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EXTENDED CESÁRO OPERATORS ON BMOA SPACES IN THE UNIT BALL

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Abstract. Let g be a holomorphic map of B, where B is the unit ball of C^n. This paper gives necessary and sufficient conditions for the extended Cesáro operators induced by g to be bounded or compact on BMOA.

1. Introduction

Let $f(z)$ be a holomorphic function on the unit disc D with the Taylor expansion $f(z) = \sum\limits_{j=0}^{\infty} a_j z^j$. The classical Cesáro operator acting on f is defined by

$$\mathscr{C}[f](z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j.$$

In the past few years, boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions have been studied by many authors. It is well known that the operator $\mathscr{C}$ is bounded on the usual Hardy spaces $H^p(D)$ for $0 < p < \infty$. Basic results facts on Hardy spaces can be found in [5]. For $1 \leqslant p < \infty$, Siskakis [19] studied the spectrum of $\mathscr{C}$. As a by-product he obtained that $\mathscr{C}$ is bounded on $H^p(D)$. For $p = 1$, the boundedness of $\mathscr{C}$ was given also by Siskakis [21] by a particularly elegant method, independent of spectrum theory. A different proof of the result can be found in [8]. After that, for $0 < p < 1$, Miao [17] proved $\mathscr{C}$ is also bounded. For $p = \infty$, the boundedness of $\mathscr{C}$ was given by Danikas and Siskakis in [4]. It has been also shown that the operator $\mathscr{C}$ ia also bounded on the Bergman space (in [22]) as well as on the weighted Bergman spaces (in [2]). But the operator $\mathscr{C}$ is not always bounded. In [24], Shi and Ren gave a sufficient and necessary condition for the operator $\mathscr{C}$ to be bounded on mixed norm spaces in the unit disc, as well as Li and Stević's papers [11, 12, 15, 16]. Recently, Siskakis and Zhao in [23] obtained sufficient and necessary conditions for Volterra type operator, which is a generalization of $\mathscr{C}$, to be bounded or compact between BMOA spaces in the unit disc. It is a natural question to ask what are the conditions for higher dimensional case.

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Let dv be the Lebesgue measure on the unit ball B of C^n normalized so that $v(B) = 1$, and $d\tau$ the Möbius invariant measure on B. Let S denote the boundary of B, and $d\sigma$ denote the normalized surface measure. The class of all holomorphic functions on B is denoted by $H(B)$. For holomorphic function f in B we write

$$\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \cdots, \frac{\partial f(z)}{\partial z_n}\right), \quad Rf(z) = \langle \nabla f(z), \bar{z}\rangle = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}$$

and

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0).$$

A little calculation shows $\mathscr{C}[f](z) = \frac{1}{z}\int_0^z f(t)(\log \frac{1}{1-t})' dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesáro operator T_g on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}.$$

It is easy to show that T_g maps $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesáro operator is bounded or compact.

Motivated by [24], some authors [9, 10, 7, 13, 14, 29] discussed the extended Cesáro operators on the mixed norm space, Bloch space, Zygmund space, Hardy space, as well as Dirichlet space in the unit ball.

The space BMOA in the unit ball consists of those functions $f \in H^2(B)$ for which the set of Möbius translates $\{f \circ \varphi_a(z) - f(a)\}$ is bounded in $H^2(B)$, that is,

$$\begin{aligned} \rho(f) &= \sup_{a\in B} \|f \circ \varphi_a(z) - f(a)\|_{H^2} \\ &\sim \sup_{a\in B} \left\{ \int_B |Rf(z)|^2 \left(1-|z|^2\right)^2 \left(1-|\varphi_a(z)|^2\right)^n d\tau(z) \right\}^{1/2} \\ &\sim \sup_{a\in B} \left\{ \int_B |\widetilde{\nabla} f(z)|^2 \left(1-|\varphi_a(z)|^2\right)^n d\tau(z) \right\}^{1/2} < \infty, \end{aligned}$$

while VMOA contains those f for which

$$\lim_{|a|\to 1^-} \|f \circ \varphi_a(z) - f(a)\|_{H^2} = 0.$$

BMOA is a Banach space under the norm $\|f\|_* = |f(0)| + \rho(f)$ and VMOA is the closure of the set of polynomials in BMOA.

