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SOME GRONWALL TYPE INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we investigate some Gronwall type inequalities on time scales, which provide explicit bounds on unknown functions. Our results unify and extend some continuous inequalities and their corresponding discrete analogues. Two applications of the main results are given in the end of this paper.

1. Introduction

In 1988 Hilger [1] introduced the calculus on time scales in order to unify the theory of continuous and discrete dynamic systems. Motivated by the paper [1], many authors have extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales. For example, we refer the reader to the literatures [2–8] and the references cited therein. In this paper, we investigate some Gronwall type inequalities on time scales, which unify and extend some continuous inequalities and their corresponding discrete analogues. The obtained inequalities can be used as important tools in the study of certain properties of dynamic equations on time scales.

2. Preliminaries on time scales

We first briefly introduce the time scales calculus, which can be found in [2, 3].

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{C}$ denotes the set of complex numbers, and $C(M,S)$ denotes the class of all continuous functions defined on set M with range in the set S. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The *forward jump operator* σ on $\mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T} \quad \text{for all} \quad t \in \mathbb{T}.$$

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In this definition we put $\inf(\emptyset) = \sup\mathbb{T}$, where $\emptyset$ is the empty set. If $\sigma(t) > t$, then we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup\mathbb{T}$, then we say that t is *right-dense*. The *backward jump operator*, *left–scattered* and *left–dense* points are defined in a similar way. The *graininess* $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left–scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} -\{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

REMARK 2.1. Clearly, we see that $\sigma(t) = t$ if $\mathbb{T} = \mathbb{R}$ and $\sigma(t) = t+1$ if $\mathbb{T} = \mathbb{Z}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \geqslant t_0$, $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number (provided it exists) such that given any $\varepsilon > 0$, there is a neighborhood U of t with

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s| \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t.

REMARK 2.2. f^Δ is the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and the usual forward difference $\boldsymbol{\Delta} f$ (defined by $\boldsymbol{\Delta} f(t) = f(t+1) - f(t)$) if $\mathbb{T} = \mathbb{Z}$.

We say that $f : \mathbb{T} \to \mathbb{R}$ is *rd–continuous* provided f is continuous at each right–dense point of $\mathbb{T}$ and has a finite left–sided limit at each left–dense point of $\mathbb{T}$. As usual, the set of rd–continuous functions is denoted by C_{rd}. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. In this case we define the *Cauchy integral* of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

We say that $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. We denote by $\mathscr{R}$ the set of all regressive and rd–continuous functions. We define the set of all positively regressive functions by $\mathscr{R}^+ = \{p \in \mathscr{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

THEOREM 2.1. *If $p \in \mathscr{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(\cdot, t_0)$ is for the unique solution of the initial value problem*

$$x^\Delta = p(t)x, \; x(t_0) = 1 \; on \; \mathbb{T}.$$

THEOREM 2.2. *If $p \in \mathscr{R}$, then*
(i) $e_p(t,t) \equiv 1$ *and* $e_0(t,s) \equiv 1$*;*
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$*;*
(iii) *if $p \in \mathscr{R}^+$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.*

REMARK 2.3. Clearly, the exponential function is given by

$$e_p(t,s) = e^{\int_s^t p(\tau)d\tau}, \; e_\alpha(t,s) = e^{\alpha(t-s)}, \; e_\alpha(t,0) = e^{\alpha t}$$

for $s, t \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function if $\mathbb{T} = \mathbb{R}$, and the exponential function is given by

$$e_p(t,s) = \prod_{\tau=s}^{t-1}[1 + p(\tau)], \; e_\alpha(t,s) = (1+\alpha)^{t-s}, \; e_\alpha(t,0) = (1+\alpha)^t$$

for $s,t \in \mathbb{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$ if $\mathbb{T} = \mathbb{Z}$.

