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SUPER–STABILITY AND STABILITY OF THE EXPONENTIAL EQUATIONS

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Abstract. In this paper, we introduce the concepts of super-stability, stability, δ-Ger-stability of the exponential equation of an operator from a normed space into a normed algebra. By using some results due to J. Baker and R. Ger, we obtain some new conclusions about these stabilities.

1. Introduction

In 1940, Ulam gave a wide-ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [1]). Among those was the question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, Hyers answered in his paper [2] the question of Ulam for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been significantly generalized. In 1950 Aoki [11] allowed growth of the form $K(\|x\|^p + \|y\|^p)$ for the norm of the Cauchy difference

$$f(x+y) - f(x) - f(y)$$

where $0 \leqslant p < 1$, and still obtained the formula

$$A(x) = \lim_{n\to\infty} \left\{ \frac{f(2^n x)}{2^n} \right\}$$

for the additive mapping approximating f, the result was rediscovered 28 years latte by Rassias [3] (cf. [12]). Since then, the stability problems of various functional equations have been investigated by a number of authors (see [4–10], [13–14]).

Throughout the paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{R}$, the sets of all complex numbers, of all natural numbers, of all real numbers, respectively, and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $(X, \|\cdot\|)$

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be a normed space over $\mathbb{F}$, $(B, \|\cdot\|)$ a normed algebra over $\mathbb{F}$ and let $M(X,B)$ be the set of all mappings from X into B. For an $f : X \to B$, the general solutions and the stability problem of the exponential operator equation (EOE)

$$f(x+y) = f(x)f(y)(\forall x, y \in X) \tag{1.0}$$

have been intensively studied (ref. [5–9]).

DEFINITION 1.1. A mapping $f : X \to B$ is said to be exponential if

$$(E(f))(x,y) := f(x+y) - f(x)f(y) = 0(\forall x, y \in X).$$

If $\|f\|_\infty := \sup_{x\in X} \|f(x)\| < \infty$, then f is said to be bounded.

Clearly, the equation (1.0) is equivalent to $E(f) = 0$. In the case that the algebra B is an abelian normed algebra with an identity e and $f(X) \subset \mathrm{Inv}B$, (1.0) is equivalent to

$$f(x+y)f(x)^{-1}f(y)^{-1} - e = 0(\forall x, y \in X).$$

DEFINITION 1.2. If every unbounded mapping $f : X \to B$ satisfying

$$\|E(f)\|_\infty := \sup_{(x,y)\in X^2} \|f(x+y) - f(x)f(y)\| < \infty \tag{1.1}$$

is exponential, then the equation (1.0) is called to be super-stable in $M(X,B)$.

The first result for the super-stability of equation (1.0) was proved by Bourgin (see [6]). Later, this problem was renewed and investigated by Baker, Lawrence and Zorzitto [7], and also by Baker [8].

Motivated by Baker [8], we prove the following generalization of the Baker's result.

THEOREM 1.3. *Suppose that B is a Banach algebra satisfying $\|ab\| = \|a\|\|b\|$. If a mapping $f : X \to B$ satisfies the inequality* (1.1)*, then either $\|f(x)\| \leqslant \frac{1+\sqrt{1+4\delta}}{2}(\forall x \in X)$ where $\delta = \|E(f)\|_\infty$, or f is exponential.*

Proof. Putting $\varepsilon = \frac{1+\sqrt{1+4\delta}}{2}$, we have $\varepsilon^2 - \varepsilon = \delta$ and $\varepsilon > 1$. Suppose that $\exists x_0 \in X$ such that $\|f(x_0)\| > \varepsilon$. Thus, $\alpha := \|f(x_0)\| - \varepsilon > 0$ and we have

$$\begin{aligned}
\|f(2x_0)\| &= \|f(x_0)^2 - (f(x_0)^2 - f(2x_0))\| \\
&\geqslant \|f(x_0)^2\| - \|f(x_0)^2 - f(2x_0)\| \\
&\geqslant \|f(x_0)\|^2 - \delta \\
&= (\varepsilon+\alpha)^2 - \delta \\
&= (\varepsilon+\alpha) + (2\varepsilon-1)\alpha + \alpha^2 \\
&> \varepsilon + 2\alpha.
\end{aligned}$$