LMOA is defined to be the spaces of $f \in H(B)$ such that

$$\|f\|^2_{\mathrm{LMOA}} = \sup_{r>0, \xi\in S} \left\{ \frac{(\log \frac{2}{r^2})^2}{r^{2n}} \int_{Q_r(\xi)} |Rf(z)|^2 (1-|z|^2) dv(z) \right\} < \infty.$$

Throughout this paper we define $d(z,w) = |1 - \langle z, w\rangle|^{1/2}$ for z and w in the closed unit ball $\overline{B}$. For $\xi \in S$ and $r, \delta > 0$ we let

$$Q_r(\xi) = \{z \in B : d(z,\xi) < r\}$$

and

$$Q(\xi,\delta)=\{\eta\in S: d(\xi,\eta)<\delta\}.$$

It is obvious that $Q(\xi,\delta)=S$ when $\delta\geqslant\sqrt{2}$. Moreover, $\sigma\big(Q(\xi,\delta)\big)\sim\delta^{2n}$ and $Q_r(\xi)\sim r^{2(n+1)}$ for all $\xi\in S$, $\delta\in(0,\sqrt{2})$ and r (Lemma 4.6,[31]).

In this paper, we discuss the extended Cesáro operator between the BMOA space on the unit ball, and give some sufficient and necessary conditions for the operator to be bounded and compact. The main results of the paper are the following:

THEOREM 1. *T_g is bounded on* BMOA *if and only if* $g\in$ LMOA.

proclaimTheorem 2 T_g is compact on BMOA if and only if

$$\lim_{r\to 0}\sup_{r>0,\xi\in S}\left\{\frac{(\log\frac{2}{r^2})^2}{r^{2n}}\int_{Q_r(\xi)}|Rg(z)|^2(1-|z|^2)dv(z)\right\}=0$$

or equivalently

$$\lim_{|a|\to 1}\big(\log\frac{2}{1-|a|}\big)^2\int_B|Rg(z)|^2(1-|z|^2)^2\big(1-|\varphi_a(z)|^2\big)^n d\tau(z)=0.$$

2. Some Lemmas

In the following, we will use the symbol C to denote a finite positive number which does not depend on variable z and may depend on some norms and parameters n,f etc, not necessarily the same at each occurrence.

In order to prove the main result, we will give some lemmas first.

LEMMA 1. [31] (Th5.9) *Let μ be a positive Borel measure on B and $0<p<\infty$. Then μ is a Carleson measure if and only if there exists a constant $C>0$ such that*

$$\int_B|f(z)|^p d\mu(z)\leqslant\int_S|f(\xi)|^p d\sigma(\xi).$$

LEMMA 2. [31] (Th5.5) *The following conditions are equivalent:*
(1) f is in BMOA *(*VMOA*);*
(2) $(1-|z|^2)|\nabla f(z)|^2dv(z)$ is a (vanishing) Carleson measure;
(3) $(1-|z|^2)|Rf(z)|^2dv(z)$ is a (vanishing) Carleson measure;
(4) $(1-|z|^2)^n|\widetilde{\nabla}f(z)|^2d\tau(z)=\frac{\widetilde{\nabla}f(z)|^2dv(z)}{1-|z|^2}$ is a (vanishing) Carleson measure.

LEMMA 3. *There is a constant $C>0$, such that for all $f\in$* BMOA *and $z\in B$, the estimate*

$$|f(z)|\leqslant C\log\frac{2}{1-|z|}\|f\|_*$$

holds.

Proof. First, recall that Bloch consists of those f in $H(B)$ with $\sup\limits_{z\in B}(1-|z|^2)|\nabla f(z)| < \infty$, and whose norm is defined by $\|f\|_{\text{Bloch}} = |f(0)| + \sup\limits_{z\in B}(1-|z|^2)|\nabla f(z)|$. For all $f \in$ Bloch and $z \in B$, we have

$$\begin{aligned}
|f(z)| &\leqslant |f(0)| + \int_0^1 \langle z, \overline{\nabla f(tz)}\rangle dt \\
&\leqslant ||f||_{\text{Bloch}}\big(1 + \int_0^{|z|} \frac{dt}{1-t^2}\big) \\
&\leqslant \big(1 + \frac{1}{2}\log\frac{4}{1-|z|^2}\big)||f||_{\text{Bloch}} \\
&\leqslant C\log\frac{2}{1-|z|}||f||_{\text{Bloch}}
\end{aligned}$$

and the conclusion follows immediately by Theorem 3.19 in [31].