THEOREM 2.3. *If $p \in \mathscr{R}$ and $a,b,c \in \mathbb{T}$, then*

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

THEOREM 2.4. *Let $t_0 \in \mathbb{T}^\kappa$ and $w : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ be continuous at (t,t), where $t \geqslant t_0$, $t \in \mathbb{T}^\kappa$ with $t > t_0$. Assume that $w^\Delta(t,\cdot)$ is rd–continuous on $[t_0,\sigma(t)]$. If for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0,\sigma(t)]$, such that*

$$|w(\sigma(t),\tau) - w(s,\tau) - w^\Delta(t,\tau)(\sigma(t)-s)| \leqslant \varepsilon|\sigma(t)-s| \ \textit{for all} \ s \in U,$$

where w^Δ denotes the derivative of w with respect to the first variable, then

$$g(t) := \int_{t_0}^t w(t,\tau)\Delta\tau$$

implies

$$g^\Delta(t) = \int_{t_0}^t w^\Delta(t,\tau)\Delta\tau + w(\sigma(t),t).$$

The following theorem is a foundational result in dynamic inequalities.

THEOREM 2.5. (Comparison Theorem). *Suppose $u,b \in \mathrm{C_{rd}}$, $a \in \mathscr{R}^+$. Then*

$$u^\Delta(t) \leqslant a(t)u(t) + b(t), \ \ t \geqslant t_0, \ t \in \mathbb{T}^\kappa$$

implies

$$u(t) \leqslant u(t_0)e_a(t,t_0) + \int_{t_0}^t e_a(t,\sigma(\tau))b(\tau)\Delta\tau, \ \ t \geqslant t_0, \ t \in \mathbb{T}^\kappa.$$

3. Main results

In this section, we investigate some Gronwall type inequalities on time scales. For convenience, we always assume that $t \geqslant t_0$, $t_0 \in \mathbb{T}^\kappa$.

THEOREM 3.1. *Assume that $u,f \in \mathrm{C_{rd}}$, $u(t)$ and $f(t)$ are nonnegative, and $c \geqslant 0$ is a constant. If $w(t,s)$ is defined as in Theorem 2.4 such that $w(t,s) \geqslant 0$ and $w^\Delta(t,s) \geqslant 0$ for $t,s \in \mathbb{T}$ with $s \leqslant t$, then*

$$u(t) \leqslant c + \int_{t_0}^t f(\tau)\left[u(\tau) + \int_{t_0}^\tau w(\tau,s)u(s)\Delta s\right]\Delta\tau, \ \ t \in \mathbb{T}^\kappa, \tag{3.1}$$

implies

$$u(t) \leqslant c\left[1 + \int_{t_0}^t f(\tau)e_{f+A}(\tau,t_0)\Delta\tau\right], \ t \in \mathbb{T}^\kappa, \tag{3.2}$$

where

$$A(t) = w(\sigma(t),t) + \int_{t_0}^{t} w^{\Delta}(t,s)\Delta s, \;\; t \in \mathbb{T}^{\kappa}. \tag{3.3}$$

Proof. Define a function $z(t)$ by the right side of (3.1). Then $z(t_0) = c$, $u(t) \leqslant z(t)$ and

$$\begin{aligned} z^{\Delta}(t) &= f(t)\left[u(t) + \int_{t_0}^{t} w(t,s)u(s)\Delta s\right] \\ &\leqslant f(t)\left[z(t) + \int_{t_0}^{t} w(t,s)z(s)\Delta s\right], \; t \in \mathbb{T}^{\kappa}. \end{aligned} \tag{3.4}$$

Define a function $v(t)$ by

$$v(t) = z(t) + \int_{t_0}^{t} w(t,s)z(s)\Delta s, \; t \in \mathbb{T}^{\kappa}. \tag{3.5}$$