Suppose that $\|f(2^n x_0)\| > \varepsilon + (n+1)\alpha$ for some positive integer n. Then

$$\begin{aligned}\|f(2^{n+1}x_0)\| &= \|f(2^n x_0)^2 - (f(2^n x_0)^2 - f(2^{n+1}x_0))\| \\ &\geqslant \|f(2^n x_0)^2\| - \|f(2^n x_0)^2 - f(2^{n+1}x_0)\| \\ &\geqslant \|f(2^n x_0)\|^2 - \delta \\ &\geqslant (\varepsilon + (n+1)\alpha)^2 - \delta \\ &= \varepsilon + 2(n+1)\alpha + \alpha^2(n+1)^2 \\ &> \varepsilon + (n+2)\alpha.\end{aligned}$$

By induction, we have $\|f(2^n x_0)\| > \varepsilon + (n+1)\alpha$ for all $n \in \mathbb{N}$. For all $x,y,z \in X$, we have that

$$\|f(x+y+z) - f(x+y)f(z)\| \leqslant \delta$$

and

$$\|f(x+y+z) - f(x)f(y+z)\| \leqslant \delta,$$

so $\|f(x+y)f(z) - f(x)f(y+z)\| \leqslant 2\delta$. Hence

$$\begin{aligned}&\|f(x+y)f(z) - f(x)f(y)f(z)\| \\ &\leqslant \|f(x+y)f(z) - f(x)f(y+z)\| + \|f(x)f(y+z) - f(x)f(y)f(z)\| \\ &\leqslant 2\delta + \|f(x)\|\delta.\end{aligned}$$

Thus, for all $x,y,z \in X$,

$$\|f(x+y) - f(x)f(y)\| \cdot \|f(z)\| \leqslant 2\delta + \|f(x)\|\delta.$$

In particular, putting in the inequality above $z = 2^n x_0$, we get

$$\|f(x+y) - f(x)f(y)\| \leqslant (2\delta + \|f(x)\|\delta)/\|f(2^n x_0)\|$$

for all $x,y \in X$ and all $n \in \mathbb{N}$. By letting $n \to \infty$, we know that $f(x+y) = f(x)f(y)$ for all $x,y \in X$, which means that f is exponential. $\square$

COROLLARY 1.4. ([8]) *If a mapping $f : X \to \mathbb{C}$ satisfies the inequality* (1.1)*, then either $|f(x)| \leqslant \frac{1+\sqrt{1+4\delta}}{2}$ $(\forall x \in X)$ where $\delta = \|E(f)\|_\infty$, or f is exponential.*

On the other hand, Ger noted that the super-stability phenomenon of the exponential equation is caused by the fact that the natural group structure in the range space is disregarded. So it seems more natural to introduce the following.

DEFINITION 1.5. Let B be a normed algebra with an identity e and $\mathrm{Inv}B$ be the set of all invertible elements of B and $\delta \geqslant 0$. If for every mapping $f : X \to \mathrm{Inv}B$ satisfying the inequality

$$\|f(x+y)f(x)^{-1}f(y)^{-1} - e\| \leqslant \delta \tag{1.2}$$

for all $(x,y) \in X^2$, there exists an exponential mapping $m : X \to \mathrm{Inv}B$, two functions Φ, $\Psi : [0,\infty) \to [0,\infty)$ such that $\|m(x)f(x)^{-1} - e\| \leqslant \Phi(\delta)$ and $\|f(x)m(x)^{-1} - e\| \leqslant \Psi(\delta)$ for all $x \in X$, then the equation (1.0) is said to be δ-Ger-stable in $M(X,B)$.

In 1996, Ger and Šemrl [9] proved the following result.

THEOREM 1.6. ([9]) *If a functional $f : X \to \mathbb{C} \setminus \{0\}$ satisfies the inequality* (1.2) *for some $\delta \in [0,1)$ and for all $(x,y) \in X^2$, then there exists an exponential functional $m : X \to \mathbb{C} \setminus \{0\}$ such that*

$$\max\left\{\left|\frac{m(x)}{f(x)} - 1\right|, \left|\frac{f(x)}{m(x)} - 1\right|\right\} \leqslant \left(1 + \frac{1}{(1-\delta)^2} - 2\sqrt{\frac{1+\delta}{1-\delta}}\right)^{\frac{1}{2}}, \tag{1.3}$$

for all $x \in X$.

From Theorem 1.6 and Definition 1.5, we get the following.

COROLLARY 1.7. *The exponential equation* (1.0) *is δ-Ger-stable in $M(X,\mathbb{C})$ whenever $\delta \in [0,1)$.*

2. Super-stability of Exponential Operator Equations

In this part, we will discuss the super-stability of the equation (1.0) for f from X into some special spaces.

If for

$$x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n,$$

we define

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}, xy = (x_1y_1, \ldots, x_ny_n), \tag{2.1}$$

then $(\mathbb{F}^n, \|\cdot\|_2)$ becomes a unital Banach algebra.

