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SOME SUBCLASSES OF MULTIVALENT FUNCTIONS INVOLVING THE EXTENDED FRACTIONAL DIFFERINTEGRAL OPERATOR

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Abstract. The object of the present paper, is to investigate various properties of several subclasses of multivalent analytic functions which are defined here by using the extended fractional differintegral operator

$$\Omega_z^{(\lambda,p)}\ (-\infty<\lambda<p+1;\ p\in\mathbb{N}).$$

1. Introduction

Let $A_n(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{k+p}z^{k+p}\quad (p,n\in\mathbb{N}=\{1,2,\ldots\};\ n>p), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$. For convenience, we write $A_1(p)=A(p)$. A function $f\in A_n(p)$ is said to be in the class $S^*_{p,n}(\alpha)$ of p-valent starlike functions of order α in U, if

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\}>\alpha\quad (0\leqslant\alpha<p;\ z\in U). \tag{1.2}$$

Furthermore, a function $f\in A_n(p)$ is said to be in the class $K_{p,n}(\alpha)$ of p-valent convex functions of order α in U, *if*

$$\operatorname{Re}\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\}>\alpha\quad (0\leqslant\alpha<p;\ z\in U). \tag{1.3}$$

From (1.2) and (1.3) it follows that

$$f(z)\in K_{p,n}(\alpha)\Leftrightarrow\frac{zf^{'}(z)}{p}\in S^*_{p,n}(\alpha)\quad (0\leqslant\alpha<p;\ z\in U). \tag{1.4}$$

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The class $S^*_{p,1}(\alpha) = S^*_p(\alpha)$ was introduced by Patil and Thakare [14] and the class $K_{p,1}(\alpha) = K_p(\alpha)$ was introduced by Owa [10]. We note that

$$S^*_{p,n}(\alpha) \subseteq S^*_p(\alpha) \subseteq S^*_p(0) = S^*_p \text{ and } K_{p,n}(\alpha) \subseteq K_p(\alpha) \subseteq K_p(0) = K_p \quad (0 \leqslant \alpha < p),$$

where S^*_p and K_p denote the subclasses of $A(p)$ consisting of functions which are p-valent starlike in U and p-valent convex in U, respectivly (see, for details, [3] see also [21]).

If f and g are analytic functions in U, we say that f is subordinate to g, written symbolically as $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g., [6], see also [7, p. 4]):

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f_i \in A(p)$ $(i = 1,2)$ given by

$$f_i(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,i} z^{k+p} \ (i = 1,2; p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} = (f_2 * f_1)(z) \quad (p \in \mathbb{N}, z \in U). \tag{1.5}$$

In our present paper, we shall also make use of the Gauss hypergeometric function ${}_2F_1$ defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \quad (a,b,c \in \mathbb{C}; c \notin \mathbb{Z}_0^- = \{0,-1,-2,...\}), \tag{1.6}$$

where $(d)_k$ denotes the Pochhammer symbol given in terms of the Gamma function Γ, by

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(k)} \begin{cases} 1 & (k = 0;\ d \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ d(d+1)...(d+k-1) & (k \in \mathbb{N};\ d \in \mathbb{C}). \end{cases}$$

We note that the series defined by (1.6) converges absolutely for $z \in U$ and hence ${}_2F_1$ represents an analytic function in U (see for details [22, Ch. 14]).

For function $f(z) \in A(p)$, the generalized Bernardi-Libera-Livingston integral operator $F_{\mu,p} : A(p) \to A(p)$ is defined by

$$\begin{aligned} F_{\mu,p}(f)(z) &= \frac{\mu + p}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \\ &= \left(z^p + \sum_{k=1}^{\infty} \frac{\mu + p}{\mu + p + k} z^{p+k} \right) * f(z) \\ &= z^p \, {}_2F_1(1, \mu + p; \mu + p + 1; z) * f(z) \ (\mu > -p;\ z \in U). \end{aligned} \tag{1.7}$$

With a view to introducing an extended fractional differintegral operator, we begin by recalling the following definitions of fractional calculus (fractional integral and fractional derivative of an arbitrary order) considered by Owa [9] (see also [11] and [20]).

DEFINITION 1. The fractional integral of order λ $(\lambda > 0)$ is defined, for a function f, analytic in a simply-connected region of the complex plane containing the origin, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.8}$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

DEFINITION 2. Under the hypothesis of Definition 1, the fractional derivative of f of order λ $(\lambda \geqslant 0)$ is defined by:

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt & (0 \leqslant \lambda < 1) \\ \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n \leqslant \lambda < n+1;\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \end{cases} \tag{1.9}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.

