Journal of
Mathematical
Inequalities
Volume 4, Number 1 (2010), 15–25

GEOMETRICALLY CONVEX FUNCTIONS AND ESTIMATION OF REMAINDER TERMS FOR TAYLOR EXPANSION OF SOME FUNCTIONS

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(*Communicated by A. Čižmešija*)

Abstract. In this paper, we establish two integral inequalities for geometrically convex functions. As consequences, we get the estimation for remainder terms of Taylor series for e^{-x}, $\sin x$ and $\cos x$.

1. Introduction

Let $I \subseteq (0,+\infty)$ be an interval and $f : I \to (0,+\infty)$ be a continuous function. f is said to be geometrically convex (or concave, resp.) on I if

$$f\left(x^{\alpha} y^{1-\alpha}\right) \leqslant (\text{or} \geqslant, \text{resp.})\ f^{\alpha}(x)\, f^{1-\alpha}(y)$$

for any $x, y \in I$ and any $\alpha \in [0,1]$.

The notion of geometrical convexity was introduced by P. Montel ([1]), in a beautiful paper where analogues of the notion of a convex function in n variables are discussed. Recently, C. P. Niculescu ([2]) discussed an attractive class of inequalities, which arise from the notion of geometrically convex functions.

The main purpose of this paper is to establish two integral inequalities for geometrically convex functions. As applications, we get the estimation for remainder terms of Taylor series for e^{-x}, $\sin x$ and $\cos x$.

The following well-known results will be used in next sections.

LEMMA 1. ([2]) *If $f : I \subseteq (0,+\infty) \to (0,+\infty)$ is a twice differentiable function, then f is a geometrically convex (or concave, resp.) if and only if*

$$x\left[f''(x)\, f(x) - \left(f'(x)\right)^2\right] + f(x)\, f'(x) \geqslant (\text{or } \leqslant, \text{ resp.})\, 0$$

for all $x \in I$.

LEMMA 2. ([2]) *If $f : [a,b] \subseteq (0,+\infty) \to (0,+\infty)$ is geometrically convex (or concave, resp.), and $g(x) = \log f(e^x)$, then g is convex (or concave, resp.) on $[\log a, \log b]$.*

LEMMA 3. ([4]) *Suppose that $f : (a,b) \subseteq (0,+\infty) \to (0,+\infty)$ is a geometrically concave function. If $g(x) = \int_x^b f(t)dt$ and $h(x) = \int_a^x f(t)dt$, then both g and h are geometrically concave on (a,b).*

Mathematics subject classification (2010): 26D15, 26D10.
Keywords and phrases: Inequality, geometrically convex function, remainder term.
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Paper JMI-04-03

2. Integral inequalities for geometrically convex functions

THEOREM 1. *Suppose that $f:[a,b]\subseteq(0,+\infty)\to(0,+\infty)$ is a geometrically convex function. Then*

$$\int_a^b f(x)\,dx \geqslant \frac{(f(a))^2}{\left(f(a)+af'_+(a)\right)a^{\frac{af'_+(a)}{f(a)}}}\cdot\left(b^{1+\frac{af'_+(a)}{f(a)}}-a^{1+\frac{af'_+(a)}{f(a)}}\right) \tag{1}$$

if $f(a)+af'_+(a)\neq 0$, and

$$\int_a^b f(x)\,dx \geqslant \frac{(f(b))^2}{\left(f(b)+bf'_-(b)\right)b^{\frac{bf'_-(b)}{f(b)}}}\cdot\left(b^{1+\frac{bf'_-(b)}{f(b)}}-a^{1+\frac{bf'_-(b)}{f(b)}}\right) \tag{2}$$

if $f(b)+bf'_-(b)\neq 0$. Inequalities (1) *and* (2) *become equality if and only if f is a power function or constant function, while inequalities* (1) *and* (2) *are reversed if f is a geometrically concave function.*

Proof. If f is a geometrically convex function, then simple computation yields that

$$\begin{aligned}\lim_{c\to a+0}\frac{\log f(c)-\log f(a)}{\log c-\log a} &= \lim_{c\to a+0}\frac{\log f\left(e^{\log c}\right)-\log f\left(e^{\log a}\right)}{\log c-\log a}\\ &= \left(\log f\left(e^t\right)\right)'_+\Big|_{t=\log a} = \frac{e^t f'_+\left(e^t\right)}{f\left(e^t\right)}\Bigg|_{t=\log a}\\ &= \frac{a\cdot f'_+(a)}{f(a)}\neq -1.\end{aligned}$$

