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ON SOME BOUNDS OF OSTROWSKI AND ČEBYŠEV TYPE

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Abstract. Making use of an identity of Dragomir and Barnett, proved in [13] [published in *J. Indian Math. Soc. (N.S.)*, **66** (1999), No. 1-4, 237-245], some new Ostrowski and Čebyšev type inequalities involving two functions have been developed. Bounds obtained for the new established Ostrowski and Čebyšev type inequalities are of interest and are better than the bounds available in the literature for these type of inequalities.

1. Introduction

In 1882, P. L. Čebyšev [4] proved that, if $f', g' \in L_\infty[a,b]$, then,

$$|T(f,g)| \leqslant \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1}$$

where for two functions $f,\ g : [a,b] \to \mathbb{R}$, the functional is

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right). \tag{2}$$

In 1935, G. Grüss [17] showed that

$$|T(f,g)| \leqslant \frac{1}{4}(M-m)(N-n), \tag{3}$$

provided m, M, n and N are real numbers satisfying the conditions,

$$m \leqslant f(x) \leqslant M,\ n \leqslant g(x) \leqslant N,$$

for all $x \in [a,b]$, where $T(f,g)$ is as defined by (2).

In 1938, Ostrowski proved the following integral inequality [20]:

Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping that is differentiable in the interior of I $(Int\, I)$, and let $a,\ b \in Int\ I$, $a < b$. If $|f'(t)| \leqslant M$, $\forall t \in (a,b)$, then,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leqslant \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M, \tag{4}$$

for all $x \in [a,b]$.

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Pachpatte in [23] proved the following results.

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $|f'|$, $|g'|$ are convex on $[a,b]$ and $f',\ g'\in L_\infty[a,b]$, then*

$$|S_s(f,g)(x)| \leqslant \left[|g(x)|\left(|f'(x)|+\|f'\|_\infty\right)+|f(x)|\left(|g'(x)|+\|g'\|_\infty\right)\right] \times\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\frac{b-a}{4}, \tag{5}$$

where

$$S_s(f,g)(x)=f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)\,dt+f(x)\int_a^b g(t)\,dt\right],$$

for all $x\in[a,b]$.

THEOREM 2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $|f'|$, $|g'|$ are convex on $[a,b]$ and $f',\ g'\in L_\infty[a,b]$, then*

$$|T(f,g)| \leqslant \frac{1}{4(b-a)^2}\int_a^b \left[|g(x)|\left(|f'(x)|+\|f'\|_\infty\right) +|f(x)|\left(|g'(x)|+\|g'\|_\infty\right)\right]E(x)\,dx, \tag{6}$$

where $E(x)=\frac{(x-a)^2+(b-x)^2}{2}$ for all $x\in[a,b]$.

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $|f'|$, $|g'|$ are convex on $[a,b]$ and $f',\ g'\in L_\infty[a,b]$, then*

$$|T(f,g)| \leqslant \frac{1}{4(b-a)^3}\int_a^b \left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|+\|f'\|_\infty\right) \times\left(\left|g'\left(\frac{a+b}{2}\right)\right|+\|g'\|_\infty\right)\right]E^2(x)\,dx, \tag{7}$$

where $E(x)=\frac{(x-a)^2+(b-x)^2}{2}$ for all $x\in[a,b]$.

During the past few years, many researchers have given considerable attention to the above results and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1]–[16], [21]–[25], and the references cited therein. Motivated by the recent results given in [3], [13], [15], [23]–[25] and [26], inequalities similar to those given by Ostrowski and Čebyšev involving two functions whose second derivatives belong to Lebesgue spaces, can be established. The analysis used in the proofs is simple and based on the use of integral identities proved in [13] and [15].

The main aim of this paper is to obtain some new Ostrowski and Čebyšev type inequalities and their bounds involving two functions whose second derivatives belong to $L_\infty[a,b]$ and $L_1[a,b]$ spaces.

