SOME SUBCLASSES OF MULTIVALENT FUNCTIONS INVOLVING
THE EXTENDED FRACTIONAL DIFIERINTEGRAL OPERATOR

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Abstract. The object of the present paper, is to investigate various properties of several subclasses of multivalent analytic functions which are defined here by using the extended fractional differintegral operator

$$\Omega_{\lambda,p}^\lambda (-\infty < \lambda < p + 1; p \in \mathbb{N}).$$

1. Introduction

Let $A_n(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (p,n \in \mathbb{N} = \{1,2,\ldots\}; n > p),$$

which are analytic and $p$-valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For convenience, we write $A_1(p) = A(p)$. A function $f \in A_n(p)$ is said to be in the class $S_{p,n}^*(\alpha)$ of $p$-valent starlike functions of order $\alpha$ in $U$, if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in U).$$

Furthermore, a function $f \in A_n(p)$ is said to be in the class $K_{p,n}(\alpha)$ of $p$-valent convex functions of order $\alpha$ in $U$, if

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in U).$$

From (1.2) and (1.3) it follows that

$$f(z) \in K_{p,n}(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_{p,n}^*(\alpha) \quad (0 \leq \alpha < p; z \in U).$$


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The class $S_{p,1}^*(\alpha) = S_{p}^*(\alpha)$ was introduced by Patil and Thakare [14] and the class $K_{p,1}(\alpha) = K_{p}(\alpha)$ was introduced by Owa [10]. We note that

$$S_{p,n}^*(\alpha) \subseteq S_{p}^*(\alpha) \subseteq S_{p}^*(0) = S_{p}^* \text{ and } K_{p,n}(\alpha) \subseteq K_{p}(\alpha) \subseteq K_{p}(0) = K_{p} \quad (0 \leq \alpha < p),$$

where $S_{p}^*$ and $K_{p}$ denote the subclasses of $A(p)$ consisting of functions which are $p$-valent starlike in $U$ and $p$-valent convex in $U$, respectively (see, for details, [3] see also [21]).

If $f$ and $g$ are analytic functions in $U$, we say that $f$ is subordinate to $g$, written symbolically as $f(z) \prec g(z)$ if there exists a Schwarz function $w$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence, (cf., e.g., [6], see also [7, p. 4]):

$$f(z) \prec g(z) (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f_i \in A(p)$ ($i = 1, 2$) given by

$$f_i(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,i} z^{k+p} \quad (i = 1, 2; p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of $f_1$ and $f_2$ by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1}a_{k+p,2} z^{k+p} = (f_2 * f_1)(z) \quad (p \in \mathbb{N}, z \in U). \quad (1.5)$$

In our present paper, we shall also make use of the Gauss hypergeometric function $\,\,_{2}F_{1}$ defined by

$$\,\,_{2}F_{1}(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!} \quad (a,b,c \in \mathbb{C}; c \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}), \quad (1.6)$$

where $(d)_k$ denotes the Pochhammer symbol given in terms of the Gamma function $\Gamma$, by

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(k)} \begin{cases} 1 & (k = 0; d \in \mathbb{C}^+ = \mathbb{C} \setminus \{0\}) \\ d(d+1)...(d+k-1) & (k \in \mathbb{N}; d \in \mathbb{C}) \end{cases}.$$ 

We note that the series defined by (1.6) converges absolutely for $z \in U$ and hence $\,\,_{2}F_{1}$ represents an analytic function in $U$ (see for details [22, Ch. 14]).

For function $f(z) \in A(p)$, the generalized Bernardi-Libera-Livingston integral operator $F_{\mu,p} : A(p) \rightarrow A(p)$ is defined by

$$F_{\mu,p}(f)(z) = \frac{\mu + p}{z^\mu} \int_{0}^{z} t^{\mu-1} f(t) dt$$

$$= \left( z^p + \sum_{k=1}^{\infty} \frac{\mu + p}{\mu + p + k} z^{p+k} \right) * f(z)$$

$$= z^p \,\,_{2}F_{1}(1, \mu + p; \mu + p + 1; z) * f(z) \quad (\mu > -p; z \in U). \quad (1.7)$$
With a view to introducing an extended fractional differintegral operator, we begin by recalling the following definitions of fractional calculus (fractional integral and fractional derivative of an arbitrary order) considered by Owa [9] (see also [11] and [20]).