Let $c\in(a,b)$ be such that $(\log f(c)-\log f(a))/(\log c-\log a)\neq -1$. For $x\in[c,b]$, let $t=(\log x-\log c)/(\log x-\log a)$. Then $0\leqslant t<1$, $(x/a)^t=x/c$, $c=a^t x^{1-t}$, $f(c)=f\left(a^t x^{1-t}\right)$, and

$$f(c)\leqslant (f(a))^t (f(x))^{1-t}. \tag{3}$$

Therefore

$$(f(c))^{\frac{1}{1-t}}(f(a))^{-\frac{t}{1-t}}\leqslant f(x),$$

$$(f(c))^{\frac{\log x-\log a}{\log c-\log a}}(f(a))^{-\frac{\log x-\log c}{\log c-\log a}}\leqslant f(x)$$

and

$$\begin{aligned}\int_c^b f(x)\,dx &\geqslant \int_c^b (f(c))^{\frac{\log x-\log a}{\log c-\log a}}(f(a))^{-\frac{\log x-\log c}{\log c-\log a}}\,dx\\ &= (f(c))^{-\frac{\log a}{\log c-\log a}}(f(a))^{\frac{\log c}{\log c-\log a}}\int_c^b (f(c))^{\frac{\log x}{\log c-\log a}}(f(a))^{-\frac{\log x}{\log c-\log a}}\,dx.\end{aligned}$$

Now, by taking $x = e^u$, we get

$$\begin{aligned}
\int_c^b f(x)\,dx &\geqslant (f(c))^{-\frac{\log a}{\log c-\log a}} (f(a))^{\frac{\log c}{\log c-\log a}} \\
&\quad \cdot \int_{\log c}^{\log b} (f(c))^{\frac{u}{\log c-\log a}} (f(a))^{-\frac{u}{\log c-\log a}}\, d(e^u) \\
&= (f(c))^{-\frac{\log a}{\log c-\log a}} (f(a))^{\frac{\log c}{\log c-\log a}} \\
&\quad \cdot \int_{\log c}^{\log b} \left[e (f(c))^{\frac{1}{\log c-\log a}} \cdot (f(a))^{-\frac{1}{\log c-\log a}} \right]^u du \\
&= (f(c))^{-\frac{\log a}{\log c-\log a}} \cdot (f(a))^{\frac{\log c}{\log c-\log a}} \\
&\quad \cdot \frac{b (f(c))^{\frac{\log b}{\log c-\log a}} (f(a))^{-\frac{\log b}{\log c-\log a}} - c (f(c))^{\frac{\log c}{\log c-\log a}} (f(a))^{-\frac{\log c}{\log c-\log a}}}{1 + \frac{\log f(c)}{\log c-\log a} - \frac{\log f(a)}{\log c-\log a}} \\
&= \frac{b (f(c))^{\frac{\log b-\log a}{\log c-\log a}} (f(a))^{\frac{\log c-\log b}{\log c-\log a}} - c \cdot f(c)}{1 + \frac{\log f(c)-\log f(a)}{\log c-\log a}} \\
&= f(c) \cdot \frac{b \left(\frac{f(c)}{f(a)} \right)^{\frac{\log b-\log c}{\log c-\log a}} - c}{1 + \frac{\log f(c)-\log f(a)}{\log c-\log a}} \\
&= f(c) \cdot \frac{b \cdot \exp\left(\frac{\log b-\log c}{\log c-\log a} \cdot (\log f(c) - \log f(a)) \right) - c}{1 + \frac{\log f(c)-\log f(a)}{\log c-\log a}}. \qquad (4)
\end{aligned}$$

Let $c \to a+0$ in inequality (4). Then

$$\begin{aligned}
\int_a^b f(x)\,dx &\geqslant f(a) \frac{b \cdot \exp\left(a (\log b - \log a) \frac{f'_+(a)}{f(a)} \right) - a}{1 + \frac{af'_+(a)}{f(a)}} \\
&= (f(a))^2 \frac{b \cdot \left(\frac{b}{a}\right)^{\frac{af'_+(a)}{f(a)}} - a}{f(a) + a \cdot f'_+(a)} \\
&= \frac{(f(a))^2}{\left(f(a) + af'_+(a)\right) a^{\frac{af'_+(a)}{f(a)}}} \cdot \left(b^{1+\frac{af'_+(a)}{f(a)}} - a^{1+\frac{af'_+(a)}{f(a)}} \right).
\end{aligned}$$

Equality (1) occurs only when equality (3) occurs, hence f is a power function or constant function obviously.