In [12] Patel and Mishra defined the extended fractional differintegral operator $\Omega_z^{(\lambda,p)} : A(p) \to A(p)$ for a function f of the form (1.1) (with $n = 1$) and for a real number λ $(-\infty < \lambda < p+1)$ by:

$$\begin{aligned} \Omega_z^{(\lambda,p)} f(z) &= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} a_{k+p} z^{k+p} \\ &= z^p\, {}_2F_1(1,p+1;p+1-\lambda;z) * f(z) \quad (-\infty < \lambda < p+1;\ z \in U). \end{aligned} \tag{1.10}$$

It is easily seen from (1.10) that (see [12])

$$z(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda)\Omega_z^{(\lambda+1,p)} f(z) + \lambda \Omega_z^{(\lambda,p)} f(z) \quad (-\infty < \lambda < p;\ z \in U). \tag{1.11}$$

We also note that

$$\Omega_z^{(0,p)} f(z) = f(z),\ \ \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p},$$

and, in general

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \quad (-\infty < \lambda < p+1;\ z \in U), \tag{1.12}$$

where $D_z^{\lambda} f(z)$ is, respectively, the fractional integral of f of order $-\lambda$ when $-\infty < \lambda < 0$ and the fractional derivative of f of order λ when $0 \leqslant \lambda < p+1$. For integral values of λ, (1.12) becomes:

$$\Omega_z^{(j,p)} f(z) = \frac{(p-j)! z^j f^{(j)}(z)}{p!} \quad (j \in \mathbb{N};\ j < p+1),$$

and

$$\begin{aligned}\Omega_z^{(-m,p)} f(z) &= \frac{p+m}{z^m}\int_0^z t^{m-1}\Omega_z^{(-m+1,p)} f(t)dt \quad (m\in\mathbb{N})\\ &= F_{1,p}\circ F_{2,p}\circ \ldots \circ F_{m,p}(f)(z)\\ &= F_{1,p}\left(\frac{z^p}{1-z}\right) * F_{2,p}\left(\frac{z^p}{1-z}\right) * \ldots * F_{m,p}\left(\frac{z^p}{1-z}\right) * f(z),\end{aligned}$$

where $F_{\mu,p}$ is the familiar integral operator defined by (1.7) (see Section 3) and $\circ$ denotes the usual composition of functions.

The fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0\leqslant\lambda<1$ was investigated by Srivastava and Aouf [17]. More recently, Srivastava and Mishra [19] obtained several interesting properties and charactaristics for certain subclasses of p-valent analytic functions involving the differintegral operator $\Omega_z^{(\lambda,p)}$ when $(-\infty<\lambda<1)$. The operator $\Omega_z^{(\lambda,1)}=\Omega_z^{\lambda}$ was introduced by Owa and Srivastava [11].

By using the extended fractional differintegral operator $\Omega_z^{(\lambda,p)}$ $(-\infty<\lambda<p+1)$, we define the following subclass of functions in $A_n(p)$.

DEFINITION 3. For fixed parameters A,B $(-1\leqslant B<A\leqslant 1), 0\leqslant\alpha<p$, and $p,n\in\mathbb{N}$ we say that a function $f\in A_n(p)$ is in the class $v_{p,n}^{\lambda}(\alpha;A,B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\alpha}\left(\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz}\,(-\infty<\lambda<p;\ z\in U,\ p,n\in\mathbb{N}). \qquad (1.13)$$

For convenience, we write

$$v_{p,n}^{\lambda}(\alpha;1,-1)=v_{p,n}^{\lambda}(\alpha)=\left\{f\in A_n(p):\operatorname{Re}\left(\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}\right)>\alpha,\ 0\leqslant\alpha<p,\ z\in U\right\}.$$

We note that the class $v_{p,1}^{\lambda}(\alpha;A,B)=v_p^{\lambda}(\alpha;A,B)$, was introduced and studied by Patel and Mishra [12]. We, further observe that:

(i) $v_{p,n}^{\lambda}(\alpha;A,B)=v_{p,n}^{\lambda}(\alpha;A+\frac{\alpha}{p}(B-A),B)$ $(0\leqslant\alpha<1;-1\leqslant B<A\leqslant 1)$;

and

(ii) $v_{p,n}^{0}(\alpha;1,-1)=\mathbf{S}_{p,n}^{*}(\alpha)$ and $v_{p,n}^{1}(\alpha;1,-1)=\mathbf{K}_{p,n}(\alpha)$.

Also we note that Srivastava et al. [18] have studied some interesting properties of the class $v_{1,1}^{\lambda}(\alpha;1,-1)=\mathbf{S}_{\lambda}(\alpha)$ $(0\leqslant\lambda<1;0\leqslant\alpha<1)$ by using the techniques of the Hadamard product.

Let us consider the first-order differential subordination

$$H(\varphi(z),z\varphi'(z))\prec h(z).$$

Then, a univalent function q is called its dominant, if $\varphi(z)\prec q(z)$ for all analytic functions φ that satisfy this differential subordination. A dominant $\widetilde{q}$ is called the best dominant, if $\widetilde{q}(z)\prec q(z)$ for all dominants q. For the general theory of the first-order differential subordination and its applications, we refer the reader to [2] and [7].

2. Preliminaries

To establish our main results, we shall require the following lemmas.

LEMMA 1. [12]. *Let* $\delta > 0$ *and the function* $f(z) \in A_n(p)$ *satisfy*

$$(1-\delta)\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p} + \delta\frac{\Omega_z^{(\lambda+1,p)}f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \quad (z \in U).$$

Then

$$\mathrm{Re}\left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\frac{1}{m}} > \rho^{\frac{1}{m}} \quad (m \in \mathbb{N};\ z \in U),$$

where

$$\rho = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\ {}_2F_1(1,1;\frac{P-\lambda}{\delta n}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{(p-\lambda)A}{p-\lambda+\delta n} & (B = 0). \end{cases}$$

The result is the best possible.