LEMMA 4. [31] (Page 49) *If f is holomorphic in B, then*

$$(1-|z|^2)|Rf(z)| \leqslant (1-|z|^2)|\nabla f(z)| \leqslant |\widetilde{\nabla} f(z)|.$$

LEMMA 5. *Let $g \in$ BMOA. Then the seminorm $\|g\|_{\text{LMOA}}$ is equivalent to the seminorm given by*

$$(\|g\|'_{\text{LMOA}})^2 = \sup_{a\in B}\big(\log\frac{2}{1-|a|}\big)^2\int_B |Rg(z)|^2(1-|z|^2)^2\big(1-|\varphi_a(z)|^2\big)^n d\tau(z).$$

Proof. On one hand, for $r \geqslant 1$, with $g \in$ BMOA and Lemma 2, the conclusion is obvious. Without loss of generality, we may assume $0 < r < 1$. For $\xi \in S$, let $a = (1-r^2)\xi$, then

$$\begin{aligned}
&\frac{(\log\frac{2}{r^2})^2}{r^{2n}}\int_{Q_r(\xi)} |Rg(z)|^2(1-|z|^2)dv(z) \\
\leqslant\ & C\big(\log\frac{2}{1-|a|}\big)^2\int_{Q_r(\xi)} |Rg(z)|^2(1-|z|^2)^2\big(1-|\varphi_a(z)|^2\big)^n d\tau(z) \\
\leqslant\ & C\big(\log\frac{2}{1-|a|}\big)^2\int_B |Rg(z)|^2(1-|z|^2)^2\big(1-|\varphi_a(z)|^2\big)^n d\tau(z)
\end{aligned}$$

where we have used the inequality $\frac{1-|z|^2}{r^2} \leqslant C\big(1-|\varphi_a(z)|^2\big)$ for $z \in Q_r(\xi)$.

Conversely, for $|a| \leqslant \frac{3}{4}$ the estimate is trivial. Now, assume $|a| > \frac{3}{4}$, $\xi = \frac{a}{|a|}$ and for $k \geqslant 0$ define $r_k = \sqrt{2^{k+1}(1-|a|)}$, $E_k = Q_{r_k}(\xi) - Q_{r_{k-1}}(\xi)$, and $E_{-1} = \phi$. Let N be the smallest positive integer such that $Q(\xi, r_N) = S$.

Since $1-\langle a,w\rangle = 1-|a|+|a|(1-\langle\xi,w\rangle)$ for $k \geqslant 1$ and $w \in E_k$, it follows that

$$|1-\langle a,w\rangle| \geqslant 2^{k-1}(1-|a|),$$

so

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = \frac{1-|a|^2}{|1-\langle a,z\rangle|^2} \leqslant \frac{C}{4^{k-1}(1-|a|)}.$$

Then

$$\begin{aligned}
&\int_B |Rg(z)|^2(1-|z|^2)^2\big(1-|\varphi_a(z)|^2\big)^n d\tau(z) \\
&= \sum_{k=0}^{N}\int_{E_k} |Rg(z)|^2(1-|z|^2)^2\big(1-|\varphi_a(z)|^2\big)^n d\tau(z) \\
&\leqslant \sum_{k=0}^{N} \frac{C}{4^{nk}(1-|a|)^n}\int_{E_k} |Rg(z)|^2(1-|z|^2)dv(z) \\
&\leqslant C\left(\sum_{k=0}^{N} \frac{1}{2^{nk}\log^2\frac{2}{2^{k+1}(1-|a|)}}\right)\|g\|^2_{\mathrm{LMOA}} \\
&\leqslant \frac{C}{\log^2\frac{1}{1-|a|}}\|g\|^2_{\mathrm{LMOA}} \\
&\leqslant \frac{C}{\log^2\frac{2}{1-|a|}}\|g\|^2_{\mathrm{LMOA}}.
\end{aligned}$$

The last inequality holds since $\log\frac{2}{1-|a|} \leqslant 2\log\frac{1}{1-|a|}$ for $|a| > \frac{3}{4}$. The proof of the lemma is completed.