Then $v(t_0) = z(t_0) = c$, $z(t) \leqslant v(t)$, $z^{\Delta}(t) \leqslant f(t)v(t)$ and $v(t)$ is nondecreasing for $t \in \mathbb{T}^{\kappa}$. Using Theorem 2.4, we obtain

$$\begin{aligned} v^{\Delta}(t) &= z^{\Delta}(t) + w(\sigma(t),t)z(t) + \int_{t_0}^{t} w^{\Delta}(t,s)z(s)\Delta s \\ &\leqslant f(t)v(t) + w(\sigma(t),t)v(t) + \int_{t_0}^{t} w^{\Delta}(t,s)v(s)\Delta s \\ &\leqslant \left[f(t) + w(\sigma(t),t) + \int_{t_0}^{t} w^{\Delta}(t,s)\Delta s\right]v(t) \\ &= [f(t) + A(t)]v(t), \; t \in \mathbb{T}^{\kappa}. \end{aligned} \tag{3.4}$$

Using Theorem 2.5 and noting $v(t_0) = c$, from (3.4), we have

$$v(t) \leqslant c e_{f+A}(t,t_0), \; t \in \mathbb{T}^{\kappa}. \tag{3.5}$$

Therefore,

$$z^{\Delta}(t) \leqslant c f(t) e_{f+A}(t,t_0), \; t \in \mathbb{T}^{\kappa}. \tag{3.5}$$

Integrating the inequality (3.5) from t_0 to t, we obtain

$$z(t) \leqslant c\left[1 + \int_{t_0}^{t} f(\tau)e_{f+A}(\tau,t_0)\Delta\tau\right], \; t \in \mathbb{T}^{\kappa}. \tag{3.6}$$

Clearly, the desired inequality (3.2) follows by using (3.6) in $u(t) \leqslant z(t)$. This completes the proof of Theorem 3.1. □

REMARK 3.1. By taking $w(t,s) = w(s)$, the inequality given in Theorem 3.1 reduces to the inequality given in [7, Theorem 1].

REMARK 3.2. The result of Theorem 3.1 holds for an arbitrary time scale. Therefore, using Theorem 3.1, we immediately obtain many results for some peculiar time scales. For example, letting $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ respectively, we can obtain Theorem 2.1(a_1) and Theorem 2.5(c_1) in [11].

THEOREM 3.2. *Assume that* $u, a, f \in \mathrm{C_{rd}}$, $u(t), a(t)$ *and* $f(t)$ *are nonnegative, and* $a(t)$ *is nondecreasing. If* $w(t,s)$ *is defined as in Theorem 2.4 such that* $w(t,s) \geqslant 0$ *and* $w^{\Delta}(t,s) \geqslant 0$ *for* $t, s \in \mathbb{T}$ *with* $s \leqslant t$, *then*

$$u(t) \leqslant a(t) + \int_{t_0}^{t} f(\tau)\left[u(\tau) + \int_{t_0}^{\tau} w(\tau,s)u(s)\Delta s\right]\Delta\tau, \quad t \in \mathbb{T}^{\kappa}, \tag{3.7}$$

implies

$$u(t) \leqslant a(t)\left[1 + \int_{t_0}^{t} f(\tau)e_{f+A}(\tau,t_0)\Delta(\tau)\right], \quad t \in \mathbb{T}^{\kappa}, \tag{3.8}$$

where $A(t)$ *is defined as in (3.3).*

Proof. Noting $a(t) \geqslant 0$ and $a(t)$ is nondecreasing, for any $\varepsilon > 0$, we observe that

$$\begin{aligned} \frac{u(t)}{a(t)+\varepsilon} &\leqslant 1 + \int_{t_0}^{t} f(\tau)\left[\frac{u(\tau)}{a(t)+\varepsilon} + \int_{t_0}^{\tau} w(\tau,s)\frac{u(s)}{a(t)+\varepsilon}\Delta s\right]\Delta\tau \\ &\leqslant 1 + \int_{t_0}^{t} f(\tau)\left[\frac{u(\tau)}{a(\tau)+\varepsilon} + \int_{t_0}^{\tau} w(\tau,s)\frac{u(s)}{a(s)+\varepsilon}\Delta s\right]\Delta\tau, \ t \in \mathbb{T}^{\kappa}. \end{aligned} \tag{3.9}$$