2. New Ostrowski and Čebyšev Type Inequalities

We consider the usual Lebesgue norms, defined as:

$$\|g\|_\infty := \operatorname*{ess\,sup}_{t\in[\alpha,\beta]} |g(t)| < \infty,$$

and

$$\|g\|_\ell := \left[\int_\alpha^\beta |g(t)|^\ell \, dt\right]^{1/\ell}, \ \ell \in [1,\infty);$$

provided that the integral and the supremum are finite. We use the following notation to simplify the details of presentation:

$$\begin{aligned} S_p(f,g)(x) = {} & f(x)g(x) - \frac{1}{b-a}\left(f(x)\int_a^b g(t)\,dt + g(x)\int_a^b f(t)\,dt\right) \\ & + MN\left(x-\frac{a+b}{2}\right)^2 - \left(x-\frac{a+b}{2}\right)[Mg(x)+Nf(x)] \\ & + \frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\left(M\int_a^b g(t)\,dt + N\int_a^b f(t)\,dt\right) \\ & + \frac{1}{(b-a)^2}\left(\int_a^b f(t)\,dt \int_a^b g(t)\,dt\right), \end{aligned} \tag{8}$$

$x \in [a,b]$, where

$$M = \frac{f(b)-f(a)}{b-a}, \qquad N = \frac{g(b)-g(a)}{b-a}, \tag{9}$$

and at the mid-point

$$\begin{aligned} S_m(f,g) = {} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) + \frac{1}{(b-a)^2}\left(\int_a^b f(t)\,dt \int_a^b g(t)\,dt\right) \\ & - \frac{1}{b-a}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt + g\left(\frac{a+b}{2}\right)\int_a^b f(t)\,dt\right]. \end{aligned} \tag{10}$$

Also,

$$\begin{aligned}
T_p(f,g) &= \frac{1}{b-a}\int_a^b S_p(f,g)(x)\,dx \\
&= \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) \\
&\quad + \frac{(b-a)^2}{12}MN - \frac{1}{b-a}M\int_a^b \left(x-\frac{a+b}{2}\right)g(x)\,dx \\
&\quad - \frac{1}{b-a}N\int_a^b \left(x-\frac{a+b}{2}\right)f(x)\,dx, \qquad (11)
\end{aligned}$$

where M and N are defined by (9).

We also use the following notations to simplify the details:

$$\begin{aligned}
\widetilde{S}(f,g)(x) &= f(x)g(x) - \frac{1}{2(b-a)}\left(f(x)\int_a^b g(t)\,dt + g(x)\int_a^b f(t)\,dt\right) \\
&\quad - \frac{1}{2}\left(x-\frac{a+b}{2}\right)\left[Mg(x)+Nf(x)\right], \qquad (12)
\end{aligned}$$

and at the mid-point

$$\begin{aligned}
\widetilde{S_m}(f,g) &= f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\quad - \frac{1}{2(b-a)}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt + g\left(\frac{a+b}{2}\right)\int_a^b f(t)\,dt\right]. \qquad (13)
\end{aligned}$$

Also

$$\begin{aligned}
\widetilde{T}(f,g) &= \frac{1}{b-a}\int_a^b \widetilde{S}(f,g)(x)\,dx \\
&= \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) \\
&\quad - \frac{1}{2(b-a)}\left[M\int_a^b \left(x-\frac{a+b}{2}\right)g(x)\,dx\right. \\
&\quad \left. + N\int_a^b \left(x-\frac{a+b}{2}\right)f(x)\,dx\right]. \qquad (14)
\end{aligned}$$