LEMMA 6. *Let $g \in H(B)$, then*

$$R[T_g f](z) = f(z)Rg(z)$$

for any $f \in H(B)$ and $z \in B$.

Proof. Suppose the holomorphic function fRg has the Taylor expansion

$$(fRg)(z) = \sum_{|\alpha|\geqslant 1} a_\alpha z^\alpha.$$

Then we have

$$\begin{aligned}
R(T_g f)(z) &= R\int_0^1 f(tz)R(tz)\frac{dt}{t} = R\int_0^1 \sum_{|\alpha|\geqslant 1} a_\alpha (tz)^\alpha\frac{dt}{t} \\
&= R\Big[\sum_{|\alpha|\geqslant 1}\frac{a_\alpha z^\alpha}{|\alpha|}\Big] = \sum_{|\alpha|\geqslant 1} a_\alpha z^\alpha = (fRg)(z).
\end{aligned}$$

3. The Proof Of Theorem 1

We prove the sufficiency first.

For a given $f \in \mathrm{BMOA}$, we will show $F = T_g(f) \in \mathrm{BMOA}$. Let $\xi \in S, 1 > r > 0$. Then

$$\begin{aligned}
&\frac{1}{r^{2n}} \int_{Q_r(\xi)} |RF(z)|^2 (1-|z|^2) dv(z) \\
&= \frac{1}{r^{2n}} \int_{Q_r(\xi)} |f(z)|^2 |Rg(z)|^2 (1-|z|^2) dv(z) \\
&\leqslant \frac{2}{r^{2n}} \int_{Q_r(\xi)} |f(z)-f(u)|^2 |Rg(z)|^2 (1-|z|^2) dv(z) \\
&+ \frac{2}{r^{2n}} \int_{Q_r(\xi)} |f(u)|^2 |Rg(z)|^2 (1-|z|^2) dv(z) \\
&= 2J_1 + 2J_2.
\end{aligned}$$

We estimate J_1 first. Consider the automorphism $\varphi_u(z)$, which maps 0 to u, where $u = (1-r^2)\xi \in B$ with $0 < r < 1$. For any $z \in Q_r(\xi)$,

$$|1-\langle z,u\rangle| = |(1-r^2)\big(1-\langle z,\xi\rangle\big) + r^2| \leqslant (1-r^2)r^2 + r^2 \leqslant 2r^2,$$

so it follows that

$$\frac{1-|u|^2}{|1-\langle z,u\rangle|^2} \geqslant \frac{(1+|u|)(1-|u|)}{4r^4} \geqslant \frac{1-|u|}{4r^4} = \frac{1}{4r^2}$$

and

$$1-|\varphi_u(z)|^2 = \frac{(1-|u|^2)(1-|z|^2)}{|1-\langle z,u\rangle|^2} \geqslant \frac{1-|z|^2}{4r^2}.$$

With Lemma 4 and the Möbius invariant of BMOA, we have

$$\begin{aligned}
J_1 &= \frac{1}{r^{2n}} \int_{Q_r(\xi)} |f(z)-f(u)|^2 \frac{|\widetilde{\nabla} g(z)|^2}{1-|z|^2} dv(z) \\
&\leqslant \frac{(4r^2)^n}{r^{2n}} \int_{Q_r(\xi)} |f(z)-f(u)|^2 |\widetilde{\nabla} g(z)|^2 \big(1-|\varphi_u(z)|^2\big)^n d\tau(z) \\
&\leqslant C \int_B |f\circ\varphi_u(w)-f(u)|^2 |\widetilde{\nabla}(g\circ\varphi_u)(w)|^2 (1-|w|^2)^n d\tau(w) \\
&\leqslant C\rho(g\circ\varphi_u)^2 \int_S |f\circ\varphi_u(\eta)-f(u)|^2 d\sigma(\eta) \\
&\leqslant C\|g\|_*^2 \|f\|_*^2.
\end{aligned}$$

Next, we estimate J_2.

$$\begin{aligned}
J_2 &= \frac{|f(u)|^2}{r^{2n}} \int_{Q_r(\xi)} |Rg(z)|^2 (1-|z|^2) dv(z) \\
&\leqslant C\|f\|_*^2 \frac{\log^2(\frac{2}{r^2})}{r^{2n}} \int_{Q_r(\xi)} |Rg(z)|^2 (1-|z|^2) dv(z) \\
&\leqslant C\|g\|_{\mathrm{LMOA}}^2,
\end{aligned}$$

where we have used Lemma 3. For the case $1 \leqslant r < \sqrt{2}$, the proof is a modification of the case $0 < r < 1$.

