qB(|A|^{-1}-|B|^{-1})$, so (i) follows from Lemma 2.5.

$(ii) \Rightarrow (iii)$: Let us define $U := -B|B|^{-1}$. Then we have $U^*U = |B|^{-1}B^*B|B|^{-1} = |B|^{-1}|B|^2|B|^{-1} = I$; hence U is an isometry. Thereby, $B = -U|B|$ and $A-B = U|A-B|$.

$(iii) \Rightarrow (ii)$: It is obvious.

THEOREM 2.8. *Let $A,B \in B(H)$ and $p,q \in \mathbb{R}$ with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Suppose that*

$$|A|A|^{-1}-B|B|^{-1}|^2 = |A|^{-1}(p|A-B|^2+q(|A|-|B|)^2)|A|^{-1}.$$

Then the following statements hold.

(i) *If* $p \geqslant 2$, *then* $A = B$.

(ii) *If* $p < 2$, *then* $((p-2)A - pB)|A-B| = 0$.

To prove the above theorem we need the following lemma.

LEMMA 2.9. *Let* $A, B \in B(H)$ *be positive operators such that* $AB + BA = tA^2$ *for some* $t \in \mathbb{R}$. *Then the following statements hold.*

(i) *If* $t < 0$, *then* $A = 0$.

(ii) *If* $t \geqslant 0$, *then* $AB = BA = \frac{1}{2}tA^2$.

Proof. Let us put $C := B - \frac{1}{2}tA$. Then it holds

$$AC + CA = A\Big(B - \frac{1}{2}tA\Big) + \Big(B - \frac{1}{2}tA\Big)A = AB + BA - tA^2 = 0,$$

wherefrom

$$AC^2 = (AC)C = (-CA)C = -C(AC) = C^2A.$$

Thus, AC^2 is positive as a product of commuting positive operators A and C^2. On the other hand, $AC^2 = -CAC = -C^*AC \leqslant 0$, since A is positive. We conclude that $AC^2 = 0$. Now we have

$$(A^{\frac{1}{2}}C)^*(A^{\frac{1}{2}}C) = CA^{\frac{1}{2}}A^{\frac{1}{2}}C = CAC = -AC^2 = 0,$$

so $A^{\frac{1}{2}}C = 0$. Hence, $AC = CA = 0$, that is, $A(B - \frac{1}{2}tA) = (B - \frac{1}{2}tA)A = 0$. So, we get $AB = BA = \frac{1}{2}tA^2$. Observe that $\frac{1}{2}tA^2$ is a positive operator as a product of commuting positive operators A and B. Thus, in the case $t < 0$ we have $\frac{1}{2}tA^2 = 0$, as A^2 is positive, wherefrom it follows $A = 0$.

Proof of Theorem 2.8. Let us put $C := A - B$. By Theorem 2.1 and Proposition 2.7 we have

$$B^*C = -B^*B|B|^{-1}|C| = -|B|^2|B|^{-1}|C| = -|B||C|,$$

from which it follows that

$$\begin{aligned} |C+B|^2 &= (C+B)^*(C+B) \\ &= |C|^2 + C^*B + B^*C + |B|^2 \\ &= |C|^2 - |C||B| - |B||C| + |B|^2. \end{aligned} \tag{14}$$

Again, by Proposition 2.7 it holds

$$|C+B|^2 = \Big(|B| + \frac{p}{q}|C|\Big)^2 = |B|^2 + \frac{p}{q}|B||C| + \frac{p}{q}|C||B| + \frac{p^2}{q^2}|C|^2. \tag{15}$$

Using (14) and (15) we get

$$\Big(\frac{p^2}{q^2}-1\Big)|C|^2+\Big(\frac{p}{q}+1\Big)|B||C|+\Big(\frac{p}{q}+1\Big)|C||B|=0,$$

from which it follows

$$\Big(\frac{p}{q}-1\Big)|C|^2+|B||C|+|C||B|=0,$$

i.e.,

$$|B||C|+|C||B|=(2-p)|C|^2.$$

Applying Lemma 2.9 we deduce that $C=0$ if $p>2$, while in the case $p\leqslant 2$ we have

$$|B||C|=|C||B|=\frac{1}{2}(2-p)|C|^2. \tag{16}$$

In particular, for $p=2$ it holds $|B||C|=0$, from which it follows $C=0$ since $|B|$ is an invertible operator.