Letting

$$v(t) = \frac{u(t)}{a(t)+\varepsilon},$$

from (3.9), we have

$$v(t) \leqslant 1 + \int_{t_0}^{t} f(\tau)\left[v(\tau) + \int_{t_0}^{\tau} w(\tau,s)v(s)\Delta s\right]\Delta\tau, \ t \in \mathbb{T}^{\kappa}. \tag{3.10}$$

Using Theorem 3.1, from (3.10), we obtain

$$v(t) \leqslant 1 + \int_{t_0}^{t} f(\tau)e_{f+A}(\tau,t_0)\Delta\tau, \ t \in \mathbb{T}^{\kappa},$$

where $A(t)$ is defined as in (3.3). Therefore,

$$u(t) \leqslant (a(t)+\varepsilon)\left[1 + \int_{t_0}^{t} f(\tau)e_{f+A}(\tau,t_0)\Delta\tau\right], \ t \in \mathbb{T}^{\kappa}. \tag{3.11}$$

Letting $\varepsilon \to 0$ in (3.11), we easily obtain the desired inequality (3.8). The proof of Theorem 3.2 is complete. $\square$

THEOREM 3.3. *If all conditions of Theorem 3.2 are satisfied, then the inequality (3.7) implies*

$$u(t) \leqslant a(t)e_{f+A}(t,t_0), \quad t \in \mathbb{T}^{\kappa}, \tag{3.12}$$

where $A(t)$ *is defined as in (3.3).*

Proof. As in the proof of Theorem 3.2, we obtain (3.10). Letting

$$y(t) = 1 + \int_{t_0}^{t} f(\tau)\left[v(\tau) + \int_{t_0}^{\tau} w(\tau,s)v(s)\Delta s\right]\Delta\tau, \ t \in \mathbb{T}^{\kappa}, \tag{3.13}$$

we have $v(t) \leqslant y(t)$, $y(t)$ is nondecreasing, and

$$\begin{aligned} y^{\Delta}(t) &= f(t)\left[v(t) + \int_{t_0}^{t} w(t,s)v(s)\Delta s\right] \\ &\leqslant f(t)\left[y(t) + \int_{t_0}^{t} w(t,s)y(s)\Delta s\right],\ t \in \mathbb{T}^{\kappa}. \end{aligned} \tag{3.14}$$

Define a function $z(t)$ by

$$z(t) = y(t) + \int_{t_0}^{t} w(t,s)y(s)\Delta s,\ \ t \in \mathbb{T}^{\kappa}. \tag{3.15}$$

Then $z(t_0) = y(t_0) = 1$, $y(t) \leqslant z(t)$, $z(t)$ is nondecreasing, and

$$\begin{aligned} z^{\Delta}(t) &= y^{\Delta}(t) + w(\sigma(t),t)y(t) + \int_{t_0}^{t} w^{\Delta}(t,s)y(s)\Delta s \\ &\leqslant f(t)z(t) + w(\sigma(t),t)y(t) + \int_{t_0}^{t} w^{\Delta}(t,s)y(s)\Delta s \\ &\leqslant \left[f(t) + w(\sigma(t),t) + \int_{t_0}^{t} w^{\Delta}(t,s)\Delta s\right]z(t) \\ &= [f(t) + A(t)]z(t),\ t \in \mathbb{T}^{\kappa}, \end{aligned} \tag{3.16}$$

where $A(t)$ is defined as in (3.3). Using Theorem 2.5 and noting $z(t_0) = 1$, from (3.16), we have

$$z(t) \leqslant e_{f+A}(t,t_0),\ t \in \mathbb{T}^{\kappa}. \tag{3.17}$$

Noting the definitions of $v(t), y(t)$ and $z(t)$, from (3.17), we easily obtain

$$u(t) \leqslant (a(t) + \varepsilon)e_{f+A}(t,t_0),\ t \in \mathbb{T}^{\kappa}. \tag{3.18}$$

Letting $\varepsilon \to 0$ in (3.18), we can obtain the desired inequality (3.12). This completes the proof of Theorem 3.3. $\square$

REMARK 3.3. By taking $w(t,s) = w(s)$, the inequalities given in Theorem 3.2 and Theorem 3.3 reduces to the inequalities given in [7, Theorem 2].