THEOREM 4. *Let $f,\ g:[a,b]\to\mathbb{R}$ be twice differentiable in (a,b). If f'' and g'' belong to the usual Lebesgue spaces $L_\infty[a,b]$ and $L_1[a,b]$, then*

$$\begin{aligned}
\left|S_p(f,g)(x)\right| &\leqslant \frac{\|f''\|_\infty\|g''\|_\infty}{4}\left[\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right]^2(b-a)^4 \\
&\leqslant \frac{(b-a)^4}{36}\left\|f''\right\|_\infty\left\|g''\right\|_\infty, \qquad (15)
\end{aligned}$$

and

$$\left|S_p(f,g)(x)\right| \leqslant \frac{9}{16}\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2 \left\|f''\right\|_1 \left\|g''\right\|_1 \leqslant \frac{9}{16}(b-a)^2 \left\|f''\right\|_1 \left\|g''\right\|_1, \tag{16}$$

for all $x \in [a,b]$, *and also*

$$\left|T_p(f,g)\right| \leqslant \frac{3(b-a)^4}{280} \left\|f''\right\|_\infty \left\|g''\right\|_\infty, \tag{17}$$

and

$$\left|T_p(f,g)\right| \leqslant \frac{9\left\|f''\right\|_1 \left\|g''\right\|_1}{16(b-a)} \int_a^b \left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2 dx, = \frac{21(b-a)^2}{64} \left\|f''\right\|_1 \left\|g''\right\|_1. \tag{18}$$

Proof. Utilizing the Montgomery identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b P(x,t) f'(t)\,dt, \tag{19}$$

for all $x \in [a,b]$, where $f : [a,b] \longrightarrow \mathbb{R}$ is differentiable on (a,b) and belongs to the usual Lebesgue spaces and the kernel $P(x,t) : [a,b]^2 \longrightarrow \mathbb{R}$ is defined as:

$$P(x,t) = \begin{cases} t-a & \text{if} \quad t \in [a,x] \\ t-b & \text{if} \quad t \in (x,b]. \end{cases} \tag{20}$$

We observe that

$$\sup_t |P(x,t)| = \left|x-\frac{a+b}{2}\right| + \frac{b-a}{2}, \tag{21}$$

$$\int_a^b \left(\sup_t |P(x,t)|\right) dx = \frac{3}{4}(b-a)^2, \tag{22}$$

$$\int_a^b P(x,t)\,dt = (b-a)\left(x-\frac{a+b}{2}\right), \tag{23}$$

$$\int_a^b |P(x,t)|\,dt = \frac{(x-a)^2+(b-x)^2}{2}, \tag{24}$$

and

$$\int_a^b \int_a^b |P(x,t)|\,|P(t,s)|\,ds\,dt = \frac{1}{2}\left[\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2 + \frac{1}{12}\right](b-a)^4, \tag{25}$$

for verification and proof of (25), see also $([13],\ p.\,239-241)$.

Dragomir and Barnett [13] used the above identity (19) for $f'(x)$ of the form

$$f'(x)=\frac{1}{b-a}\int_a^b f'(t)\,dt+\frac{1}{b-a}\int_a^b P(x,t)\,f''(t)\,dt, \tag{26}$$

which is equivalent to

$$f'(x)=\frac{f(b)-f(a)}{b-a}+\frac{1}{b-a}\int_a^b P(x,t)\,f''(t)\,dt. \tag{27}$$

Using (27) in the right membership of (19), the authors obtained the representation,

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\quad+\frac{1}{b-a}\int_a^b P(x,t)\left(\frac{f(b)-f(a)}{b-a}+\frac{1}{b-a}\int_a^b P(t,s)\,f''(s)\,ds\right)dt. \end{aligned} \tag{28}$$

Using (9) and (23) in (28), we have

$$\begin{aligned} & f(x)-M\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\qquad=\frac{1}{(b-a)^2}\int_a^b\int_a^b P(x,t)\,P(t,s)\,f''(s)\,dsdt. \end{aligned} \tag{29}$$

