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INEQUALITIES INVOLVING INVERSE CIRCULAR AND INVERSE HYPERBOLIC FUNCTIONS II

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Abstract. Inequalities connecting inverse circular and inverse hyperbolic functions are obtained. Also, bounds for the inverse hyperbolic sine function are established. Some of the results presented in this paper are derived from the inequalities satisfied by particular bivariate means which belong to the family of the Schwab-Borchardt means.

1. Introduction

The goal of this paper is to obtain inequalities which connect the inverse circular functions $\sin^{-1} t$ and $\tan^{-1} t$ with the inverse hyperbolic functions $\sinh^{-1} t$ and $\tanh^{-1} t$. Also, the bounds for $\sinh^{-1} t$ are given. This paper is a continuation of the author's earlier work [1] and is organized as follows. In Section 2 we recall definitions of some means of two variables which belong to the family of means introduced by Schwab and Borchard. The main results of this note are presented in Section 3.

2. Some means of two variables

For later use, recall definitions of some bivariate means of two positive variables x and y. In what follows we will assume that $x \neq y$. The arithmetic mean A of x and y is defined by $A = \frac{x+y}{2}$. The logarithmic mean L, two Seiffert means P and T (see [6] and [7], respectively) and a mean M introduced in [2] are defined as follows

$$\begin{aligned} L &= \frac{z}{\tanh^{-1} z} A, \qquad & P &= \frac{z}{\sin^{-1} z} A, \\ T &= \frac{z}{\tan^{-1} z} A, \qquad & M &= \frac{z}{\sinh^{-1} z} A, \end{aligned} \tag{2.1}$$

where

$$z = \frac{x-y}{x+y}. \tag{2.2}$$

All the means listed in (2.1) belong to the class of Schwab-Borchardt means which have been studied extensively in [2] and [3]. Some of the results presented in those papers will be utilized in the next section.

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3. Main results

We are in a position to prove the first result of this paper.

THEOREM 3.1. *Let* $0<|t|<1$. *Then the following inequalities*

$$\left(\frac{\sinh^{-1}t}{t}\right)^2<\frac{\tan^{-1}t}{t}<\frac{\sinh^{-1}t}{t}<1<\frac{\sin^{-1}t}{t}<\left(\frac{\sin^{-1}t}{t}\right)^2<\frac{\tanh^{-1}t}{t}, \tag{3.1}$$

$$1<\frac{\sin^{-1}t}{t}\,\frac{\sinh^{-1}t}{t}, \tag{3.2}$$

and

$$1<\frac{\tan^{-1}t}{t}\,\frac{\tanh^{-1}t}{t}, \tag{3.3}$$

hold true.

Proof. The first and the last inequalities in (3.1) are established in [1] (see (2.2) and (2.1), respectively). The second, third, and fourth inequalities in (3.1) can be established as follows. We let $x=1+t$ and $y=1-t$ $(0<|t|<1)$ in (2.2) to obtain $z=t$ and $A=1$. It follows from (2.10) in [2] that

$$P<A<M<T.$$

This in conjunction with (2.1) gives

$$\frac{t}{\sin^{-1}t}<1<\frac{t}{\sinh^{-1}t}<\frac{t}{\tan^{-1}t}.$$

Hence the desired inequalities follow. The fifth inequality in (3.1) is an obvious consequence of the previous one. For the proof of (3.2) we utilize the following inequality $PM<A^2$ (see [3, (3.16)]). Applying the second and the fourth formulas in (2.1) one obtains the inequality in question. Finally, inequality (3.3) can be established in an analogous manner using $LT<A^2$ (see [3, (3.16)]) together with the first and third formulas in (2.1). The proof is complete. □

COROLLARY 3.1. *The following inequalities*

$$2<\frac{\sin^{-1}t}{t}+\frac{\sinh^{-1}t}{t} \tag{3.4}$$

and

$$2<\frac{\tan^{-1}t}{t}+\frac{\tanh^{-1}t}{t} \tag{3.5}$$

are valid for all values of t *which satisfy* $0<|t|<1$.

Proof. The inequalities (3.4) and (3.5) follow from (3.2) and (3.3), respectively, by using the inequality of the arithmetic and geometric means. □

Our next result reads as follows.