LEMMA 2. [4]. *Let* $h(z)$ *be analytic and convex (univalent) function in* U *with* $h(0) = 1$. *Also let the function* ϕ *given by*

$$\phi(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + ... \tag{2.1}$$

be analytic in U. *If*

$$\phi(z) + \frac{z\phi'(z)}{\delta} \prec h(z) \quad (\mathrm{Re}(\delta) \geqslant 0;\ \delta \neq 0;\ z \in U),$$

then

$$\phi(z) \prec \psi(z) = \frac{\delta}{n}z^{-\frac{\delta}{n}}\int_0^z t^{\frac{\delta}{n}-1}h(t)dt \prec h(z) \quad (z \in U), \tag{2.2}$$

and ϕ *is the best dominant of* (2.2).

LEMMA 3. [5]. *Let* $\zeta \neq 0$ *be a real number,* $\dfrac{a}{\zeta} > 0$ *and* $0 \leqslant \beta < 1$. *Suppose also that the function* Ψ *given by*

$$\Psi(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + ...,$$

is analytic in U and that

$$\Psi(z) \prec 1 + \frac{aM}{n\zeta + a}z \qquad (n \in \mathbb{N};\ z \in U),$$

where

$$M = \frac{(1-\beta)|\zeta|(1+\frac{n\zeta}{a})}{|1-\zeta+\zeta\beta| + \sqrt{1+(1+\frac{n\zeta}{a})^2}}.$$

If the function $\theta(z) = 1 + e_n z^n + e_{n+1} z^{n+1} + \ldots$ *is analytic in* U *and satisfies the following subordination condition :*

$$\Psi(z)\left[1-\zeta+\zeta\left\{(1-\beta)\theta(z)+\beta\right\}\right] \prec 1+Mz \quad (z \in U),$$

then

$$\operatorname{Re}(\theta(z)) > 0 \qquad (z \in U).$$

LEMMA 4. [15]. *Let* ϕ *be analytic in* U *with* $\phi(0) = 1$ *and* $\phi(z) \neq 0$ *for* $0 < |z| < 1$, *and let* $A, B \in \mathbb{C}$ *with* $A \neq B, |B| \leqslant 1$.

(*i*) *Let* $B \neq 0$ *and* $\gamma \in \mathbb{C}^*$ *satisfy either*

$$\left|\frac{\gamma(A-B)}{B} - 1\right| \leqslant 1 \quad or \quad \left|\frac{\gamma(A-B)}{B} + 1\right| \leqslant 1.$$

If ϕ *satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\phi(z) \prec (1+Bz)^{\gamma\left(\frac{A-B}{B}\right)}$$

and this is the best dominant.

(*ii*) *Let* $B = 0$ *and* $\gamma \in \mathbb{C}^*$ *be such that* $|\gamma A| < \pi$, *and if* ϕ *satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec 1 + Az,$$

then

$$\phi(z) \prec e^{\gamma A z}$$

and this is the best dominant.

3. Main results

Unless otherwise mentioned, we assume throughout this paper that: $-1 \leqslant B < A \leqslant 1$, $-\infty < \lambda < p, p, n \in \mathbb{N}$ and the powers understood as principle values.

THEOREM 1. *Let* $-\infty < \lambda < p, p, n \in \mathbb{N}, \nu \in \mathbb{C}^*$ *and* $A, B \in \mathbb{C}$ *with* $A \neq B$ *and* $|B| \leqslant 1$. *Suppose that*

$$\left|\frac{\nu(p-\lambda)(A-B)}{B} - 1\right| \leqslant 1 \;\; or \;\; \left|\frac{\nu(p-\lambda)(A-B)}{B} + 1\right| \leqslant 1, \;\; if \;\; B \neq 0,$$

$$|\nu A| < \frac{\pi}{p-\lambda}, \;\; if \;\; B = 0.$$

If $f(z) \in A_n(p)$ *with* $\Omega_z^{(\lambda,p)} f(z) \neq 0$ *for all* $z \in \overset{*}{U} \equiv U \backslash \{0\}$, *then*

$$\frac{\Omega^{(\lambda+1,p)} f(z)}{\Omega_z^{(\lambda,p)} f(z)} \prec \frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} \right)^{\nu} \prec q(z),$$

where

$$q(z) = \begin{cases} (1+Bz)^{\nu(p-\lambda)\left(\frac{A-B}{B}\right)} & \text{if } B \neq 0, \\ e^{\nu(p-\lambda)Az} & \text{if } B = 0, \end{cases}$$

is the best dominant.

Proof. If we let

$$\varphi(z) = \left(\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} \right)^{\nu} \quad (z \in U), \tag{3.1}$$

then φ is analytic in U, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for $z \in U$. Taking the logarithmic derivatives in both sides of (3.1), multiplying by z and using the identity (1.11), we have

$$1 + \frac{z\varphi'(z)}{\nu(p-\lambda)\varphi(z)} = \frac{\Omega_z^{(\lambda+1,p)} f(z)}{\Omega_z^{(\lambda,p)} f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now, the assertions of Theorem 1 follows by using Lemma 4 for $\gamma = \nu(p-\lambda)$, which completes the proof of Theorem 1.