**DEFINITION 1.** The fractional integral of order \( \lambda \) (\( \lambda > 0 \)) is defined, for a function \( f \), analytic in a simply-connected region of the complex plane containing the origin, by

\[
D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt,
\]

where the multiplicity of \((z-t)^{\lambda-1}\) is removed by requiring \(\log(z-t)\) to be real when \((z-t) > 0\).

**DEFINITION 2.** Under the hypothesis of Definition 1, the fractional derivative of \( f \) of order \( \lambda \) (\( \lambda \geq 0 \)) is defined by:

\[
D_z^{\lambda} f(z) = \begin{cases} 
\frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\lambda} dt & (0 \leq \lambda < 1) \\
\frac{d^n}{dz^n} D_z^{-n-\lambda} f(z) & (n \leq \lambda < n+1; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),
\end{cases}
\]

where the multiplicity of \((z-t)^{-\lambda}\) is removed as in Definition 1.

In [12] Patel and Mishra defined the extended fractional differintegral operator \( \Omega_z^{(\lambda,p)} : \Lambda(p) \rightarrow \Lambda(p) \) for a function \( f \) of the form (1.1) (with \( n = 1 \)) and for a real number \( \lambda \) (\( -\infty < \lambda < p+1 \)) by:

\[
\Omega_z^{(\lambda,p)} f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)}{\Gamma(p+1) \Gamma(k+p+1-\lambda)} a_k z^{k+p} = z^p \, \mathcal{F}_1(1, p+1; p+1-\lambda; z) \ast f(z) \quad (\infty < \lambda < p+1; z \in U). \quad (1.10)
\]

It is easily seen from (1.10) that (see [12])

\[
z(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda) \Omega_z^{(\lambda+1,p)} f(z) + \lambda \Omega_z^{(\lambda,p)} f(z) \quad (\infty < \lambda < p; z \in U). \quad (1.11)
\]

We also note that

\[
\Omega_z^{(0,p)} f(z) = f(z), \quad \Omega_z^{(1,p)} f(z) = \frac{z f'(z)}{p},
\]

and, in general

\[
\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^\lambda D_z^{\lambda} f(z) \quad (\infty < \lambda < p+1; z \in U), \quad (1.12)
\]

where \( D_z^{\lambda} f(z) \) is, respectively, the fractional integral of \( f \) of order \(-\lambda\) when \(-\infty < \lambda < 0\) and the fractional derivative of \( f \) of order \( \lambda \) when \( 0 \leq \lambda < p+1 \). For integral values of \( \lambda \), (1.12) becomes:

\[
\Omega_z^{(j,p)} f(z) = \frac{(p-j)! z^j f^{(j)}(z)}{p!} \quad (j \in \mathbb{N}; j < p+1),
\]
and

$$\Omega_z^{(-m,p)} f(z) = \frac{p + m}{z^m} \int_0^z t^{m-1} \Omega_z^{(-m+1,p)} f(t) \, dt \quad (m \in \mathbb{N})$$

$$= F_{1,p} \circ F_{2,p} \circ \ldots \circ F_{m,p}(f)(z)$$

$$= F_{1,p} \left( \frac{z^p}{1-z} \right) * F_{2,p} \left( \frac{z^p}{1-z} \right) * \ldots * F_{m,p} \left( \frac{z^p}{1-z} \right) * f(z),$$

where $F_{\mu,p}$ is the familiar integral operator defined by (1.7) (see Section 3) and $\circ$ denotes the usual composition of functions.

The fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0 \leq \lambda < 1$ was investigated by Srivastava and Aouf [17]. More recently, Srivastava and Mishra [19] obtained several interesting properties and characteristics for certain subclasses of p-valent analytic functions involving the differintegral operator $\Omega_z^{(\lambda,p)}$ when $(-\infty < \lambda < 1)$. The operator $\Omega_z^{(\lambda,1)} = \Omega_z^{\lambda}$ was introduced by Owa and Srivastava [11].

By using the extended fractional differintegral operator $\Omega_z^{(\lambda,p)} (-\infty < \lambda < p + 1)$, we define the following subclass of functions in $A_n(p)$.