Similarly we can prove inequality (2).

The proof of Theorem 1 is completed. □

THEOREM 2. *Let* $0 < a < b$ *and* $f : [a,b] \to (0,+\infty)$ *be a geometrically convex function. Then*

$$\int_a^b f(x)\,dx \leqslant \begin{cases} \frac{bf(b)-af(a)}{\log(bf(b))-\log(af(a))} \cdot \log \frac{b}{a}, & af(a) \neq bf(b), \\ af(a) \cdot \log \frac{b}{a}, & af(a) = bf(b). \end{cases} \tag{5}$$

The equality occurs only when f *is a power function or constant function. Inequality* (5) *is reversed if* f *is geometrically concave.*

Proof. If f is a geometrically convex function and $\alpha = (\log x - \log a)/(\log b - \log a)$, then

$$\int_a^b f(x)\,dx = \int_0^1 f\left(a^{1-\alpha} b^{\alpha}\right) \cdot a^{1-\alpha} b^{\alpha} \cdot \log \frac{b}{a} d\alpha.$$

If $af(a) \neq bf(b)$, then

$$\begin{aligned} \int_a^b f(x)\,dx &\leqslant \int_0^1 f^{1-\alpha}(a) \cdot f^{\alpha}(b) \cdot a^{1-\alpha} b^{\alpha} \cdot \log \frac{b}{a} d\alpha \\ &= af(a) \cdot \log \frac{b}{a} \cdot \int_0^1 \left(\frac{bf(b)}{af(a)} \right)^{\alpha} d\alpha \\ &= \frac{af(a)}{\log\left(\frac{bf(b)}{af(a)}\right)} \cdot \log \frac{b}{a} \cdot \left. \left(\frac{bf(b)}{af(a)} \right)^{\alpha} \right|_0^1 \\ &= \frac{af(a)}{\log(bf(b) - af(a))} \cdot \log \frac{b}{a} \cdot \left[\frac{bf(b)}{af(a)} - 1 \right] \\ &= \frac{bf(b) - af(a)}{\log(bf(b)) - \log(af(a))} \cdot \log \frac{b}{a}. \end{aligned}$$

If $af(a) = bf(b)$, then we clearly see that

$$\int_a^b f(x)\,dx \leqslant af(a) \log \frac{b}{a}. \qquad \square$$

3. Estimation of Remainder Term of Taylor Series for e^{-x}

THEOREM 3. *For* $x > 0$, $n \in \mathbf{N}$, *and* $T_n(x) = e^{-x} - 1 + x - x^2/2! + \cdots + (-1)^{n-1} x^n/n!$, *we have the following results.*

i) *If* $0 < x < (n+2)/(n+1)$, *then*

$$\frac{2(n+1)}{n + x + \sqrt{(n+2+x)^2 - 4x}} \cdot \frac{x^{n+1}}{(n+1)!} \leqslant |T_n(x)| \leqslant \frac{n+2}{n+2+x} \cdot \frac{x^{n+1}}{(n+1)!}.$$

ii) *If* $x \geqslant (n+2)/(n+1)$*, then*

$$\frac{2(n+1)}{n+x+\sqrt{(n+2+x)^2-4x}}\cdot\frac{x^{n+1}}{(n+1)!}\leqslant |T_n(x)|$$

$$\leqslant \frac{2(n+1)}{n+1+x+\sqrt{(x+n+1)^2-4x}}\cdot\frac{x^{n+1}}{(n+1)!}.$$