Putting $\lambda = 0$ in Theorem 1, we obtain the following result.

COROLLARY 1. *Let* $p,n \in \mathbb{N}, \nu \in \mathbb{C}^*$ *and let* $A,B \in \mathbb{C}$ *with* $A \neq B$ *and* $|B| \leqslant 1$. *Suppose that*

$$\left| \frac{\nu p(A-B)}{B} - 1 \right| \leqslant 1 \quad \text{or} \quad \left| \frac{\nu p(A-B)}{B} + 1 \right| \leqslant 1, \ \text{if } B \neq 0,$$

$$|\nu A| < \frac{\pi}{p}, \ \text{if } B = 0.$$

If $f(z) \in A_n(p)$ with $f(z) \neq 0$ for all $z \in \overset{*}{U}$, then

$$\frac{zf'(z)}{f(z)} \prec p\frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{f(z)}{z^p} \right)^{\nu} \prec q(z),$$

where

$$q(z)=\begin{cases}(1+Bz)^{\nu p\left(\frac{A-B}{B}\right)} & \text{if } B\neq 0,\\ e^{\nu pAz} & \text{if } B=0,\end{cases}$$

is the best dominant.

THEOREM 2. *Let* $\delta>0,\ 0\leqslant\alpha<p,\ p,n\in\mathbb{N}$ *and let the function* $f(z)\in A_n(p)$ *satisfy the following subordination condition:*

$$(1-\delta)\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}+\delta\frac{\Omega_z^{(\lambda+1,p)}f(z)}{z^p}\prec 1+M_1z \quad (z\in U), \tag{3.2}$$

where

$$M_1=\frac{\delta(p-\alpha)\left(1+\frac{\delta n}{p-\lambda}\right)}{|(p-\lambda)-\delta(p-\alpha)|+\sqrt{(p-\lambda)^2+(p-\lambda+\delta n)^2}}. \tag{3.3}$$

Then $f(z)\in v_{p,n}^{\lambda}(\alpha)$.

Proof. Put

$$\varphi(z)=\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p} \quad (z\in U). \tag{3.4}$$

Then $\varphi(z)$ is of the form (2.1) and is analytic in U. From Lemma 1 with $A=M_1$, $B=0$ and $m=1$, we have

$$\varphi(z)\prec 1+\frac{(p-\lambda)M_1}{p-\lambda+\delta n}z \quad (z\in U),$$

which is equivalent to

$$|\varphi(z)-1|<\frac{(p-\lambda)M_1}{p-\lambda+\delta n}=N<1 \quad (z\in U). \tag{3.5}$$

Set

$$P(z)=\frac{1}{p-\alpha}\left(\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}-\alpha\right) \quad (0\leqslant\alpha<p;\ z\in U). \tag{3.6}$$

Using the identity (1.11) followed by (3.4), we obtain

$$\frac{\Omega_z^{(\lambda+1,p)}f(z)}{z^p}=\left[\left(1-\frac{p-\alpha}{p-\lambda}\right)+\left(\frac{p-\alpha}{p-\lambda}\right)P(z)\right]\varphi(z). \tag{3.7}$$

In view of (3.7), the hypothesis (3.2) can be written as follows:

$$\left|\left(1-\delta\frac{p-\alpha}{p-\lambda}\right)\varphi(z)+\frac{\delta(p-\alpha)}{p-\lambda}P(z)\varphi(z)-1\right|<M_1=\frac{p-\lambda+\delta n}{p-\lambda}N. \tag{3.8}$$

We need to show that (3.8) yields

$$\operatorname{Re}\{P(z)\} > 0 \quad (z \in U). \tag{3.9}$$

If we suppose that $\operatorname{Re}\{P(z)\} \not> 0$ $(z \in U)$, then there exists a point $z_0 \in U$ such that $P(z_0) = ix$ for some $x \in \mathbb{R}$. To prove (3.9), it is sufficient to obtain a contradiction from the following inequality:

$$W = \left| \left(1 - \frac{\delta(p-\alpha)}{p-\lambda}\right)\varphi(z_0) + \frac{\delta(p-\alpha)}{p-\lambda}P(z_0)\varphi(z_0) - 1 \right| \geqslant M_1.$$

Let $\varphi(z_0) = u + iv$. Then, by using (3.4) and the triangle inequality, we obtain that

$$\begin{aligned} W^2 &= \left| \left(1 - \frac{\delta(p-\alpha)}{p-\lambda}\right)\varphi(z_0) + \frac{\delta(p-\alpha)}{p-\lambda}P(z_0)\varphi(z_0) - 1 \right|^2 \\ &= (u^2+v^2)\left(\frac{\delta(p-\alpha)}{p-\lambda}\right)^2 x^2 + 2\frac{\delta(p-\alpha)}{p-\lambda}vx + \left| \left(1 - \frac{\delta(p-\alpha)}{p-\lambda}\right)\varphi(z_0) - 1 \right|^2 \\ &\geqslant (u^2+v^2)\left(\frac{\delta(p-\alpha)}{p-\lambda}\right)^2 x^2 + 2\frac{\delta(p-\alpha)}{p-\lambda}vx + \left(\frac{\delta(p-\alpha)}{p-\lambda} - \left|1 - \frac{\delta(p-\alpha)}{p-\lambda}\right| N\right)^2. \end{aligned}$$