**DEFINITION 3.** For fixed parameters $A, B \ (-1 \leq B < A \leq 1), 0 \leq \alpha < p,$ and $p, n \in \mathbb{N}$ we say that a function $f \in A_n(p)$ is in the class $v_{p,n}^\lambda(\alpha; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p - \alpha} \left( \frac{z(\Omega_z^{(\lambda,p)} f(z))'}{\Omega_z^{(\lambda,p)} f(z)} - \alpha \right) < \frac{1 + A}{1 + B} (\lambda < p; \ z \in U, \ p, n \in \mathbb{N}). \quad (1.13)$$

For convenience, we write

$$v_{p,n}^\lambda(\alpha; 1, -1) = v_{p,n}^\lambda(\alpha) = \left\{ f \in A_n(p) : \text{Re} \left( \frac{z(\Omega_z^{(\lambda,p)} f(z))'}{\Omega_z^{(\lambda,p)} f(z)} \right) > \alpha, \ 0 \leq \alpha < p, \ z \in U \right\}.$$  

We note that the class $v_{p,1}^\lambda(\alpha; A, B) = v_{p}^\lambda(\alpha; A, B)$, was introduced and studied by Patel and Mishra [12]. We, further observe that:

(i) $v_{p,n}^\lambda(\alpha; A, B) = v_{p,n}^\lambda(\alpha; A + \frac{\alpha}{p}(B - A), B)$ ($0 \leq \alpha < 1; -1 \leq B < A \leq 1$);

and

(ii) $v_{0,n}^0(\alpha; 1, -1) = S_{p,n}^*(\alpha)$ and $v_{1,n}^1(\alpha; 1, -1) = K_{p,n}(\alpha)$.

Also we note that Srivastava et al. [18] have studied some interesting properties of the class $v_{1,1}^\lambda(\alpha; 1, -1) = S_{\lambda}(\alpha)$ ($0 \leq \lambda < 1; 0 \leq \alpha < 1$) by using the techniques of the Hadamard product.

Let us consider the first-order differential subordination

$$H(\varphi(z), z\varphi'(z)) \prec h(z).$$

Then, a univalent function $q$ is called its dominant, if $\varphi(z) \prec q(z)$ for all analytic functions $\varphi$ that satisfy this differential subordination. A dominant $\tilde{q}$ is called the best dominant, if $\tilde{q}(z) \prec q(z)$ for all dominants $q$. For the general theory of the first-order differential subordination and its applications, we refer the reader to [2] and [7].
2. Preliminaries

To establish our main results, we shall require the following lemmas.

**Lemma 1.** [[12]. Let $\delta > 0$ and the function $f(z) \in A_n(p)$ satisfy

$$(1 - \delta) \frac{\Omega_z^{(\lambda, p)} f(z)}{z^p} + \delta \frac{\Omega_z^{(\lambda + 1, p)} f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \quad (z \in U).$$

Then

$$\Re \left( \frac{\Omega_z^{(\lambda, p)} f(z)}{z^p} \right)^{\frac{1}{n}} > \rho^{\frac{1}{n}} \quad (m \in \mathbb{N}; z \in U),$$

where

$$\rho = \begin{cases} \frac{A}{B} & (B \neq 0) \\ 1 - \frac{(p - \lambda) A}{p - \lambda + B n} & (B = 0). \end{cases}$$

The result is the best possible.

**Lemma 2.** [[4]. Let $h(z)$ be analytic and convex (univalent) function in $U$ with $h(0) = 1$. Also let the function $\phi$ given by

$$\phi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ... \quad (2.1)$$

be analytic in $U$. If

$$\phi(z) + \frac{z \phi'(z)}{\delta} < h(z) \quad (\Re(\delta) \geq 0; \delta \neq 0; z \in U),$$

then

$$\phi(z) \prec \psi(z) = \frac{\delta}{n} z^{-\frac{\delta}{n}} \int_0^z t^{\frac{\delta}{n} - 1} h(t) dt \prec h(z) \quad (z \in U), \quad (2.2)$$

and $\phi$ is the best dominant of (2.2).