For $p<2$, by Proposition 2.7 (ii) and by using (16), we obtain

$$B|C|=\frac{1}{2}(2-p)B|B|^{-1}|C|^2=\frac{1}{2}(p-2)C|C|.$$

Hence,

$$((p-2)A-pB)|A-B|=(pC-2(C+B))|C|=2\Big(\frac{1}{2}(p-2)C-B\Big)|C|=0$$

and the theorem is proved.

Notice that Theorem 2.8 fully describes the case of equality in (6) (and also in (9), (10)) when $p\geqslant 2$. (Namely, the equality holds precisely when $A=B$.) To get a complete characterization of the case of equality in (6) when $1<p<2$, we set one more condition on operators A and B; that is, $(p-2)A-pB$ or $|A-B|$ is an invertible operator.

COROLLARY 2.10. *Let $A,B\in B(H)$ and $p,q\in\mathbb{R}$ with $1<p<2$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $(p-2)A-pB$ be invertible. Then*

$$|A|A|^{-1}-B|B|^{-1}|^2=|A|^{-1}(p|A-B|^2+q(|A|-|B|)^2)|A|^{-1}$$

if and only if $A=B$.

Proof. It follows immediately from Theorem 2.8.

COROLLARY 2.11. *Let $A,B\in B(H)$ and $p,q\in\mathbb{R}$ with $1<p<2$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $|A-B|$ be invertible. Then*

$$|A|A|^{-1}-B|B|^{-1}|^2=|A|^{-1}(p|A-B|^2+q(|A|-|B|)^2)|A|^{-1}$$

if and only if $B=\frac{p-2}{p}A$.

Proof. Let us suppose that $B = \frac{p-2}{p}A$. By Theorem 2.1 it is enough to show that $(p-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1})$. Indeed,

$$(p-1)(A-B)|A|^{-1} = (p-1)\frac{2}{p}A|A|^{-1} = \left(2-\frac{2}{p}\right)A|A|^{-1},$$

$$B(|A|^{-1} - |B|^{-1}) = \frac{p-2}{p}A\left(|A|^{-1} - \frac{p}{2-p}|A|^{-1}\right) = \left(2-\frac{2}{p}\right)A|A|^{-1}.$$

The converse follows immediately from Theorem 2.8.

Concluding remarks

(a) Applying Theorem 1, Corollary 1 and Theorem 2 of [5] on the operators $(A-B)|A|^{-1}$ and $B(|B|^{-1} - |A|^{-1})$ we obtain the following upper and lower estimates for $|A|A|^{-1} - B|B|^{-1}|^2$.

(i) If $1 < p \leqslant 2$ then

$$|A|A|^{-1} - B|B|^{-1}|^2 + |(1-p)(A-B)|A|^{-1} - B(|B|^{-1} - |A|^{-1})|^2$$
$$\leqslant |A|^{-1}(p|A-B|^2 + q(|A| - |B|)^2)|A|^{-1}, \tag{17}$$
$$|A|A|^{-1} - B|B|^{-1}|^2 + |(A-B)|A|^{-1} - (1-q)B(|B|^{-1} - |A|^{-1})|^2$$
$$\geqslant |A|^{-1}(p|A-B|^2 + q(|A| - |B|)^2)|A|^{-1}. \tag{18}$$

(ii) If $p > 2$ then the reverse inequalities of (17) and (18) hold.

(iii) If $p < 1$ then (18) and the reverse inequality of (17) hold.

Note that (17) and the reverse inequality of (18) are refinements of (6).

Furthermore, if $p = q = 2$ then the equality holds in both (17) and (18). Also, in the case $p \neq 2$, the equality holds in both (17) and (18) if and only if $(p-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1})$. Therefore, the case of equality in both (17) and (18) when $1 < p < 2$, and in their reverse inequalities when $p > 2$ is also described in Proposition 2.7, Theorem 2.8, Corollary 2.10 and Corollary 2.11. When $p < 1$ the case of equality in both (18) and the reverse inequality of (17) can be characterized in a similar way.

(b) The proofs of inequalities (3) and (4) are based on the norm-valued triangle inequality. These inequalities cannot be fully generalized to the operator-valued case (see Remark 2.4). The reason for this obviously lies in the fact that the operator-valued triangle inequality $|A+B| \leqslant |A| + |B|$ need not hold (see [11, p. 4] or [8]). However, there are some other kinds of the operator-valued triangle inequalities (see, for instance, [19, 1]; see also [20, 2, 3] where the equality conditions have been investigated) which could serve to generalize (3) and (4) to the operator-valued case.

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