Fix $0 < r_0 < 1$, and set $u = (1 - r_0^2)\xi$. Then for any $z \in Q_r(\xi) - Q_{r_0}(\xi)$, the case $z \in Q_{r_0}(\xi)$ is reduced to the case $0 < r < 1$. Since

$$|1 - \langle z, u\rangle| = |(1 - r_0^2)(1 - \langle z, \xi\rangle) + r_0^2| \leqslant (1 - r_0^2)r^2 + r_0^2 \leqslant r^2,$$

it follows that

$$\frac{1 - |u|^2}{|1 - \langle z, u\rangle|^2} \geqslant \frac{(1 + |u|)(1 - |u|)}{r^4} \geqslant \frac{1 - |u|}{r^4} = \frac{r_0^2}{r^4}$$

and

$$1 - |\varphi_u(z)|^2 = \frac{(1 - |u|^2)(1 - |z|^2)}{|1 - \langle z, u\rangle|^2} \geqslant \frac{r_0^2(1 - |z|^2)}{r^4}.$$

Therefore

$$\begin{aligned}
J_1' &= \frac{1}{r^{2n}} \int_{Q_r(\xi) - Q_{r_0}(\xi)} |f(z) - f(u)|^2 \frac{|\widetilde{\nabla} g(z)|^2}{1 - |z|^2} dv(z) \\
&\leqslant \frac{(r^4)^n}{(r_0^2)^n \times r^{2n}} \int_{Q_r(\xi) - Q_{r_0}(\xi)} |f(z) - f(u)|^2 |\widetilde{\nabla} g(z)|^2 (1 - |\varphi_u(z)|^2)^n d\tau(z) \\
&\leqslant C \int_B |f \circ \varphi_u(w) - f(u)|^2 |\widetilde{\nabla}(g \circ \varphi_u)(w)|^2 (1 - |w|^2)^n d\tau(w) \\
&\leqslant C\rho(g \circ \varphi_u)^2 \int_S |f \circ \varphi_u(\eta) - f(u)|^2 d\sigma(\eta) \\
&\leqslant C\|g\|_*^2 \|f\|_*^2
\end{aligned}$$

and

$$\begin{aligned}
J_2' &= \frac{|f(u)|^2}{r^{2n}} \int_{Q_r(\xi)} |Rg(z)|^2 (1 - |z|^2) dv(z) \\
&\leqslant C\|f\|_*^2 \frac{\log^2(\frac{2}{r_0^2})}{r^{2n}} \int_{Q_r(\xi)} |Rg(z)|^2 (1 - |z|^2) dv(z) \\
&\leqslant C.
\end{aligned}$$

The last inequality holds because of $g \in \mathrm{BMOA}$, and the desired result follows.

Now, we prove the necessity.

Suppose $T_g : \mathrm{BMOA} \to \mathrm{BMOA}$ is bounded. The result follows by $g \in \mathrm{BMOA}$ for $r \geqslant 1$. For the case $0 < r < 1$, $\xi \in S$ we put $f_a(z) = \log \frac{1}{1 - \langle z, a\rangle}$, where $a = (1 - r^2)\xi$. Since $r^2 \leqslant |1 - \langle z, u\rangle| \leqslant 2r^2$ there is a constant C such that

$$\frac{1}{C} \log \frac{2}{r^2} \leqslant |f_a(z)| \leqslant C \log \frac{2}{r^2}$$

for all $z \in Q_r(\xi)$. It follows that

$$\begin{aligned}
\frac{(\log \frac{2}{r^2})^2}{r^{2n}} &\int_{Q_r(\xi)} |Rg(z)|^2 (1-|z|^2) dv(z) \\
&\leqslant \frac{C^2}{r^{2n}} \int_{Q_r(\xi)} |f_a(z)|^2 |Rg(z)|^2 (1-|z|^2) dv(z) \\
&\leqslant \frac{C^2}{r^{2n}} \int_{Q_r(\xi)} |RT_g(f_a)(z)|^2 (1-|z|^2) dv(z) \\
&\leqslant C \|T_g(f_a)\|_*^2 \\
&\leqslant C \|T_g\|^2 \|f_a\|_*^2 \\
&\leqslant C \|T_g\|^2.
\end{aligned}$$