**Lemma 3.** [[5]. Let $\zeta \neq 0$ be a real number, $\frac{a}{\zeta} > 0$ and $0 \leq \beta < 1$. Suppose also that the function $\Psi$ given by

$$\Psi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ..., \quad (n \in \mathbb{N}; z \in U),$$

is analytic in $U$ and that

$$\Psi(z) \prec 1 + \frac{aM}{n\zeta + a} z \quad (n \in \mathbb{N}; z \in U),$$

where

$$M = \frac{(1 - \beta)|\zeta| \left(1 + \frac{n\zeta}{a}\right)}{|1 - \zeta + \zeta \beta| + \sqrt{1 + (1 + \frac{n\zeta}{a})^2}}.$$
If the function $\theta(z) = 1 + en\zeta^n + en+1\zeta^{n+1} + \ldots$ is analytic in $U$ and satisfies the following subordination condition:

$$\Psi(z)[1 - \z + \z [(1 - \beta)\theta(z) + \beta]] < 1 + Mz \quad (z \in U),$$

then

$$\Re(\theta(z)) > 0 \quad (z \in U).$$

**Lemma 4.** [15]. Let $\phi$ be analytic in $U$ with $\phi(0) = 1$ and $\phi(z) \neq 0$ for $0 < |z| < 1$, and let $A, B \in \mathbb{C}$ with $A \neq B, |B| \leq 1$.

(i) Let $B \neq 0$ and $\gamma \in \mathbb{C}^*$ satisfy either

$$\left|\frac{\gamma(A - B)}{B} - 1\right| \leq 1 \quad \text{or} \quad \left|\frac{\gamma(A - B)}{B} + 1\right| \leq 1.$$ 

If $\phi$ satisfies

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} < \frac{1 + Az}{1 + Bz},$$

then

$$\phi(z) < (1 + Bz)^{\gamma(A - B)}$$

and this is the best dominant.

(ii) Let $B = 0$ and $\gamma \in \mathbb{C}^*$ be such that $|\gamma A| < \pi$, and if $\phi$ satisfies

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} < 1 + Az,$$

then

$$\phi(z) < e^{\gamma Az}$$

and this is the best dominant.

### 3. Main results

Unless otherwise mentioned, we assume throughout this paper that: $-1 \leq B < A \leq 1, -\infty < \lambda < p, p, n \in \mathbb{N}$ and the powers understood as principle values.

**Theorem 1.** Let $-\infty < \lambda < p, p, n \in \mathbb{N}, \nu \in \mathbb{C}^*$ and $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that

$$\left|\frac{\nu(p - \lambda)(A - B)}{B} - 1\right| \leq 1 \quad \text{or} \quad \left|\frac{\nu(p - \lambda)(A - B)}{B} + 1\right| \leq 1,$$

if $B \neq 0$,

$$|\nu A| < \frac{\pi}{p - \lambda}, \quad \text{if} \quad B = 0.$$
If \( f(z) \in A_n(p) \) with \( \Omega_z^{(\lambda, p)} f(z) \neq 0 \) for all \( z \in U \equiv U \setminus \{0\} \), then

\[
\frac{\Omega_z^{(\lambda + 1, p)} f(z)}{\Omega_z^{(\lambda, p)} f(z)} \prec \frac{1 + Az}{1 + Bz}
\]

implies

\[
\left( \frac{\Omega_z^{(\lambda, p)} f(z)}{z^p} \right)^\nu \prec q(z),
\]

where

\[
q(z) = \begin{cases} 
(1 + Bz)^{v(p - \lambda) \left( \frac{A - B}{B} \right)} & \text{if } B \neq 0, \\
e^{v(p - \lambda)Az} & \text{if } B = 0,
\end{cases}
\]

is the best dominant.

**Proof.** If we let

\[
\phi(z) = \left( \frac{\Omega_z^{(\lambda, p)} f(z)}{z^p} \right)^\nu (z \in U), \quad (3.1)
\]

then \( \phi \) is analytic in \( U \), \( \phi(0) = 1 \) and \( \phi(z) \neq 0 \) for \( z \in U \). Taking the logarithmic derivatives in both sides of (3.1), multiplying by \( z \) and using the identity (1.11), we have

\[
1 + \frac{z \phi'(z)}{v(p - \lambda) \phi(z)} = \frac{\Omega_z^{(\lambda + 1, p)} f(z)}{\Omega_z^{(\lambda, p)} f(z)} \prec \frac{1 + Az}{1 + Bz}.
\]

Now, the assertions of Theorem 1 follows by using Lemma 4 for \( \gamma = v(p - \lambda) \), which completes the proof of Theorem 1.

Putting \( \lambda = 0 \) in Theorem 1, we obtain the following result.