Using Theorem 3.2 and Theorem 3.3, we easily establish the following corollaries.

COROLLARY 3.1. *Let $\mathbb{T} = \mathbb{R}$ and assume that $u(t), a(t), f(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $a(t)$ is nondecreasing. If $w(t,s)$ and its partial derivative $\frac{\partial}{\partial t}w(t,s)$ are real–valued nonnegative continuous functions for $t,s \in \mathbb{R}_+$ with $s \leqslant t$, then the inequality*

$$u(t) \leqslant a(t) + \int_0^t f(\tau)\left[u(\tau) + \int_0^{\tau} w(\tau,s)u(s)ds\right]d\tau,\ \ t \in \mathbb{R}_+, \tag{3.19}$$

implies

$$u(t) \leqslant a(t)\left[1 + \int_0^t f(\tau)\exp\left(\int_s^{\tau} (f(s) + \overline{A}(s)ds\right)d\tau\right],\ \ t \in \mathbb{R}_+, \tag{3.20}$$

where

$$\overline{A}(t) = w(t,t) + \int_0^t \frac{\partial}{\partial t}w(t,s)ds. \tag{3.21}$$

REMARK 3.4. Letting $w(t,s) = w(s)$ in Corollary 3.1, we immediately obtain Theorem 1.7.4 in [9].

COROLLARY 3.2. *Assume that the conditions of Corollary 3.1 are satisfied. Then the inequality (3.19) implies*

$$u(t) \leqslant a(t)\exp\left(\int_0^t (f(s)+\overline{A}(s)ds\right), \quad t \in \mathbb{R}_+, \tag{3.22}$$

where $\overline{A}(t)$ is defined as in (3.21).

COROLLARY 3.3. *Let $\mathbb{T}=\mathbb{Z}$ and assume that $u(t), a(t)$ and $f(t)$ are nonnegative functions defined for $t \in \mathbb{N}_0$, and $a(t)$ is nondecreasing. If $w(t,s)$ and $\mathbf{\Delta}_1 w(t,s)$ are real–valued nonnegative functions for $t,s \in \mathbb{N}_0$ with $s \leqslant t$, then the inequality*

$$u(t) \leqslant a(t)+\sum_{s=0}^{t-1} f(s)\left[u(s)+\sum_{\tau=0}^{s-1} w(s,\tau)u(\tau)\right], \quad t \in \mathbb{N}_0, \tag{3.23}$$

implies

$$u(t) \leqslant a(t)\left[1+\sum_{s=0}^{t-1} f(s)\prod_{\tau=0}^{s-1}(1+f(\tau)+\widetilde{A}(\tau))\right], \; t \in \mathbb{N}_0, \tag{3.24}$$

where $\mathbf{\Delta}_1 w(t,s)=w(t+1,s)-w(t,s)$ for $t,s \in \mathbb{N}_0$ with $s \leqslant t$, and

$$\widetilde{A}(t)=w(t+1,t)+\sum_{s=0}^{t-1}\mathbf{\Delta}_1 w(t,s), \; t \in \mathbb{N}_0. \tag{3.25}$$

REMARK 3.5. Let $w(t,s)=w(s)$ in Corollary 3.3. We easily obtain Theorem 2.4.2 in [10].

COROLLARY 3.4.. *Assume that the conditions of Corollary 3.3 are satisfied. Then the inequality (3.23) implies*

$$u(t) \leqslant a(t)\prod_{s=0}^{t-1}\left(f(s)+\widetilde{A}(s)\right), \quad t \in \mathbb{N}_0, \tag{3.26}$$

where $\widetilde{A}(t)$ is defined as in (3.25).