Setting

$$\begin{aligned} \Psi(x) = W^2 - M_1^2 &= (u^2+v^2)\left(\frac{\delta(p-\alpha)}{p-\lambda}\right)^2 x^2 + 2\frac{\delta(p-\alpha)}{p-\lambda}vx \\ &\quad + \left(\frac{\delta(p-\alpha)}{p-\lambda} - \left|1 - \frac{\delta(p-\alpha)}{p-\lambda}\right| N\right)^2 - \left(\frac{p-\lambda+\delta n}{p-\lambda}\right) N^2, \end{aligned}$$

we note that (3.9) holds true if $\Psi(x) \geqslant 0$ for any $x \in \mathbb{R}$. Since

$$(u^2+v^2)\left(\frac{\delta(p-\alpha)}{p-\lambda}\right)^2 > 0,$$

the inequality $\Psi(x) \geqslant 0$ holds true if the discriminant $\Delta \leqslant 0$; that is,

$$\begin{aligned} \Delta = 4 \Bigg[&\left(\frac{\delta(p-\alpha)}{p-\lambda}\right)^2 v^2 - \left(\frac{\delta(p-\alpha)}{p-\lambda}\right)^2 (u^2+v^2) \\ &\times \left\{ \left(\frac{\delta(p-\alpha)}{p-\lambda} - \left|1 - \frac{\delta(p-\alpha)}{p-\lambda}\right| N\right)^2 - \left(\frac{p-\lambda+\delta n}{p-\lambda}\right) N^2 \right\} \Bigg] \leqslant 0, \end{aligned}$$

which is equivalent to

$$v^2 \left[1 - \left(\frac{\delta(p-\alpha)}{p-\lambda} - \left|1 - \frac{\delta(p-\alpha)}{p-\lambda}\right| N\right)^2 + \left(\frac{p-\lambda+\delta n}{p-\lambda}\right)^2 N^2 \right]$$

$$\leqslant u^2\left[\left(\frac{\delta(p-\alpha)}{p-\lambda}-\left|1-\frac{\delta(p-\alpha)}{p-\lambda}\right|N\right)^2-\left(\frac{p-\lambda+\delta n}{p-\lambda}\right)^2N^2\right].$$

Putting $\varphi(z_0)-1=\xi e^{i\theta}$ for some real $\theta\in\mathbb{R}$, we get

$$\frac{v^2}{u^2}=\frac{\xi^2\sin^2\theta}{(1+\xi\cos\theta)^2}.$$

Since the above expression attains its maximum value at $\cos\theta=-\xi$, by using (3.5), we obtain

$$\begin{aligned}\frac{v^2}{u^2}&\leqslant\frac{\xi^2}{1-\xi^2}\leqslant\frac{N^2}{1-N^2}\\&\leqslant\frac{\left[\left(\frac{\delta(p-\alpha)}{p-\lambda}-\left|1-\frac{\delta(p-\alpha)}{p-\lambda}\right|N\right)^2-\left(\frac{p-\lambda+\delta n}{p-\lambda}\right)^2N^2\right]}{\left[1-\left(\frac{\delta(p-\alpha)}{p-\lambda}-\left|1-\frac{\delta(p-\alpha)}{p-\lambda}\right|N\right)^2+\left(\frac{p-\lambda+\delta n}{p-\lambda}\right)^2N^2\right]},\end{aligned}$$

which yields $\Delta\leqslant 0$. Therefore, $W\geqslant M_1$, which contradicts (3.8). Hence $\operatorname{Re}\{P(z)\}>0$ $(z\in U)$. This proves that $f(z)\in v^{\lambda}_{p,n}(\alpha)$. This completes the proof of Theorem 2.

Taking $\lambda=0$ in Theorem 2, we obtain

COROLLARY 2. *Let $\delta>0,\ 0\leqslant\alpha<p,\ p\in\mathbb{N}$. If $f(z)\in A_n(p)$ such that $\frac{f(z)}{z^p}\neq 0$ $(z\in U)$ and satisfies the following differential subordination:*

$$(1-\delta)\frac{f(z)}{z^p}+\delta\frac{f'(z)}{pz^{p-1}}\prec 1+M_1z\ (\delta>0;\ z\in U),$$

where

$$M_1=\frac{\delta(p-\alpha)\left(1+\frac{\delta n}{p}\right)}{|p-\delta(p-\alpha)|+\sqrt{p^2+(p+\delta n)^2}}.$$

Then $f(z)\in S^{*}_{p,n}(\alpha), 0\leqslant\alpha<p$.