**Corollary 1.** Let \( p, n \in \mathbb{N}, v \in \mathbb{C}^* \) and let \( A, B \in \mathbb{C} \) with \( A \neq B \) and \( |B| \leq 1 \). Suppose that

\[
\left| \frac{vp(A - B)}{B} - 1 \right| \leq 1 \quad \text{or} \quad \left| \frac{vp(A - B)}{B} + 1 \right| \leq 1, \quad \text{if } B \neq 0,
\]

\[
|vA| < \frac{\pi}{p}, \quad \text{if } B = 0.
\]

If \( f(z) \in A_n(p) \) with \( f(z) \neq 0 \) for all \( z \in U \), then

\[
\frac{zf'(z)}{f(z)} \prec p \frac{1 + Az}{1 + Bz}
\]

implies

\[
\left( \frac{f(z)}{z^p} \right)^\nu \prec q(z),
\]
where
\[ q(z) = \begin{cases} (1 + Bz)^{\nu p} & \text{if } B \neq 0, \\ e^{\nu p Az} & \text{if } B = 0, \end{cases} \]
is the best dominant.

**Theorem 2.** Let \( \delta > 0, 0 \leq \alpha < p, p, n \in \mathbb{N} \) and let the function \( f(z) \in A_{\nu}(p) \) satisfy the following subordination condition:
\[
(1 - \delta) \frac{\Omega_{\lambda}^{(\lambda,p)} f(z)}{z^p} + \delta \frac{\Omega_{\lambda}^{(\lambda+1,p)} f(z)}{z^p} \prec 1 + M_1 z \quad (z \in U),
\]
where
\[
M_1 = \delta (p - \alpha) \left( 1 + \frac{\delta n}{p - \alpha} \right) \frac{1}{|(p - \lambda) - \delta (p - \alpha)| + \sqrt{(p - \lambda)^2 + (p - \lambda + \delta n)^2}}. \tag{3.3}
\]
Then \( f(z) \in V_{\nu,\lambda}^{\alpha}(\alpha) \).

**Proof.** Put
\[
\phi(z) = \frac{\Omega_{\lambda}^{(\lambda,p)} f(z)}{z^p} \quad (z \in U). \tag{3.4}
\]
Then \( \phi(z) \) is of the form (3.1) and is analytic in \( U \). From Lemma 1 with \( A = M_1, B = 0 \) and \( m = 1 \), we have
\[
\phi(z) \prec 1 + \frac{(p - \lambda) M_1}{p - \lambda + \delta n} z \quad (z \in U),
\]
which is equivalent to
\[
|\phi(z) - 1| < \frac{(p - \lambda) M_1}{p - \lambda + \delta n} = N < 1 \quad (z \in U). \tag{3.5}
\]
Set
\[
P(z) = \frac{1}{p - \alpha} \left( \frac{z (\Omega_{\lambda}^{(\lambda,p)} f(z))'}{(\Omega_{\lambda}^{(\lambda,p)} f(z))'} - \alpha \right) \quad (0 \leq \alpha < p; \ z \in U). \tag{3.6}
\]
Using the identity (1.11) followed by (3.4), we obtain
\[
\frac{\Omega_{\lambda}^{(\lambda+1,p)} f(z)}{z^p} = \left[ \left( 1 - \frac{p - \alpha}{p - \lambda} \right) + \left( \frac{p - \alpha}{p - \lambda} \right) P(z) \right] \phi(z). \tag{3.7}
\]
In view of (3.7), the hypothesis (3.2) can be written as follows:
\[
\left| \left( 1 - \frac{p - \alpha}{p - \lambda} \right) \phi(z) + \frac{\delta (p - \alpha)}{p - \lambda} P(z) \phi(z) - 1 \right| < M_1 = \frac{p - \lambda + \delta n}{p - \lambda} N. \tag{3.8}
\]
We need to show that (3.8) yields

\[ \text{Re} \{P(z)\} > 0 \quad (z \in U). \tag{3.9} \]

If we suppose that \( \text{Re} \{P(z)\} \neq 0 \quad (z \in U) \), then there exists a point \( z_0 \in U \) such that \( P(z_0) = ix \) for some \( x \in \mathbb{R} \). To prove (3.9), it is sufficient to obtain a contradiction from the following inequality:

\[ W = \left| \left( 1 - \frac{\delta(p - \alpha)}{p - \lambda} \right) \varphi(z_0) + \frac{\delta(p - \alpha)}{p - \lambda} P(z_0) \varphi(z_0) - 1 \right| \geq M_1. \]