Proof. Let $n=2m-1$ if n is an odd number, and $n=2m$ if n is an even number. According to the Taylor series expansion of e^{-x}, we have

$$T_{2m-1}(x)=e^{-x}-1+x-\frac{x^2}{2!}+\cdots+\frac{x^{2m-1}}{(2m-1)!}\geqslant 0,$$

$$T_{2m}(x)=e^{-x}-1+x-\frac{x^2}{2!}+\cdots-\frac{x^{2m}}{(2m)!}\leqslant 0,$$

$$-T_{2m}(x)=-e^{-x}+1-x+\frac{x^2}{2!}-\cdots+\frac{x^{2m}}{(2m)!}\geqslant 0.$$

If $f(x)=e^{-x}, x\in(0,+\infty)$, then f is a geometrically concave function by Lemma 1 and a simple computation. On the other hand,

$$\int_0^x e^{-t}dt=1-e^{-x},\quad \int_0^x\left(1-e^{-t}\right)dt=e^{-x}-1+x,$$

$$\int_0^x\left(e^{-t}-1+x\right)dt=-e^{-x}+1-x+\frac{x^2}{2},\quad \cdots.$$

Hence both T_{2m-1} and $-T_{2m}$ are geometrically concave functions by Lemma 3.

For $0<a<x$, according to Theorem 1, we get

$$\int_a^x T_{2m-1}(x)dx\leqslant\frac{T_{2m-1}(x)}{(1+\eta)x^{\eta}}\cdot\left(x^{1+\eta}-a^{1+\eta}\right), \tag{6}$$

where

$$\eta=\frac{xT'_{2m-1}(x)}{T_{2m-1}(x)}=\frac{x\left(-e^{-x}+1-x+\cdots+\frac{x^{2m-2}}{(2m-2)!}\right)}{T_{2m-1}(x)}>0.$$

By taking $a\to 0+$ in (6), we get

$$-e^{-x}+1-x+\frac{x^2}{2!}+\cdots+\frac{x^{2m}}{(2m)!}\leqslant\frac{xT_{2m-1}(x)}{1+\frac{xT'_{2m-1}(x)}{T_{2m-1}(x)}}.$$

It follows that

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!}\leqslant\frac{xT^2_{2m-1}(x)}{T_{2m-1}(x)+x\left(-e^{-x}+1-x+\cdots+\frac{x^{2m-2}}{(2m-2)!}\right)},$$

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!}\leqslant\frac{xT^2_{2m-1}(x)}{T_{2m-1}(x)+x\left(-T_{2m-1}(x)+\frac{x^{2m-1}}{(2m-1)!}\right)},$$

$$(x-1)\,T_{2m-1}^2(x)+\left(-\frac{x^{2m}}{(2m-1)!}+\frac{x^{2m}}{(2m)!}-\frac{x^{2m+1}}{(2m)!}\right)T_{2m-1}(x)$$
$$+\frac{x^{4m}}{(2m)!\,(2m-1)!}\leqslant xT_{2m-1}^2(x),$$

and

$$T_{2m-1}^2(x)+\frac{(2m-1+x)\,x^{2m}}{(2m)!}T_{2m-1}(x)-\frac{x^{4m}}{(2m)!\,(2m-1)!}\geqslant 0.$$

Therefore

$$\begin{aligned}
T_{2m-1}(x) &\geqslant \frac{-\frac{(2m-1+x)x^{2m}}{(2m)!}+\sqrt{\frac{(2m-1+x)^2x^{4m}}{((2m)!)^2}+\frac{4x^{4m}}{(2m)!(2m-1)!}}}{2}\\
&=\frac{-(2m-1+x)+\sqrt{(2m-1+x)^2+8m}}{2}\cdot\frac{x^{2m}}{(2m)!}\\
&=\frac{4m}{\sqrt{(2m-1+x)^2+8m}+2m-1+x}\cdot\frac{x^{2m}}{(2m)!}\\
&=\frac{4m}{2m-1+x+\sqrt{(2m+1+x)^2-4x}}\cdot\frac{x^{2m}}{(2m)!}.
\end{aligned} \tag{7}$$

Since $-T_{2m}$ is a geometrically concave function, by making use of the same method as in Theorem 1, we get

$$-T_{2m}(x)\geqslant\frac{4m+2}{2m+x+\sqrt{(2m+2+x)^2-4x}}\cdot\frac{x^{2m+1}}{(2m+1)!}. \tag{8}$$