REMARK 1. (i) We note that this result (with $n=1$) also obtained by Patel et al. [13, Corollary 3];

(ii) Putting $p=1$ in Corollary 2, we obtain the result obtained by Liu [5, Theorem 2.1].

THEOREM 3. *Let $\delta>0,\ 0\leqslant\alpha<p,\ p,n\in\mathbb{N}$ and $\mu\geqslant 0$. If $f(z)\in A_n(p)$ such that $\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\neq 0$ $(z\in U)$ and satisfies the following differential subordination:*

$$(1-\delta)\left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\mu}+\delta\frac{(\Omega_z^{(\lambda,p)}f(z))'}{pz^{p-1}}\left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\mu-1}\prec 1+M_2z\ (z\in U), \tag{3.10}$$

where

$$M_2 = \begin{cases} \dfrac{(p-\alpha)\delta\left(1+\frac{\delta n}{\mu p}\right)}{|p-(p-\alpha)\delta|+\sqrt{p^2+\left(p+\frac{\delta n}{\mu}\right)^2}} & (\mu > 0), \\ \dfrac{p-\alpha}{p}\delta & (\mu = 0). \end{cases} \tag{3.11}$$

Then $f(z) \in v^{\lambda}_{p,n}(\alpha)$.

Proof. If $\mu = 0$, then the condition (3.10) is equivalent to

$$\left|\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)} - p\right| < p-\alpha \quad (z \in U),$$

which, in turn, implies that $f(z) \in v^{\lambda}_{p,n}(\alpha)$.

So, we let $\mu > 0$ and suppose that

$$\varphi(z) = \left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\mu} \quad (z \in U). \tag{3.12}$$

Choosing the principal value in (3.12), we note that φ is of the form (2.1) and is analytic in U. Differentiating (3.12) with respect to z, we obtain

$$(1-\delta)\left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\mu} + \delta\frac{(\Omega_z^{(\lambda,p)}f(z))'}{pz^{p-1}}\left(\frac{\Omega_z^{(\lambda,p)}f(z)}{z^p}\right)^{\mu-1} = \varphi(z) + \frac{\delta}{\mu p}z\varphi'(z) \ (z \in U), \tag{3.13}$$

which, in view of Lemma 2 (with $A = M_2$ and $B = 0$), yields

$$\varphi(z) \prec 1 + \frac{\mu p}{\mu p + \delta n} M_2 z \ (z \in U).$$

Also, with the aid of $(3.12), (3.10)$ can be written as follows:

$$\varphi(z)\left\{1-\delta+\delta\left[\left(1-\frac{\alpha}{p}\right)P(z)+\frac{\alpha}{p}\right]\right\} \prec 1 + M_2 z \ (z \in U),$$

where $P(z)$ is given by (3.6). Therefore, by Lemma 3, we find that

$$\operatorname{Re}\{P(z)\} > 0 \quad (z \in U),$$

that is, that

$$\operatorname{Re}\left\{\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}\right\} > \alpha \quad (0 \leqslant \alpha < p;\ z \in U).$$

This completes the proof of Theorem 3.

Putting $\lambda = 0$ in Theorem 3, we obtain the following result.

COROLLARY 3. *Let* $\delta > 0$, $0 \leqslant \alpha < p$, $p,n \in \mathbb{N}$ *and* $\mu \geqslant 0$. *If* $f(z) \in A_n(p)$ *such that* $\frac{f(z)}{z^p} \neq 0$ $(z \in U)$ *and satisfies the following differential subordination :*

$$(1-\delta)\left(\frac{f(z)}{z^p}\right)^{\mu} + \delta \frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\mu} \prec 1 + M_2 z \ (z \in U),$$

where M_2 is given as in Theorem 3, then $f(z) \in S^*_{p,n}(\alpha)$.

REMARK 2. (i) We note that this result (with $n = 1$) also obtained by Patel et al. [13, Corollary 4];

(ii) Putting $p = 1$ in Corollary 3, we obtain the result obtained by Liu [5, Theorem 2.2].

Putting $\lambda = 0$ and $\delta = 1$ in Theorem 3 we obtain the following result:

COROLLARY 4. *Let* $\mu \geqslant 0$, $0 \leqslant \alpha < p$ *and* $p,n \in \mathbb{N}$. *If* $f(z) \in A_n(p)$ *such that* $\frac{f(z)}{z^p} \neq 0$ $(z \in U)$ *and satisfies the inequality:*

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z^p}\right)^{\mu} - p\right| < \frac{(p-\alpha)(p\mu+n)}{\mu\alpha + \sqrt{p^2\mu^2 + (p\mu+n)^2}} \qquad (z \in U),$$

then $f(z) \in S^*_{p,n}(\alpha)$.

REMARK 3. (i) Putting $p = 1$ in Corollary 4, we get the result obtained by Liu [5, Corollary 2.1];

(ii) Putting $p = \mu = 1$ in Corollary 4, we obtain the result obtained by Mocanu and Oros [8, Corollary 2.2].