Let \( \varphi(z_0) = u + iv \). Then, by using (3.4) and the triangle inequality, we obtain that

\[
\begin{align*}
W^2 &= \left| \left( 1 - \frac{\delta(p - \alpha)}{p - \lambda} \right) \varphi(z_0) + \frac{\delta(p - \alpha)}{p - \lambda} P(z_0) \varphi(z_0) - 1 \right|^2 \\
&= (u^2 + v^2) \left( \frac{\delta(p - \alpha)}{p - \lambda} \right)^2 x^2 + 2 \frac{\delta(p - \alpha)}{p - \lambda} v x + \left| \left( 1 - \frac{\delta(p - \alpha)}{p - \lambda} \right) \varphi(z_0) - 1 \right|^2 \\
&\geq (u^2 + v^2) \left( \frac{\delta(p - \alpha)}{p - \lambda} \right)^2 x^2 + 2 \frac{\delta(p - \alpha)}{p - \lambda} v x + \left( \frac{\delta(p - \alpha)}{p - \lambda} - \left| 1 - \frac{\delta(p - \alpha)}{p - \lambda} \right| N \right)^2.
\end{align*}
\]

Setting

\[ \Psi(x) = W^2 - M_1^2 = (u^2 + v^2) \left( \frac{\delta(p - \alpha)}{p - \lambda} \right)^2 x^2 + 2 \frac{\delta(p - \alpha)}{p - \lambda} v x + \left( \frac{\delta(p - \alpha)}{p - \lambda} - \left| 1 - \frac{\delta(p - \alpha)}{p - \lambda} \right| N \right)^2 - \left( \frac{p - \lambda + \delta n}{p - \lambda} \right) N^2, \]

we note that (3.9) holds true if \( \Psi(x) \geq 0 \) for any \( x \in \mathbb{R} \). Since

\[ (u^2 + v^2) \left( \frac{\delta(p - \alpha)}{p - \lambda} \right)^2 > 0, \]

the inequality \( \Psi(x) \geq 0 \) holds true if the discriminant \( \Delta \leq 0 \); that is,

\[
\Delta = 4 \left[ \left( \frac{\delta(p - \alpha)}{p - \lambda} \right)^2 v^2 - \left( \frac{\delta(p - \alpha)}{p - \lambda} \right)^2 (u^2 + v^2) \right] \times \left\{ \left( \frac{\delta(p - \alpha)}{p - \lambda} - \left| 1 - \frac{\delta(p - \alpha)}{p - \lambda} \right| N \right)^2 - \left( \frac{p - \lambda + \delta n}{p - \lambda} \right) N^2 \right\} \leq 0,
\]

which is equivalent to

\[
v^2 \left[ 1 - \left( \frac{\delta(p - \alpha)}{p - \lambda} - \left| 1 - \frac{\delta(p - \alpha)}{p - \lambda} \right| N \right)^2 + \left( \frac{p - \lambda + \delta n}{p - \lambda} \right) N^2 \right]
\]
Putting $\phi(z_0) - 1 = \xi e^{i\theta}$ for some real $\theta \in \mathbb{R}$, we get
\[
\frac{\xi^2}{u^2} = -\frac{\xi^2 \sin^2 \theta}{(1 + \xi \cos \theta)^2}.
\]
Since the above expression attains its maximum value at $\cos \theta = -\xi$, by using (3.5), we obtain
\[
\frac{v^2}{u^2} \leq \frac{\xi^2}{1 - \xi^2} \leq \frac{N^2}{1 - N^2}
\]
which yields $\Delta \leq 0$. Therefore, $W \geq M_1$, which contradicts (3.8). Hence $\text{Re} \{P(z)\} > 0 \ (z \in U)$. This proves that $f(z) \in \mathcal{V}_{p,n}^\lambda(\alpha)$. This completes the proof of Theorem 2.