Similarly

$$\begin{aligned} & g(x)-N\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(t)\,dt \\ &\qquad=\frac{1}{(b-a)^2}\int_a^b\int_a^b P(x,t)\,P(t,s)\,g''(s)\,dsdt. \end{aligned} \tag{30}$$

Multiplying the left and right sides of the identities (29) and (30), we get

$$\begin{aligned} f(x)\,g(x) &- \frac{1}{b-a}\left(f(x)\int_a^b g(t)\,dt+g(x)\int_a^b f(t)\,dt\right) \\ &+ MN\left(x-\frac{a+b}{2}\right)^2-\left(x-\frac{a+b}{2}\right)\left[Mg(x)+Nf(x)\right] \\ &+ \frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\left(M\int_a^b g(t)\,dt+N\int_a^b f(t)\,dt\right) \\ &+ \frac{1}{(b-a)^2}\left(\int_a^b f(t)\,dt\int_a^b g(t)\,dt\right) \\ = \frac{1}{(b-a)^4}&\left(\int_a^b\int_a^b P(x,t)\,P(t,s)\,f''(s)\,dsdt\right) \\ &\times\left(\int_a^b\int_a^b P(x,t)\,P(t,s)\,g''(s)\,dsdt\right). \end{aligned} \tag{31}$$

Using (8) in (31), we have

$$S_p(f,g)(x) = \frac{1}{(b-a)^4}\left(\int_a^b\int_a^b P(x,t)P(t,s)f''(s)\,dsdt\right) \times \left(\int_a^b\int_a^b P(x,t)P(t,s)g''(s)\,dsdt\right), \tag{32}$$

from which applying the modulus on both sides gives the inequality

$$\left|S_p(f,g)(x)\right| \leqslant \frac{1}{(b-a)^4}\left(\int_a^b\int_a^b |P(x,t)|\,|P(t,s)|\,\left|f''(s)\right|dsdt\right) \times \left(\int_a^b\int_a^b |P(x,t)|\,|P(t,s)|\,\left|g''(s)\right|dsdt\right). \tag{33}$$

For $f'',\ g'' \in L_\infty[a,b]$, from (33), we have

$$\begin{aligned}\left|S_p(f,g)(x)\right| &\leqslant \frac{1}{(b-a)^4}\left(\sup_s\left|f''(s)\right|\right)\left(\int_a^b\int_a^b |P(x,t)|\,|P(t,s)|\,dsdt\right) \\ &\quad\times\left(\sup_s\left|g''(s)\right|\right)\left(\int_a^b\int_a^b |P(x,t)|\,|P(t,s)|\,dsdt\right) \\ &= \frac{\|f''\|_\infty\|g''\|_\infty}{(b-a)^4}\left(\int_a^b\int_a^b |P(x,t)|\,|P(t,s)|\,dsdt\right)^2. \end{aligned}\tag{34}$$

From (25) and (34), we deduce the first part of the inequality (15). We know that $\left|x-\frac{a+b}{2}\right| \leqslant \frac{b-a}{2}$ for all $x\in[a,b]$, so the second part of the inequality (15) is obvious.

Using (21), (22) and (33) for $f'',\ g'' \in L_1[a,b]$, we have

$$\begin{aligned}\left|S_p(f,g)(x)\right| &\leqslant \frac{1}{(b-a)^4}\left(\sup_t|P(x,t)|\int_a^b\sup_s|P(t,s)|\int_a^b f''(s)\,dsdt\right) \\ &\quad\times\left(\sup_t|P(x,t)|\int_a^b\sup_s|P(t,s)|\int_a^b g''(s)\,dsdt\right) \\ &\leqslant \frac{1}{(b-a)^4}\left(\sup_t|P(x,t)|\right)^2\left(\int_a^b\sup_s|P(t,s)|\,dt\right)^2\left\|f''\right\|_1\left\|g''\right\|_1 \\ &= \frac{1}{(b-a)^4}\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2\left(\frac{3}{4}(b-a)^2\right)^2\left\|f''\right\|_1\left\|g''\right\|_1 \\ &= \frac{9}{16}\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2\left\|f''\right\|_1\left\|g''\right\|_1, \end{aligned}\tag{35}$$

we have the first part of the inequality (16), the second part of the (16) is obvious.