THEOREM 2.1. *If an operator $f = (f_1, \ldots, f_n) : X \to \mathbb{F}^n$ satisfies* (1.1)*, then either there exists a k_0 such that*

$$\sup_{x\in X} |f_{k_0}(x)| \leqslant \frac{1+\sqrt{1+4\delta}}{2},$$

where $\delta = \|E(f)\|_\infty$, or f is exponential.

Proof. From (1.1), for all $(x,y) \in X^2$, we have

$$\|f(x+y) - f(x)f(y)\|_2^2 = \sum_{i=1}^{n} |f_k(x+y) - f_k(x)f_k(y)|^2 \leqslant \delta^2,$$

hence $|f_k(x+y) - f_k(x)f_k(y)| \leqslant \delta$ for all $(x,y) \in X^2$ and for $1 \leqslant k \leqslant n$. By Corollary 1.4, for each k, either $\sup_{x\in X} |f_k(x)| \leqslant \frac{1+\sqrt{1+4\delta}}{2}$ or f_k is exponential. If for any k there exists an $x_k \in X$ such that $|f_k(x_k)| > \frac{1+\sqrt{1+4\delta}}{2}$, then each f_k is exponential and so is f. $\square$

Denote by $M_n(\mathbb{F})$ the Banach algebra of all n by n matrices over $\mathbb{F}$ with the norm

$$\|[a_{i,j}]\|_2 = \left(\sum_{i,j=1}^{n} |a_{i,j}|^2\right)^{\frac{1}{2}}. \tag{2.2}$$

THEOREM 2.2. *If an operator $f=[f_{i,j}]:X\to M_n(\mathbb{F})$ satisfies* (1.1), *then either there exist i_0 and j_0 such that*

$$|f_{i_0,j_0}(x)|\leqslant\frac{1+\sqrt{1+4\delta}}{2}$$

where $\delta=\|E(f)\|_\infty$, or f is exponential.

Proof. Similar to the proof of Theorem 2.1. □

For $x=(x_n)_{n\in\mathbb{N}},y=(y_n)_{n\in\mathbb{N}}\in l^p(\mathbb{F})$ define $xy=(x_ny_n)_{n\in\mathbb{N}}$. Then the Banach space $l^p(\mathbb{F})$ becomes a Banach algebra with the norm $\|\cdot\|_p$, where $1\leqslant p\leqslant\infty$.

THEOREM 2.3. *Let $1\leqslant p\leqslant\infty$. If $f=\{f_n\}_{n\in\mathbb{N}}:X\to l^p(\mathbb{F})$ satisfies* (1.1), *then either there exists a k_0 such that*

$$\sup_{x\in X}|f_{k_0}(x)|\leqslant\frac{1+\sqrt{1+4\delta}}{2},$$

where $\delta=\|E(f)\|_\infty$, or f is exponential.

Proof. Since $f(x)=(f_n(x))_{n\in\mathbb{N}}$, we know from (1.1) that for all $x,y\in X$,

$$|f_k(x+y)-f_k(x)f_k(y)|\leqslant\|f(x+y)-f(x)f(y)\|_p\leqslant\delta$$

then for any $k\in\mathbb{N}$ and for all $(x,y)\in X^2$. It follows from Corollary 1.4 that for any $k\in\mathbb{N}$, either $\sup_{x\in X}|f_k(x)|\leqslant\frac{1+\sqrt{1+4\delta}}{2}$ or f_k is exponential. If for every k, there exists an $x_k\in X$ such that $|f_k(x_k)|>\frac{1+\sqrt{1+4\delta}}{2}$, then each f_k is exponential and so is f. □

THEOREM 2.6. *Let $\mathscr{A}$ be a unital semi-simple abelian Banach algebra with the character space $\Phi_{\mathscr{A}}$. If an operator $f:X\to\mathscr{A}$ satisfies* (1.1), *then either there exists a $\varphi_0\in\Phi_{\mathscr{A}}$ such that*

$$\sup_{x\in X}|(\varphi_0\circ f)(x)|\leqslant\frac{1+\sqrt{1+4\delta}}{2},$$

where $\delta=\|E(f)\|_\infty$, or f is exponential.