THEOREM 3.4. *Assume that $u,b,f \in \mathrm{C_{rd}}$, $u(t), b(t)$ and $f(t)$ are nonnegative. If $w(t,s)$ is defined as in Theorem 2.4 such that $w(t,s) \geqslant 0$ and $w^{\Delta}(t,s) \geqslant 0$ for $t,s \in \mathbb{T}$ with $s \leqslant t$, then*

$$u(t) \leqslant b(t)+\int_{t_0}^t f(\tau)\left[u(\tau)+\int_{t_0}^{\tau} w(\tau,s)u(s)\Delta s\right]\Delta\tau, \quad t \in \mathbb{T}^{\kappa}, \tag{3.27}$$

implies

$$u(t) \leqslant b(t)+H(t)\left[1+\int_{t_0}^t f(\tau)e_{f+A}(\tau,t_0)\Delta\tau\right], \quad t \in \mathbb{T}^{\kappa}, \tag{3.28}$$

where $A(t)$ is defined as in (3.3) and

$$H(t)=\int_{t_0}^t f(\tau)\left[b(\tau)+\int_{t_0}^{\tau} w(\tau,s)b(s)\Delta s\right]\Delta\tau, \quad t \in \mathbb{T}^{\kappa}. \tag{3.29}$$

Proof. Define a function $z(t)$ by

$$z(t)=\int_{t_0}^{t} f(\tau)\left[u(\tau)+\int_{t_0}^{\tau} w(\tau,s)u(s)\Delta s\right]\Delta\tau,\ \ t\in\mathbb{T}^{\kappa}. \tag{3.30}$$

Then from (3.27), $u(t)\leqslant b(t)+z(t)$ and using this in (3.30), we have

$$\begin{aligned} z(t) &\leqslant \int_{t_0}^{t} f(\tau)\left[b(\tau)+z(\tau)+\int_{t_0}^{\tau} w(\tau,s)(b(s)+z(s))\Delta s\right]\Delta\tau \\ &= H(t)+\int_{t_0}^{t} f(\tau)\left[z(\tau)+\int_{t_0}^{\tau} w(\tau,s)z(s)\Delta s\right]\Delta\tau,\ t\in\mathbb{T}^{\kappa}, \end{aligned} \tag{3.31}$$

where $H(t)$ is defined as in (3.29). Clearly $H\in \mathrm{C_{rd}}$, and $H(t)$ is nonnegative and nondecreasing in t, $t\in\mathbb{T}^{\kappa}$. Therefore, for any $\varepsilon>0$, it follows from (3.31) that

$$\begin{aligned} \frac{z(t)}{H(t)+\varepsilon} &\leqslant 1+\int_{t_0}^{t} f(\tau)\left[\frac{z(\tau)}{H(t)+\varepsilon}+\int_{t_0}^{\tau} w(\tau,s)\frac{z(s)}{H(t)+\varepsilon}\Delta s\right]\Delta\tau \\ &\leqslant 1+\int_{t_0}^{t} f(\tau)\left[\frac{z(\tau)}{H(\tau)+\varepsilon}+\int_{t_0}^{\tau} w(\tau,s)\frac{z(s)}{H(s)+\varepsilon}\Delta s\right]\Delta\tau,\ t\in\mathbb{T}^{\kappa}. \end{aligned} \tag{3.32}$$

By Theorem 3.1, from (3.32), we have

$$\frac{z(t)}{H(t)+\varepsilon}\leqslant 1+\int_{t_0}^{t} f(\tau)e_{f+A}(\tau,t_0)\Delta\tau,\ t\in\mathbb{T}^{\kappa},$$

where $A(t)$ is defined as in (3.3). Hence

$$z(t)\leqslant (H(t)+\varepsilon)\left[1+\int_{t_0}^{t} f(\tau)e_{f+A}(\tau,t_0)\Delta\tau\right],\ t\in\mathbb{T}^{\kappa}. \tag{3.33}$$

Letting $\varepsilon\to 0$ in (3.33) and noting $u(t)\leqslant b(t)+z(t)$, we easily obtain the desired inequality (3.28). The proof of Theorem 3.4 is complete. □

REMARK 3.6. In Theorem 3.4, letting $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$ respectively, we can obtain Theorem 2.1(a_2) and Theorem 2.5(c_2) in [11].