Putting $\lambda = 0$ and $\delta = \frac{1}{p-\alpha}$, $0 \leqslant \alpha < p$ in Theorem 3 we obtain the following result :

COROLLARY 5. *Let* $\mu \geqslant 0$, $0 \leqslant \alpha < p$ *and* $p,n \in \mathbb{N}$. *If* $f(z) \in A_n(p)$ *such that* $\frac{f(z)}{z^p} \neq 0$ $(z \in U)$ *and satisfies the inequality:*

$$\left|(p-\alpha-1)\left(\frac{f(z)}{z^p}\right)^{\mu} + \frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\mu} + \alpha - p\right|$$
$$< \frac{(p-\alpha)[p\mu(p-\alpha)+n]}{(p-1) + \sqrt{p^2\mu^2(p-\alpha)^2 + [p\mu(p-\alpha)+n]^2}} \qquad (z \in U),$$

then $f(z) \in S^*_{p,n}(\alpha)$.

REMARK 4. (i) Putting $p = 1$ in Corollary 5, we get the result obtained by Liu [5, Corollary 2.2];

(ii) Putting $p = \mu = 1$ in Corollary 5, we obtain the result obtained by Mocanu and Oros [8, Corollary 2.4].

4. Properties involving the operator $\Omega_z^{(\lambda,p)}f(z)$

THEOREM 4. *If* $f(z)\in v_{p,n}^{\lambda}(\alpha;A,B)$, *then for all* $s,t\in\mathbb{C}$ *with* $|s|\leqslant 1,|t|\leqslant 1$ *and* $s\neq t$ *the next subordination holds:*

$$\frac{t^p\Omega_z^{(\lambda,p)}f(zs)}{s^p\Omega_z^{(\lambda,p)}f(zt)}\prec\begin{cases}\left(\dfrac{1+Bzs}{1+Bzt}\right)^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B\neq 0\\ \exp\left[(p-\alpha)Az(s-t)\right], & \text{if } B=0.\end{cases}\tag{4.1}$$

Proof. If $f\in v_{p,n}^{\lambda}(\alpha;A,B)$, from (1.13) it follows that

$$\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}\prec\frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz}\equiv k(z).\tag{4.2}$$

It is easy to check that the function k defined by (4.2) and the function h given by

$$h(z)=H(z;s,t)=\int_0^z\left(\frac{s}{1-su}-\frac{t}{1-tu}\right)du$$

are convex in U, and by combining a general subordination theorem [16, Theorem 4.1] with (4.2) we deduce

$$\left(\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}-p\right)*h(z)\prec\frac{(p-\alpha)(A-B)z}{1+Bz}*h(z).\tag{4.3}$$

A simple computation shows that, if $\varphi(z)$ is an analytic functioin in the unit disc U with $\varphi(0)=0$, then

$$\varphi(z)*h(z)=\int_{tz}^{sz}\frac{\varphi(u)}{u}du\tag{4.4}$$

and thus, from (4.3) and (4.4) we have

$$\int_{tz}^{sz}\left(\frac{(\Omega_z^{(\lambda,p)}f(u))'}{\Omega_z^{(\lambda,p)}f(u)}-\frac{p}{u}\right)du\prec(p-\alpha)(A-B)\int_{tz}^{sz}\frac{du}{1+Bu}.$$

This last subordination implies

$$\exp\left[\int_{tz}^{sz}\left(\frac{(\Omega_z^{(\lambda,p)}f(u))'}{\Omega_z^{(\lambda,p)}f(u)}-\frac{p}{u}\right)du\right]\prec\exp\left[(p-\alpha)(A-B)\int_{tz}^{sz}\frac{du}{1+Bu}\right],$$

which repesents the conclusion (4.1).

COROLLARY 6. *If $f \in \nu_{p,n}^{\lambda}(\alpha;A,B)$, then for all $|z| = r < 1$, the next inequalities hold:*

$$\left|\Omega_z^{(\lambda,p)} f(z)\right| \leqslant \begin{cases} r^p (1+Br)^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B \neq 0 \\ r^p \exp\left[(p-\alpha)Ar\right], & \text{if } B = 0, \end{cases} \tag{4.5}$$

$$\left|\Omega_z^{(\lambda,p)} f(z)\right| \geqslant \begin{cases} r^p (1-Br)^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B \neq 0 \\ r^p \exp\left[-(p-\alpha)Ar\right], & \text{if } B = 0, \end{cases} \tag{4.6}$$

and

$$\left|\arg\left(\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p}\right)\right| \leqslant \begin{cases} \frac{(p-\alpha)(A-B)}{|B|} \sin^{-1}(|B|\, r), & \text{if } B \neq 0 \\ (p-\alpha)Ar, & \text{if } B = 0. \end{cases} \tag{4.7}$$

All of the estimates asserted here are sharp.

Proof. Taking $s = 1$ and $t = 0$ in (4.1), and using the definition of subordination, we obtain

$$\left(\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p}\right) = \begin{cases} (1+Bw(z))^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\ \exp\left[(p-\alpha)Aw(z)\right], & \text{if } B = 0, \end{cases} \tag{4.8}$$

where w is an analytic function in U, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$. According to the well-known Schwarz's Theorem, we have $|w(z)| \leqslant |z|$ for all $z \in U$.