Proof. For any $\varphi\in\Phi_{\mathscr{A}}$, by (1.1) we have that for all $(x,y)\in X^2$,

$$\begin{aligned}|(\varphi\circ f)(x+y)-(\varphi\circ f)(x)(\varphi\circ f)(y)|&=|\varphi(f(x+y)-f(x)f(y))|\\&\leqslant\|\varphi\|\cdot\|f(x+y)-f(x)f(y)\|\\&\leqslant\delta.\end{aligned}$$

Thus by Corollary 1.4, for any $\varphi\in\Phi_{\mathscr{A}}$, either $\sup\limits_{x\in X}|(\varphi\circ f)(x)|\leqslant\frac{1+\sqrt{1+4\delta}}{2}$ or $\varphi\circ f$ is exponential. Suppose that for any $\varphi\in\Phi_{\mathscr{A}}$, there exists an $x_\varphi\in X$ such that

$$|(\varphi\circ f)(x_\varphi)|>\frac{1+\sqrt{1+4\delta}}{2},$$

then $\varphi\circ f$ is exponential. Thus, for all $(x,y)\in X^2$, we have $\varphi((E(f))(x,y))=0$ for any $\varphi\in\Phi_{\mathscr{A}}$. Note that $\mathscr{A}$ is semi-simple, so $(E(f))(x,y)=0$ for all $(x,y)\in X^2$, hence f is exponential. □

3. Ger-Stability of Exponential Operator Equations

Let $e=(1,1,\ldots,1)$ be the identity of the algebra $\mathbb{F}^n$ discussed in Section 2. Denote by $\mathrm{Inv}\,\mathbb{F}^n$ the set of all invertible elements in $\mathbb{F}^n$. Clearly, for $x=(x_1,\ldots,x_n)\in \mathrm{Inv}\,\mathbb{F}^n$, the inverse of x is given by $x^{-1}=(x_1^{-1},\ldots,x_n^{-1})$. Also, $\mathbb{F}^n$ becomes a unital Banach algebra over $\mathbb{F}$ with respect to the norm $\|\cdot\|_\infty$.

THEOREM 3.1. *If an operator $f=(f_1,\ldots,f_n):X\to \mathrm{Inv}\,\mathbb{F}^n$ satisfies*

$$\sup_{x,y\in X}\|f(x+y)f(x)^{-1}f(y)^{-1}-e\|_\infty\leqslant\delta \tag{3.1}$$

for some $\delta\in[0,1)$, then there exists an exponential operator $m:X\to \mathrm{Inv}\,\mathbb{F}^n$ such that

$$\max\{\|f(x)m(x)^{-1}-e\|_\infty,\|m(x)f(x)^{-1}-e\|_\infty\}\leqslant M_\delta \tag{3.2}$$

for all $x\in X$, where $M_\delta=\left(1+\frac{1}{(1-\delta)^2}-2\sqrt{\frac{1+\delta}{1-\delta}}\right)^{\frac{1}{2}}$. Hence, the exponential equation (1.0) is δ-Ger-stable in $M(X,\mathbb{F}^n)$ whenever $\delta\in[0,1)$.

Proof. Since $f(x)=(f_1(x),\ldots,f_n(x))$ and $f(x)^{-1}=(f_1(x)^{-1},\ldots,f_n(x)^{-1})$, we know from (3.1) that

$$\sup_{x,y\in X}|f_k(x+y)f_k(x)^{-1}f_k(y)^{-1}-1|\leqslant\delta,\ \text{ for } 1\leqslant k\leqslant n.$$

By Theorem 1.6, we have for any k, there is an exponential operator $m_k:X\to\mathbb{F}\setminus\{0\}$ such that

$$\max\left\{\left|\frac{f_k(x)}{m_k(x)}-1\right|,\left|\frac{m_k(x)}{f_k(x)}-1\right|\right\}\leqslant M_\delta$$

for any $x\in X$. Put $m(x)=(m_1(x),\ldots,m_n(x))$. Then $m:X\to\mathrm{Inv}\,\mathbb{F}^n$ is an exponential operator satisfying

$$\|f(x)m(x)^{-1}-e\|_\infty=\max_{1\leqslant k\leqslant n}\left|\frac{f_k(x)}{m_k(x)}-1\right|\leqslant M_\delta,$$

$$\|f(x)m(x)^{-1}-e\|_\infty=\max_{1\leqslant k\leqslant n}\left|\frac{m_k(x)}{f_k(x)}-1\right|\leqslant M_\delta,$$

for all $x\in X$. $\square$

EXAMPLE 3.2. Define $f:\mathbb{C}\to\mathbb{C}\setminus\{0\}$ by $f(x)=2e^x$, then

$$\sup_{x,y\in\mathbb{C}}\left|f(x+y)f(x)^{-1}f(y)^{-1}-1\right|=\sup_{x,y\in\mathbb{C}}\left|\frac{2e^{x+y}}{4e^xe^y}-1\right|=\frac{1}{2}.$$

Take $\delta=\frac{1}{2}$, then $M_\delta=5-2\sqrt{3}$. Clearly, $m(x)=e^x$ is the exponential function from $\mathbb{C}$ to $\mathbb{C}\setminus\{0\}$ satisfying

$$\left|\frac{f(x)}{m(x)}-1\right|=1,\ \text{and}\ \left|\frac{m(x)}{f(x)}-1\right|=\frac{1}{2}.$$

Moreover, $\max\{1,\frac{1}{2}\} \leqslant 5-2\sqrt{3}$.