Taking $\lambda = 0$ in Theorem 2, we obtain

**Corollary 2.** Let $\delta > 0$, $0 \leq \alpha < p$, $p \in \mathbb{N}$. If $f(z) \in A_n(p)$ such that $\frac{f(z)}{z^p} \neq 0 \ (z \in U)$ and satisfies the following differential subordination:

\[
(1 - \delta) \frac{f(z)}{z^p} + \delta \frac{f'(z)}{pz^{p-1}} < 1 + M_1z \ (\delta > 0; \ z \in U),
\]
where
\[
M_1 = \frac{\delta (p - \alpha) \left( 1 + \frac{\delta n}{p} \right)}{|p - \delta (p - \alpha)| + \sqrt{p^2 + (p + \delta n)^2}}.
\]
Then $f(z) \in \mathcal{S}_{p,n}^\lambda(\alpha), 0 \leq \alpha < p$.

**Remark 1.** (i) We note that this result (with $n = 1$) also obtained by Patel et al. [13, Corollary 3];

(ii) Putting $p = 1$ in Corollary 2, we obtain the result obtained by Liu [5, Theorem 2.1].

**Theorem 3.** Let $\delta > 0$, $0 \leq \alpha < p$, $p,n \in \mathbb{N}$ and $\mu \geq 0$. If $f(z) \in A_n(p)$ such that $\frac{\Omega_{z}^{(\lambda,p)} f(z)}{z^p} \neq 0 \ (z \in U)$ and satisfies the following differential subordination:

\[
(1 - \delta) \left( \frac{\Omega_{z}^{(\lambda,p)} f(z)}{z^p} \right)^\mu + \delta \frac{\left( \frac{\Omega_{z}^{(\lambda,p)} f(z)}{z^p} \right)'}{pz^{p-1}} \left( \frac{\Omega_{z}^{(\lambda,p)} f(z)}{z^p} \right)^{\mu-1} < 1 + M_2z \ (z \in U),
\]

(3.10)
where

\[
M_2 = \begin{cases} 
\frac{(p - \alpha)\delta \left(1 + \frac{\delta \mu}{\mu}\right)}{|p - (p - \alpha)\delta| + \sqrt{p^2 + \left(p + \frac{\delta \mu}{\mu}\right)^2}} & (\mu > 0), \\
\frac{p - \alpha}{p}\delta & (\mu = 0).
\end{cases}
\]  
(3.11)

Then \( f(z) \in \nu_{p,n}^\lambda(\alpha). \)

**Proof.** If \( \mu = 0, \) then the condition (3.10) is equivalent to

\[
\left| \frac{z(\Omega_z^{(\lambda,p)} f(z))'}{\Omega_z^{(\lambda,p)} f(z)} - p \right| < p - \alpha \quad (z \in U),
\]

which, in turn, implies that \( f(z) \in \nu_{p,n}^\lambda(\alpha). \)

So, we let \( \mu > 0 \) and suppose that

\[
\varphi(z) = \left( \frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} \right)^\mu (z \in U).
\]  
(3.12)

Choosing the principal value in (3.12), we note that \( \varphi \) is of the form (2.1) and is analytic in \( U. \) Differentiating (3.12) with respect to \( z, \) we obtain

\[
(1 - \delta) \left( \frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} \right)^\mu + \delta \left( \frac{\Omega_z^{(\lambda,p)} f(z)}{pz^p-1} \right)^\mu-1 = \varphi(z) + \delta \frac{\mu\varphi'}{\mu p} z \varphi (z) (z \in U),
\]  
(3.13)

which, in view of Lemma 2 (with \( A = M_2 \) and \( B = 0)\), yields

\[
\varphi(z) < 1 + \frac{\mu p}{\mu p + \delta \mu} M_2 z (z \in U).
\]

Also, with the aid of (3.12), (3.10) can be written as follows:

\[
\varphi(z) \left\{ 1 - \delta + \delta \left[ \left(1 - \frac{\alpha}{p}\right) P(z) + \frac{\alpha}{p} \right] \right\} < 1 + M_2 z \quad (z \in U),
\]

where \( P(z) \) is given by (3.6). Therefore, by Lemma 3, we find that

\[ \text{Re} \{P(z)\} > 0 \quad (z \in U), \]

that is, that

\[ \text{Re} \left\{ \frac{z(\Omega_z^{(\lambda,p)} f(z))^'}{\Omega_z^{(\lambda,p)} f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; \ z \in U). \]

This completes the proof of Theorem 3.