For $0<a<x$, according to Theorem 2, we get

$$\int_a^x T_{2m-1}(t)\,dt\geqslant\frac{xT_{2m-1}(x)-aT_{2m-1}(a)}{\log(xT_{2m-1}(x))-\log(aT_{2m-1}(a))}\cdot\log\frac{x}{a}$$

and

$$\int_a^x T_{2m-1}(t)\,dt\geqslant(xT_{2m-1}(x)-aT_{2m-1}(a))\cdot\frac{\log x-\log a}{\log(xT_{2m-1}(x))-\log(aT_{2m-1}(a))}.$$

Let $a \to 0+$. Using the L'Hospital rule, we get

$$\int_0^x T_{2m-1}(t)\,dt \geqslant xT_{2m-1}(x) \cdot \lim_{a\to 0+} \frac{-1/a}{-\frac{T_{2m-1}(a)+aT'_{2m-1}(a)}{T_{2m-1}(a)}},$$

$$-e^{-x}+1-x+\frac{x^2}{2!}+\cdots+\frac{x^{2m}}{(2m)!} \geqslant \frac{xT_{2m-1}(x)}{1+\lim_{a\to 0+}\frac{aT'_{2m-1}(a)}{T_{2m-1}(a)}},$$

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!} \geqslant \frac{xT_{2m-1}(x)}{2+\lim_{a\to 0+}\frac{aT''_{2m-1}(a)}{T'_{2m-1}(a)}},$$

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!} \geqslant \frac{xT_{2m-1}(x)}{2m+\lim_{a\to 0+}\frac{ae^{-a}}{1-e^{-a}}}$$

and

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!} \geqslant \frac{xT_{2m-1}(x)}{2m+1}.$$

Therefore

$$\frac{(2m+1)x^{2m}}{(2m)!} \geqslant (2m+1+x)\,T_{2m-1}(x)$$

and

$$T_{2m-1}(x) \leqslant \frac{(2m+1)}{(2m+1+x)} \cdot \frac{x^{2m}}{(2m)!}. \tag{9}$$

Using the same method as above, we can get

$$-T_{2m}(x) \leqslant \frac{(2m+2)}{(2m+2+x)} \cdot \frac{x^{2m+1}}{(2m+1)!}. \tag{10}$$

Since T_{2m-1} is a geometrically concave function, according to Lemma 1 we have

$$x\left(T_{2m-1}(x)\cdot T''_{2m-1}(x)-\left(T'_{2m-1}(x)\right)^2\right)+T_{2m-1}(x)\cdot T'_{2m-1}(x) \leqslant 0.$$

A simple computation yields that

$$T'_{2m-1}(x) = -T_{2m-1}(x)+\frac{x^{2m-1}}{(2m-1)!},$$

and

$$T''_{2m-1}(x) = T_{2m-1}(x)+\frac{x^{2m-2}}{(2m-2)!}-\frac{x^{2m-1}}{(2m-1)!}.$$

Therefore

$$-T_{2m-1}^{2}(x)+\frac{2mx^{2m-1}+x^{2m}}{(2m-1)!}T_{2m-1}(x)-\frac{x^{4m-1}}{((2m-1)!)^{2}}\leqslant 0,$$
$$T_{2m-1}^{2}(x)-\frac{2mx^{2m-1}+x^{2n}}{(2m-1)!}T_{2m-1}(x)+\frac{x^{4m-1}}{((2m-1)!)^{2}}\geqslant 0$$

and

$$\begin{aligned} T_{2m-1} &\leqslant \frac{2m+x-\sqrt{(x+2m)^{2}-4x}}{2}\cdot\frac{x^{2m-1}}{(2m-1)!} \\ &= \frac{2x}{2m+x+\sqrt{(2m+x)^{2}-4x}}\cdot\frac{x^{2m-1}}{(2m-1)!} \\ &= \frac{4m}{2m+x+\sqrt{(2m+x)^{2}-4x}}\cdot\frac{x^{2m}}{(2m)!}. \end{aligned}\tag{11}$$

Similarly, we can get

$$-T_{2m}(x)\leqslant\frac{4m+2}{2m+1+x+\sqrt{(2m+1+x)^{2}-4x}}\cdot\frac{x^{2m+1}}{(2m+1)!}.\tag{12}$$