For any two elements A,B of the vector space $M_n(\mathbb{C})$, we define that $AB=[a_{ij}b_{ij}]$ and $\|A\|_\infty = \max\limits_{1\leqslant i,j\leqslant n}|a_{ij}|$. Then $(M_n(\mathbb{C}),\|\cdot\|_\infty)$ becomes a unital Banach algebra with the identity

$$E=\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

Clearly,

$$\operatorname{Inv} M_n(\mathbb{C}) = \left\{[a_{ij}] \in M_n(\mathbb{C}) : a_{ij} \neq 0 (\forall i,j)\right\}.$$

COROLLARY 3.3. *The exponential equation* (1.0) *is* δ*-Ger-stable in* $M(X,\mathbb{F}^n)$ *whenever* $\delta \in [0,1)$.

Denote by $\operatorname{Inv} l^\infty(\mathbb{F})$ the set of all invertible elements in the unital abelian Banach algebra $l^\infty(\mathbb{F})$. Let $e=(1,1,\dots)$ be the identity of $l^\infty(\mathbb{F})$. For every $x=\{x_n\}_{n\in\mathbb{N}} \in l^\infty(\mathbb{F})$ in $\operatorname{Inv} l^\infty(\mathbb{F})$, we have $x^{-1}=\{x_n^{-1}\}_{n\in\mathbb{N}}$.

THEOREM 3.4. *If an operator* $f=\{f_n\}_{n\in\mathbb{N}} : X \to \mathit{Inv}\, l^\infty(\mathbb{F})$ *satisfies*

$$\sup_{x,y\in X} \|f(x+y)f(x)^{-1}f(y)^{-1}-e\|_\infty \leqslant \delta, \tag{3.4}$$

for some $\delta \in [0,1)$*, then there exists an exponential operator* $m : X \to \mathit{Inv}\, l^\infty(\mathbb{F})$ *such that*

$$\max\{\|f(x)m(x)^{-1}-e\|_\infty, \|m(x)f(x)^{-1}-e\|_\infty\} \leqslant M_\delta, \tag{3.5}$$

for any $x\in X$*, where* $M_\delta = \left(1+\frac{1}{(1-\delta)^2}-2\sqrt{\frac{1+\delta}{1-\delta}}\right)^{\frac{1}{2}}$. *Hence, the exponential equation* (1.0) *is* δ*-Ger-stable in* $M(X,l^\infty(\mathbb{F})$ *whenever* $\delta\in[0,1)$.

Proof. By (3.4), we have

$$\sup_{x,y\in X} |f_k(x+y)f_k(x)^{-1}f_k(y)^{-1}-1| \leqslant \sup_{x,y\in X} \|f(x+y)f(x)^{-1}f(y)^{-1}-e\|_\infty \leqslant \delta,$$

for any $k\in\mathbb{N}$. By Theorem 1.6, we have for any $k\in\mathbb{N}$, there is an exponential operator $m_k : X \to \mathbb{C}\setminus\{0\}$ such that

$$\max\left\{\left|\frac{f_k(x)}{m_k(x)}-1\right|, \left|\frac{m_k(x)}{f_k(x)}-1\right|\right\} \leqslant M_\delta, \tag{3.6}$$

for all $x\in X$. Put $m(x)=(m_1(x),\dots,m_n(x),\dots)$, then for all $x\in X$, we have from (3.6) that

$$|m_k(x)| \leqslant (1+M_\delta)|f_k(x)| \leqslant (1+M_\delta)\|f(x)\|_\infty < \infty (\forall k\in\mathbb{N}).$$

Thus, we get a mapping $m : X \to \operatorname{Inv} l^\infty(\mathbb{F})$, which is an exponential operator satisfying:

$$\|f(x)m(x)^{-1}-e\|_\infty = \sup_{k\in\mathbb{N}} |f_k(x)m_k(x)^{-1}-1| \leqslant M_\delta$$

$$\|m(x)f(x)^{-1}-e\|_\infty = \sup_{k\in\mathbb{N}} |m_k(x)f_k(x)^{-1}-1| \leqslant M_\delta$$

for all $x\in X$. This completes the proof. $\square$

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