Integrating (32) with respect to x from a to b and dividing with $(b-a)$, we have

$$\frac{1}{b-a}\int_a^b S_p(f,g)(x)\,dx = \frac{1}{(b-a)^5}\int_a^b\left[\int_a^b\int_a^b P(x,t)P(t,s)f''(s)\,dsdt \times \int_a^b\int_a^b P(x,t)P(t,s)g''(s)\,dsdt\right]dx, \tag{36}$$

which gives the identity

$$T_p(f,g) = \frac{1}{(b-a)^5}\int_a^b \left[\int_a^b\int_a^b P(x,t)P(t,s)f''(s)\,dsdt \right.$$
$$\left. \times \int_a^b\int_a^b P(x,t)P(t,s)g''(s)\,dsdt\right] dx. \tag{37}$$

Applying properties of the modulus, we get from (37),

$$\left|T_p(f,g)\right| \leqslant \frac{1}{(b-a)^5}\int_a^b \left[\left(\int_a^b\int_a^b |P(x,t)||P(t,s)|\left|f''(s)\right| dsdt\right)\right.$$
$$\left. \times \left(\int_a^b\int_a^b |P(x,t)||P(t,s)|\left|g''(s)\right| dsdt\right)\right] dx. \tag{38}$$

From (25), we have

$$\int_a^b \left(\int_a^b\int_a^b |P(x,t)||P(t,s)|\,dsdt\right) dx$$
$$= \int_a^b \frac{1}{2}\left[\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right](b-a)^4\,dx$$
$$= \frac{1}{10}(b-a)^5, \tag{39}$$

and so

$$\int_a^b \left[\left(\int_a^b\int_a^b |P(x,t)||P(t,s)|\,dsdt\right)\left(\int_a^b\int_a^b |P(x,t)||P(t,s)|\,dsdt\right)\right] dx$$
$$= \frac{(b-a)^8}{4}\int_a^b \left[\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right]^2 dx$$
$$= \frac{3(b-a)^9}{280}. \tag{40}$$

When f'', $g'' \in L_\infty[a,b]$, then from (38) and (40), we have

$$\left|T_p(f,g)\right| \leqslant \frac{3(b-a)^4}{280}\left\|f''\right\|_\infty \left\|g''\right\|_\infty,$$

so that the inequality (17) results.

For f'', $g'' \in L_1[a,b]$, using the integral mean of (35) and from (21), (22) and (38), we get

$$\left|T_p(f,g)\right| \leqslant \frac{9\left\|f''\right\|_1\left\|g''\right\|_1}{16(b-a)}\int_a^b \left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2 dx. \tag{41}$$

This completes the proof of first part of the inequality (18).

Now,

$$\int_a^b \left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2 dx = \frac{2}{3}\left(1-\frac{1}{2^3}\right)(b-a)^3 = \frac{7}{12}(b-a)^3. \tag{42}$$

From (41) and (42), we have the second part of the inequality (18). □

COROLLARY 1. *Under the assumptions of Theorem 4, we have the mid-point inequalities*

$$|S_m(f,g)| \leqslant \frac{49(b-a)^4}{9216}\left\|f''\right\|_\infty \left\|g''\right\|_\infty, \tag{43}$$

and

$$|S_m(f,g)| \leqslant \frac{9(b-a)^2}{64}\left\|f''\right\|_1 \left\|g''\right\|_1. \tag{44}$$

Proof. By putting $x=\frac{a+b}{2}$ in the first parts of the inequalities (15) and (16) respectively, we get the above mentioned mid-point inequalities. □

REMARK 1. If f'', $g'' \in L_p[a,b]$, using Hölder's Inequality in (33) and (38), we can get another expression. However, the details are omitted.