In Theorem 3.4, letting $w(t,s)=w(s)$, we can obtain the following result.

COROLLARY 3.4. *Assume that $u,b,f,w\in \mathrm{C_{rd}}$, $u(t),b(t),f(t)$ and $w(t)$ are nonnegative. Then*

$$u(t)\leqslant b(t)+\int_{t_0}^{t} f(\tau)\left[u(\tau)+\int_{t_0}^{\tau} w(s)u(s)\Delta s\right]\Delta\tau,\ \ t\in\mathbb{T}^{\kappa}, \tag{3.34}$$

implies

$$u(t)\leqslant b(t)+\tilde{H}(t)\left[1+\int_{t_0}^{t} f(\tau)e_{f+w}(\tau,t_0)\Delta\tau\right],\ \ t\in\mathbb{T}^{\kappa}, \tag{3.35}$$

where

$$\tilde{H}(t)=\int_{t_0}^{t} f(\tau)\left[b(\tau)+\int_{t_0}^{\tau} w(s)b(s)\Delta s\right]\Delta\tau,\ \ t\in\mathbb{T}^{\kappa}. \tag{3.36}$$

REMARK 3.7. Using our main results, we can obtain many dynamic inequalities for some peculiar time scales. Due to limited space, their statements are omitted here.

4. Applications

In this section, we present some applications of Theorem 3.1 to investigate certain properties of solutions of dynamic equation on time scales.

Consider the the following initial value problem

$$u^{\Delta}(t) = f(t)\left(u(t) + \int_{t_0}^{t} w(t,s)u(s)\Delta s\right), \quad u(t_0) = C,\ t \in \mathbb{T}^{\kappa}, \tag{4.1}$$

where $f(t)$ and $w(t,s)$ are as defined in Theorem 3.1, and C is a constant.

THEOREM 4.1. *Assume $u(t)$ is a solution of IVP (4.1). Then*

$$|u(t)| \leqslant |C|\left[1 + \int_{t_0}^{t} f(\tau)e_{f+A}(\tau,t_0)\Delta\tau\right],\ t \in \mathbb{T}^{\kappa}, \tag{4.2}$$

where $A(t)$ is as defined in Theorem 3.1.

Proof. Obviously, the solution $u(t)$ of IVP (4.1) satisfies the following equivalent equation

$$u(t) = C + \int_{t_0}^{t} f(\tau)\left(u(\tau) + \int_{t_0}^{t} w(\tau,s)u(s)\Delta s\right)\Delta\tau,\ t \in \mathbb{T}^{\kappa}. \tag{4.3}$$

It follows from (4.3) that

$$|u(t)| \leqslant |C| + \int_{t_0}^{t} f(\tau)\left(|u(\tau)| + \int_{t_0}^{t} w(\tau,s)|u(s)|\Delta s\right)\Delta\tau,\ t \in \mathbb{T}^{\kappa}. \tag{4.4}$$

Using Theorem 3.1 in (4.4), we immediately obtain (4.2). This completes the proof of Theorem 4.1. □

THEOREM 4.2. *The IVP (4.1) has at most one solution.*

Proof. Let $u_1(t)$ and $u_2(t)$ be two solutions of IVP (4.1). Then we have

$$|u_1(t) - u_2(t)| \leqslant \int_{t_0}^{t} f(\tau)\left[|u_1(\tau) - u_2(\tau)| + \int_{t_0}^{\tau} w(\tau,s)|u_1(s) - u_2(s)|\Delta s\right]\Delta\tau,\ t \in \mathbb{T}^{\kappa}. \tag{4.5}$$

Using Theorem 3.1 in (4.5), we have $|u_1(t) - u_2(t)| \equiv 0$, $t \in \mathbb{T}^{\kappa}$. Therefore, $u_1(t) = u_2(t)$, i,e., the IVP (4.1) has at most one solution. The proof of Theorem 4.2 is complete. □

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