THEOREM 3.2. *Let* $0 < |t| < 1$ *and let* $\mu > 0$. *Then*

$$2 < \left(\frac{\sin^{-1} t}{t}\right)^2 + \sqrt{1-t^2}\,\frac{\sin^{-1} t}{t}, \tag{3.6}$$

and

$$\left(\frac{\sin^{-1} t}{t}\right)^{2\mu} + \left(\sqrt{1-t^2}\frac{\sin^{-1} t}{t}\right)^{\mu} < \left(\frac{t}{\sin^{-1} t}\right)^{2\mu} + \left(\frac{1}{\sqrt{1-t^2}}\,\frac{t}{\sin^{-1} t}\right)^{\mu}. \tag{3.7}$$

Proof. In order to establish the inequality (3.6) we use the result of Wu and Srivastava

$$2 < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x},$$

$0 < |x| < \frac{\pi}{2}$ (see [9, Lemma 3]). Letting $\sin x = t$ we have $x = \sin^{-1} t$ and $\frac{1}{\tan x} = \frac{\sqrt{1-t^2}}{t}$. Inequality (3.7) follows from the following one

$$\left(\frac{x}{\sin x}\right)^{2\mu} + \left(\frac{x}{\tan x}\right)^{\mu} < \left(\frac{\sin x}{x}\right)^{2\mu} + \left(\frac{\tan x}{x}\right)^{\mu}$$

$(0 < |x| < \frac{\pi}{2})$ which has been established in [4, (2.15)]. Putting $\sin x = t$ we obtain the desired result. □

Recently Wilker's inequality [8]

$$2 < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}$$

$(0 < |x| < \pi/2)$ has attracted attention of several researchers (see, e.g., [4], [5], [9], [10]). Its refinements are obtained in [5] and [10]. Those results can be used to obtain new inequalities involving inverse circular functions. We omit further details.

The hyperbolic counterparts of the inequalities (3.6) and (3.7) together with the bounds for the inverse hyperbolic sine function are contained in the following.

THEOREM 3.3. *Let* $t \neq 0$ *and let* $\mu > 0$. *Then*

$$2 < \left(\frac{\sinh^{-1} t}{t}\right)^2 + \sqrt{1+t^2}\frac{\sinh^{-1} t}{t}, \tag{3.8}$$

$$\left(\frac{\sinh^{-1} t}{t}\right)^{2\mu} + \left(\sqrt{1+t^2}\frac{\sinh^{-1} t}{t}\right)^{\mu} < \left(\frac{t}{\sinh^{-1} t}\right)^{2\mu} + \left(\frac{1}{\sqrt{1+t^2}}\,\frac{t}{\sinh^{-1} t}\right)^{\mu}, \tag{3.9}$$

$$\frac{3}{2+x} < \frac{6}{1+x+\sqrt{8(1+x)}} < \frac{\sinh^{-1} t}{t} < \left(\frac{2}{1+x}\right)^{2/3} < x^{-1/3}, \tag{3.10}$$

where $x = \sqrt{1+t^2}$.

Proof. In order to establish the inequality (3.8) we utilize the following result [4, (2.13)]

$$2 < \left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}$$

$(x \neq 0)$. Letting $\sinh x = t$ we have $x = \sinh^{-1} t$ and $\tanh x = \frac{t}{\sqrt{1+t^2}}$. Hence the inequality in question follows. For the proof of (3.9) we employ the following result [4, (2.16)]

$$\left(\frac{x}{\sinh x}\right)^{2\mu} + \left(\frac{x}{\tanh x}\right)^{\mu} < \left(\frac{\sinh x}{x}\right)^{2\mu} + \left(\frac{\tanh x}{x}\right)^{\mu}$$

$(x \neq 0)$. Letting above $\sinh x = t$ we obtain the desired result. The bounds for $\sinh^{-1} t$ follow from

$$(\cosh x)^{1/3} < \left(\frac{1+\cosh x}{2}\right)^{2/3} < \frac{\sinh x}{x} < \frac{1+\cosh x + \sqrt{8(1+\cosh x)}}{6} < \frac{2+\cosh x}{3},$$

$x \neq 0$ (see [4, (2.7), (2.8)] by using the substitution $\sinh x = t$. This completes the proof. $\square$

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