THEOREM 5. *Let $f,g:[a,b]\to\mathbb{R}$ be twice differentiable in (a,b). If f, g, f'', $g''\in L_\infty[a,b]$ and $L_1[a,b]$, then*

$$\begin{aligned}\left|\widetilde{S}(f,g)(x)\right| &\leqslant \frac{|g(x)|\,\|f''\|_\infty+|f(x)|\,\|g''\|_\infty}{4}\left[\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right](b-a)^2 \\ &\leqslant \frac{(b-a)^2}{12}\left(|g(x)|\left\|f''\right\|_\infty+|f(x)|\left\|g''\right\|_\infty\right),\end{aligned} \tag{45}$$

and

$$\begin{aligned}\left|\widetilde{S}(f,g)(x)\right| &\leqslant \frac{3\left(|g(x)|\,\|f''\|_1+|f(x)|\,\|g''\|_1\right)}{8}\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right) \\ &\leqslant \frac{3(b-a)}{8}\left(|g(x)|\left\|f''\right\|_1+|f(x)|\left\|g''\right\|_1\right),\end{aligned} \tag{46}$$

for all $x\in[a,b]$. Further,

$$\left|\widetilde{T}(f,g)\right| \leqslant \frac{(b-a)^2}{20}\left(\|g\|_\infty\left\|f''\right\|_\infty+\|f\|_\infty\left\|g''\right\|_\infty\right), \tag{47}$$

and

$$\left|\widetilde{T}(f,g)\right| \leqslant \frac{3\left(\|g\|_\infty \|f''\|_1 + \|f\|_\infty \|g''\|_1\right)}{8(b-a)} \int_a^b \left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right) dx,$$
$$= \frac{9(b-a)}{32}\left(\|g\|_\infty \left\|f''\right\|_1 + \|f\|_\infty \left\|g''\right\|_1\right). \quad (48)$$

Proof. Multiplying both sides of (29) and (30) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting by using (9) and (12), we have:

$$f(x)g(x) - \frac{1}{2(b-a)}\left(f(x)\int_a^b g(t)\,dt + g(x)\int_a^b f(t)\,dt\right)$$
$$-\frac{1}{2}\left(x - \frac{a+b}{2}\right)\left[Mg(x) + Nf(x)\right]$$
$$= \frac{1}{2(b-a)^2}\left(g(x)\int_a^b\int_a^b P(x,t)P(t,s)f''(s)\,dsdt\right.$$
$$\left. + f(x)\int_a^b\int_a^b P(x,t)P(t,s)g''(s)\,dsdt\right), \quad (49)$$

which implies

$$\widetilde{S}(f,g)(x) = \frac{1}{2(b-a)^2}\left(g(x)\int_a^b\int_a^b P(x,t)P(t,s)f''(s)\,dsdt\right.$$
$$\left. + f(x)\int_a^b\int_a^b P(x,t)P(t,s)g''(s)\,dsdt\right). \quad (50)$$

Using properties of the modulus, we observe that

$$\left|\widetilde{S}(f,g)(x)\right| \leqslant \frac{1}{2(b-a)^2}\left(|g(x)|\int_a^b\int_a^b |P(x,t)||P(t,s)|\left|f''(s)\right| dsdt\right.$$
$$\left. + |f(x)|\int_a^b\int_a^b |P(x,t)||P(t,s)|\left|g''(s)\right| dsdt\right), \quad (51)$$

for $f'',\ g'' \in L_\infty[a,b]$, we have

$$\left|\widetilde{S}(f,g)(x)\right| \leqslant \frac{|g(x)|\,\|f''\|_\infty + |f(x)|\,\|g''\|_\infty}{2(b-a)^2}\int_a^b\int_a^b |P(x,t)||P(t,s)|\,dsdt. \quad (52)$$

Using (25) in (52), we deduce the first part of the inequality (45), the last part of (45) is obvious since $\left|x - \frac{a+b}{2}\right| \leqslant \frac{b-a}{2}$.