This completes the proof of the theorem.

4. The Proof of Theorem 2

Now, suppose g satisfies the condition. We will show that T_g is compact. It is well known that the operator T_g is compact on BMOA if and only if every bounded sequence $\{f_m\}$ in BMOA with $f_m \to 0$ uniformly on compact subsets of B has a subsequence $\{f_{m_k}\}$ such that $T_g f_{m_k} \to 0$ in BMOA. For any $\varepsilon > 0$, there is some $r \in (0,1)$ such that

$$\big(\log \frac{2}{1-|a|}\big)^2 \int_B |Rg(z)|^2 (1-|z|^2)^2 \big(1-|\varphi_a(z)|^2\big)^n d\tau(z) < \varepsilon, r \leqslant |a| < 1.$$

So we get

$$\begin{aligned}
\|T_g f_m\| &\leqslant \sup_{|z|\leqslant r} \int_B |f_m(z)|^2 |Rg(z)|^2 (1-|z|^2)^2 \big(1-|\varphi_a(z)|^2\big)^n d\tau(z) \\
&+ \sup_{|z|> r} \int_B |f_m(z)|^2 |Rg(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\tau(z) \\
&< 3\varepsilon
\end{aligned}$$

if m is sufficiently large.

Conversely, we assume T_g is compact. Let $0 < r_j < 1$ with $r_j \to 0$, and let $u_j = (1-r_j^2)\xi$ with $\xi \in S$. Denote

$$f_j(z) = \log \frac{1}{1-\langle z, u_j\rangle}, \quad f_0(z) = \log \frac{1}{1-\langle z, \xi\rangle}, \quad q_j(z) = \log \frac{1-\langle z, \xi\rangle}{1-\langle z, u_j\rangle}.$$

As in the proof of Theorem 1, we have for any j

$$
\begin{aligned}
\frac{(\log\frac{2}{r_j^2})^2}{r_j^{2n}}\int_{Q_{r_j}(\xi)} &|Rg(z)|^2(1-|z|^2)dv(z)\\
&\leqslant \frac{C}{r_j^{2n}}\int_{Q_{r_j}(\xi)}|f_j(z)|^2|Rg(z)|^2(1-|z|^2)dv(z)\\
&\leqslant \frac{2C}{r_j^{2n}}\int_{Q_{r_j}(\xi)}|q_j(z)|^2|Rg(z)|^2(1-|z|^2)dv(z)\\
&+\frac{2C}{r_j^{2n}}\int_{Q_{r_j}(\xi)}|f_0(z)|^2|Rg(z)|^2(1-|z|^2)dv(z)\\
&\leqslant C\|T_g(q_j)\|_* + \frac{2C}{r_j^{2n}}\int_{Q_{r_j}(\xi)}|R(T_g(f_0))|^2(1-|z|^2)dv(z).
\end{aligned}
$$

We just need to show the two terms above converge to 0.

It is easy to check that q_j is uniformly bounded and $q_j \to 0$ uniformly on compact sets, so the compactness of T_g implies $\lim_{j\to\infty}\|T_g(q_j)\|_* = 0$.

For the second term, since $\mathrm{VMOA}^{**} = \mathrm{BMOA}$, $T_g^{**} = T_g$, and T_g compact on BMOA implies weakly compact on VMOA, it follows that $T_g(\mathrm{BMOA}) \subset \mathrm{VMOA}$, so $T_g(f_0) \in \mathrm{VMOA}$. By Lemma 2, $|R(T_g(f_0))|^2(1-|z|^2)dv(z)$ is a vanishing Carleson measure, so the second term converges to 0. The desired conclusion follows.

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