(i) If $B > 0$, then from (4.8) are find that

$$\begin{aligned} \left|\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p}\right| &= \exp\left[\frac{(p-\alpha)(A-B)}{B} \log|1+Bw(z)|\right] \\ &= |1+Bw(z)|^{\frac{(p-\alpha)(A-B)}{B}} \leqslant (1+Br)^{\frac{(p-\alpha)(A-B)}{B}}. \end{aligned}$$

(ii) If $B < 0$, we easily obtain that

$$\begin{aligned} \left|\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p}\right| &= \left|(1+Bw(z))^{-1}\right|^{\frac{(p-\alpha)(A-B)}{-B}} \\ &\leqslant \left[(1+Br)^{-1}\right]^{\frac{(p-\alpha)(A-B)}{-B}} = (1+Br)^{\frac{(p-\alpha)(A-B)}{B}}. \end{aligned}$$

This proves the inequality (4.5) for $B \neq 0$, and similarly we may prove the other inequalities in (4.6) and (4.7).

Now, for $|z| = r$ and $B \neq 0$, from (4.8) we see that

$$\begin{aligned} \left|\arg\left(\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p}\right)\right| &= \frac{(p-\alpha)(A-B)}{|B|} \left|\arg(1+Bw(z))\right| \\ &\leqslant \frac{(p-\alpha)(A-B)}{|B|} \sin^{-1}(|B|\, r), \end{aligned}$$

and for $B = 0$, (4.7) is a direct consequence of (4.8).

It is easy to verify that all of the esimates from this corollary are sharp, being attained by the function f_0 defined in U by

$$\Omega_z^{(\lambda,p)} f_0(z) = \begin{cases} z^p(1+Bz)^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\ \exp\left[(p-\alpha)Az\right], & \text{if } B = 0. \end{cases} \tag{4.9}$$

COROLLARY 7. *If* $f \in v_{p,n}^{\lambda}(\alpha; A, B)$, *then for all* $|z| = r < 1$, *the next inequalities hold :*

$$\left|\left(\Omega_z^{(\lambda,p)} f(z)\right)'\right| \leqslant \begin{cases} r^{p-1}\left\{p + [\alpha B + (p-\alpha)A]\, r\right\}(1+Br)^{\frac{(p-\alpha)(A-B)}{B}-1}, & \text{if } B \neq 0, \\ r^{p-1}\left[p + (p-\alpha)Ar\right]\exp\left[(p-\alpha)Ar\right], & \text{if } B = 0, \end{cases} \tag{4.10}$$

$$\left|\left(\Omega_z^{(\lambda,p)} f(z)\right)'\right| \geqslant \begin{cases} r^{p-1}\left\{p - [\alpha B + (p-\alpha)A]\, r\right\}(1-Br)^{\frac{(p-\alpha)(A-B)}{B}-1}, & \text{if } B \neq 0, \\ r^{p-1}\left[p - (p-\alpha)\,|A|\, r\right]\exp\left[-(p-\alpha)Ar\right], & \text{if } B = 0, \end{cases} \tag{4.11}$$

and

$$\left|\arg\left(\frac{\left(\Omega_z^{(\lambda,p)} f(z)\right)'}{z^{p-1}}\right)\right| \leqslant \begin{cases} \frac{(p-\alpha)(A-B)}{|B|}\sin^{-1}(|B|\, r) + \sin^{-1}\left[\dfrac{(p-\alpha)(A-B)r}{p-[\alpha B+(p-\alpha)A]Br^2}\right], \\ \qquad\qquad\qquad\qquad \text{if } B \neq 0, \\ (p-\alpha)Ar + \sin^{-1}\left[\frac{A(p-\alpha)r}{p}\right], \quad \text{if } B = 0. \end{cases} \tag{4.12}$$

All of the estimates asserted here are sharp.

Proof. If we define the function g by

$$g(z) = \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)'}{\Omega_z^{(\lambda,p)} f(z)} \quad (z \in U), \tag{4.13}$$

then g is analytic in U with $g(0) = p$ and

$$g(z) \prec \frac{p + [pB + (A-B)(p-\alpha)]z}{1+Bz}.$$

It is known from [1] that the function g satisfies the following sharp inequalities:

$$\frac{p - [\alpha B + (p-\alpha)A]r}{1-Bz} \leqslant |g(z)| \leqslant \frac{p + [\alpha B + (p-\alpha)A]r}{1+Br}, \quad |z| = r < 1, \tag{4.14}$$

$$\left|g(z) - \frac{p - [\alpha B + (p-\alpha)A]Br^2}{1-B^2r^2}\right| \leqslant \frac{(A-B)(p-\alpha)r}{1-B^2r^2}, \quad |z| = r < 1, \tag{4.15}$$

and

$$|\arg g(z)| \leqslant \sin^{-1}\frac{(A-B)(p-\alpha)r}{p-[\alpha B+(p-\alpha)A]Br^2}, \quad |z| = r < 1. \tag{4.16}$$

Applying to the function g given by (4.13) the inequalities $(4.14),(4.15)$ and (4.16), in conjunction with the estimates given by Corollary 6, we deduce the relations $(4.10),(4.11)$ and (4.12), All of the estimates are sharp for the the function f_0 defined by (4.9).

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