Now for $f'', g'' \in L_1[a,b]$, we have from (21), (22) and (51)

$$\left|\widetilde{S}(f,g)(x)\right| \leqslant \frac{1}{2(b-a)^2}\left[|g(x)|\left(\sup_t |P(x,t)|\right)\left(\int_a^b \sup_s |P(t,s)|\,dt\right)\left\|f''\right\|_1 + |f(x)|\left(\sup_t |P(x,t)|\right)\left(\int_a^b \sup_s |P(t,s)|\,dt\right)\left\|g''\right\|_1\right]$$
$$= \frac{3\left(|g(x)|\,\|f''\|_1 + |f(x)|\,\|g''\|_1\right)}{8}\left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right). \tag{53}$$

This completes the proof of the first part of the desired inequality (46), the last part (46) is obvious as $\left|x - \frac{a+b}{2}\right| \leqslant \frac{b-a}{2}$.

Integrating (50) over x from a to b, and dividing by $(b-a)$, we have

$$\left|\widetilde{T}(f,g)\right| \leqslant \frac{1}{2(b-a)^3}\int_a^b \left(|g(x)|\int_a^b\int_a^b |P(x,t)|\,|P(t,s)|\left|f''(s)\right| ds dt + |f(x)|\int_a^b\int_a^b |P(x,t)|\,|P(t,s)|\left|g''(s)\right| ds dt\right) dx. \tag{54}$$

For $f, g, f'', g'' \in L_\infty[a,b]$, we have by applying properties of the modulus on (54) and using (39),

$$\left|\widetilde{T}(f,g)\right| \leqslant \frac{(b-a)^2}{20}\left(\|g\|_\infty \left\|f''\right\|_\infty + \|f\|_\infty \left\|g''\right\|_\infty\right). \tag{55}$$

This gives us (47).

Now for $f, g, f'', g'' \in L_1[a,b]$, applying properties of the modulus on (54), using (21) and (22) gives

$$\left|\widetilde{T}(f,g)\right| \leqslant \frac{\|g\|_\infty \|f''\|_1 + \|f\|_\infty \|g''\|_1}{2(b-a)^3} \times \int_a^b \left(\sup_t |P(x,t)|\right)\left(\int_a^b \sup_s |P(t,s)|\,dt\right) dx$$
$$= \frac{3\left(\|g\|_\infty \|f''\|_1 + \|f\|_\infty \|g''\|_1\right)}{8(b-a)}\int_a^b \left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right) dx, \tag{56}$$

which completes the proof of the first part of inequality (48).

Using (22) in (56), we deduce the second part of the inequality (48). □

COROLLARY 2. *Under the assumptions of Theorem 5, we have the mid-point inequalities*

$$\left|\widetilde{S}_m(f,g)\right| \leqslant \frac{7(b-a)^2}{192}\left[\left|g\left(\frac{a+b}{2}\right)\right|\left\|f''\right\|_\infty + \left|f\left(\frac{a+b}{2}\right)\right|\left\|g''\right\|_\infty\right], \tag{57}$$

and

$$\left|\widetilde{S}_m(f,g)\right| \leqslant \frac{3(b-a)}{16}\left[\left|g\left(\frac{a+b}{2}\right)\right|\left\|f''\right\|_1 + \left|f\left(\frac{a+b}{2}\right)\right|\left\|g''\right\|_1\right]. \tag{58}$$

Proof. By putting $x = \frac{a+b}{2}$ in the first parts of the inequalities (45) and (46) respectively, we get the above mentioned mid-point inequalities. □

REMARK 2. If f'', $g'' \in L_p[a,b]$, using Hölder's inequality in (51) and (54), produces an expression whose details are omitted.

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