If $0<x\leqslant(2n+1)/2n$, then we can find

$$\frac{(2m+1)}{(2m+1+x)}\cdot\frac{x^{2m}}{(2m)!}\geqslant\frac{4m}{2m+x+\sqrt{(2m+x)^{2}-4x}}\cdot\frac{x^{2m}}{(2m)!},\tag{13}$$

while inequality (13) is reversed if $x\geqslant(2n+1)/2n$. If $0<x\leqslant(2n+2)/(2n+1)$, we can find

$$\frac{(2m+2)}{(2m+2+x)}\cdot\frac{x^{2m+1}}{(2m+1)!}\geqslant\frac{4m+2}{2m+1+x+\sqrt{(2m+1+x)^{2}-4x}}\cdot\frac{x^{2m+1}}{(2m+1)!},\tag{14}$$

while inequality (14) is reversed if $x\geqslant(2n+2)/(2n+1)$.

According to inequalities (7), (8), (9), (10), (11), (12), (13) and (14) we know that Theorem 3 is true. $\square$

4. Estimation of Remainder Term of Taylor Series for $\sin x$ and $\cos x$

THEOREM 4. *If* $n\in\mathbf{N}$, $x\in(0,\pi/2)$, $S_n(x)=\sin x-x+x^3/3!-\cdots+(-1)^n x^{2n-1}/(2n-1)!$ *and* $P_n(x)=\cos x-1+x^2/2!-\cdots+(-1)^{n+1}x^{2n}/(2n)!$, *then*

$$\frac{x^{2n+1}}{(2n+1)!}\cdot\frac{2n(2n+1)}{2n(2n+1)+x^{2}}\leqslant|S_n(x)|\leqslant\frac{x^{2n+1}}{(2n+1)!}\cdot\frac{(2n+2)(2n+3)}{(2n+2)(2n+3)+x^{2}}$$

and

$$\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n+1)(2n+2)}{(2n+1)(2n+2)+x^2}\leqslant|P_n(x)|\leqslant\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n+3)(2n+4)}{(2n+3)(2n+4)+x^2}.$$

Proof. Let $x\in(0,\pi/2)$, and

$$f_1(x)=-\sin x+x-\frac{x^3}{3!}+\cdots+\frac{x^{4n-3}}{(4n-3)!},\quad f_2(x)=\cos x-1+\frac{x^2}{2!}-\cdots+\frac{x^{4n-2}}{(4n-2)!},$$
$$f_3(x)=\sin x-x+\frac{x^3}{3!}-\cdots+\frac{x^{4n-1}}{(4n-1)!},\quad f_4(x)=-\cos x+1-\frac{x^2}{2!}+\cdots+\frac{x^{4n}}{(4n)!},$$
$$f_5(x)=-\sin x+x-\frac{x^3}{3!}-\cdots+\frac{x^{4n+1}}{(4n+1)!},\quad f_6(x)=\cos x-1+\frac{x^2}{2!}-\cdots+\frac{x^{4n+2}}{(4n+2)!},$$
$$f_7(x)=\sin x-x+\frac{x^3}{3!}-\cdots+\frac{x^{4n+3}}{(4n+3)!},\quad f_8(x)=-\cos x+1-\frac{x^2}{2!}+\cdots+\frac{x^{4n+4}}{(4n+4)!}.$$

We can show that $\sin:x\in(0,\pi/2)\mapsto\sin x$ and $\cos:x\in(0,\pi/2)\mapsto\cos x$ are geometrically concave functions according to Lemma 1 and simple computations. From Lemma 3, we clearly see that the following functions are geometrically concave on $(0,\pi/2)$:

$$\int_0^x\sin t\,dt=1-\cos x,\quad\int_0^x(1-\cos t)\,dt=x-\sin x,$$
$$\int_0^x(t-\sin t)\,dt=\cos x-1+\frac{x^2}{2},\cdots.$$

Hence functions $f_i\,(i=1,2,\cdots,8)$ are geometrically concave on $\left(0,\frac{\pi}{2}\right)$. For $0<a<x$ and $i=1,\cdots,7,8$, Theorem 2 leads to

$$\int_a^x f_i(t)\,dt\geqslant\frac{xf_i(x)-af_i(a)}{\log(xf_i(x))-\log(af_i(a))}\cdot(\log x-\log a).$$

Let $a\to0+$. Then we have

$$\int_0^x f_i(t)\,dt\geqslant\frac{xf_i(x)}{1+\lim\limits_{a\to0+}\frac{a\cdot f_i'(a)}{f_i(a)}}$$

and

$$f_{i+1}(x)\geqslant\frac{xf_i(x)}{1+\lim\limits_{a\to0+}\frac{a\cdot f_i'(a)}{f_i(a)}}.\tag{15}$$

By making use of L'Hospital rule in inequality (15), we get

$$f_{i+1}(x)\geqslant\frac{xf_i(x)}{4n+i+1}.$$

Therefore

$$f_3(x) \geqslant \frac{xf_1(x)}{4n+1}, \quad f_2(x) \geqslant \frac{xf_1(x)}{4n},$$
$$f_4(x) \geqslant \frac{xf_3(x)}{4n+2}, \quad f_5(x) \geqslant \frac{xf_4(x)}{4n+3},$$

$$f_3(x) \geqslant \frac{xf_2(x)}{4n+1} \geqslant \frac{x^2 f_1(x)}{4n(4n+1)} = \frac{x^2}{4n(4n+1)}\left(-f_3(x) + \frac{x^{4n-1}}{(4n-1)!}\right),$$

$$4n(4n+1) f_3(x) \geqslant -x^2 f_3(x) + \frac{x^{4n+1}}{(4n-1)!},$$

and

$$f_3(x) \geqslant \frac{x^{4n+1}}{(4n+1)!} \cdot \frac{4n(4n+1)}{4n(4n+1)+x^2}. \tag{16}$$

Furthermore, we get

$$\frac{xf_3(x)}{4n+2} \leqslant f_4(x) \leqslant \frac{4n+3}{x} f_5(x) = \frac{4n+3}{x}\left(-f_3(x) + \frac{x^{4n+1}}{(4n+1)!}\right),$$
$$x^2 f_3(x) \leqslant -(4n+2)(4n+3) f_3(x) + (4n+2)(4n+3)\frac{x^{4n+1}}{(4n+1)!}$$

and

$$f_3(x) \leqslant \frac{x^{4n+1}}{(4n+1)!} \cdot \frac{(4n+2)(4n+3)}{(4n+2)(4n+3)+x^2}. \tag{17}$$

According to (16) and (17), we have

$$\frac{x^{4n+1}}{(4n+1)!} \cdot \frac{4n(4n+1)}{4n(4n+1)+x^2} \leqslant f_3(x) \leqslant \frac{x^{4n+1}}{(4n+1)!} \cdot \frac{(4n+2)(4n+3)}{(4n+2)(4n+3)+x^2}.$$

Similarly, we can get

$$\frac{x^{4n+2}}{(4n+2)!} \cdot \frac{(4n+1)(4n+2)}{(4n+1)(4n+2)+x^2} \leqslant f_4(x) \leqslant \frac{x^{4n+2}}{(4n+2)!} \cdot \frac{(4n+3)(4n+4)}{(4n+3)(4n+4)+x^2},$$

$$\frac{x^{4n+3}}{(4n+3)!} \cdot \frac{(4n+2)(4n+3)}{(4n+2)(4n+3)+x^2} \leqslant f_5(x) \leqslant \frac{x^{4n+3}}{(4n+3)!} \cdot \frac{(4n+4)(4n+5)}{(4n+4)(4n+5)+x^2},$$

$$\frac{x^{4n+4}}{(4n+4)!} \cdot \frac{(4n+3)(4n+4)}{(4n+3)(4n+4)+x^2} \leqslant f_6(x) \leqslant \frac{x^{4n+4}}{(4n+4)!} \cdot \frac{(4n+5)(4n+6)}{(4n+5)(4n+6)+x^2},$$

so the proof of Theorem 4 is completed. □

Acknowledgements. This work was supported by the NSF of Zhejiang Radio & TV University under Grant No. XKT-07G19 and China Central Radio and TV University under Grant No. GEQ1633.

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(Received November 4, 2008)

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