Putting \( \lambda = 0 \) in Theorem 3, we obtain the following result.
**Corollary 3.** Let $\delta > 0$, $0 \leq \alpha < p$, $p, n \in \mathbb{N}$ and $\mu \geq 0$. If $f(z) \in A_n(p)$ such that $\frac{f(z)}{z^p} \neq 0$ $(z \in U)$ and satisfies the following differential subordination:

$$(1 - \delta) \left( \frac{f(z)}{z^p} \right)^\mu + \delta \frac{zf'(z)}{pf(z)} \left( \frac{f(z)}{z^p} \right)^\mu < 1 + M_2z$$(z \in U),$$

where $M_2$ is given as in Theorem 3, then $f(z) \in S^*_{p,n}(\alpha)$.

**Remark 2.** (i) We note that this result (with $n = 1$) also obtained by Patel et al. [13, Corollary 4];
(ii) Putting $p = 1$ in Corollary 3, we obtain the result obtained by Liu [5, Theorem 2.2].

Putting $\lambda = 0$ and $\delta = 1$ in Theorem 3 we obtain the following result:

**Corollary 4.** Let $\mu \geq 0$, $0 \leq \alpha < p$ and $p, n \in \mathbb{N}$. If $f(z) \in A_n(p)$ such that $\frac{f(z)}{z^p} \neq 0$ $(z \in U)$ and satisfies the inequality:

$$\left| \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z^p} \right)^\mu - p \right| < \frac{(p - \alpha)(p\mu + n)}{\mu\alpha + p^2\mu^2 + (p\mu + n)^2}$$

$(z \in U),$

then $f(z) \in S^*_{p,n}(\alpha)$.

**Remark 3.** (i) Putting $p = 1$ in Corollary 4, we get the result obtained by Liu [5, Corollary 2.1];
(ii) Putting $p = \mu = 1$ in Corollary 4, we obtain the result obtained by Mocanu and Oros [8, Corollary 2.2].

Putting $\lambda = 0$ and $\delta = \frac{1}{p - \alpha}$, $0 \leq \alpha < p$ in Theorem 3 we obtain the following result:

**Corollary 5.** Let $\mu \geq 0$, $0 \leq \alpha < p$ and $p, n \in \mathbb{N}$. If $f(z) \in A_n(p)$ such that $\frac{f(z)}{z^p} \neq 0$ $(z \in U)$ and satisfies the inequality:

$$\left| (p - \alpha - 1) \left( \frac{f(z)}{z^p} \right)^\mu + \frac{zf'(z)}{pf(z)} \left( \frac{f(z)}{z^p} \right)^\mu + \alpha - p \right| < \frac{(p - \alpha)[p(\mu(p - \alpha) + n)]}{(p - 1) + \sqrt{p^2\mu^2(p - \alpha)^2 + [p\mu(p - \alpha) + n]^2}}$$

$(z \in U),$

then $f(z) \in S^*_{p,n}(\alpha)$.

**Remark 4.** (i) Putting $p = 1$ in Corollary 5, we get the result obtained by Liu [5, Corollary 2.2];
(ii) Putting $p = \mu = 1$ in Corollary 5, we obtain the result obtained by Mocanu and Oros [8, Corollary 2.4].
4. Properties involving the operator $\Omega_{\lambda}^{(\lambda,p)} f(z)$

**Theorem 4.** If $f(z) \in \mathcal{V}_{p,n}(\alpha;A,B)$, then for all $s, t \in \mathbb{C}$ with $|s| \leq 1, |t| \leq 1$ and $s \neq t$ the next subordination holds:

$$
\frac{t^p \Omega_{\lambda}^{(\lambda,p)} f(z)}{s^p \Omega_{\lambda}^{(\lambda,p)} f(z)} \preceq \begin{cases} 
\left( \frac{1 + Bzs}{1 + Bzt} \right)^{\frac{(p - \alpha)(A - B)}{B}}, & \text{if } B \neq 0 \\
\exp[(p - \alpha)Az(s - t)], & \text{if } B = 0.
\end{cases}
$$

*(4.1)*

**Proof.** If $f \in \mathcal{V}_{p,n}(\alpha;A,B)$, from (1.13) it follows that

$$
\frac{z(\Omega_{\lambda}^{(\lambda,p)} f(z))'}{\Omega_{\lambda}^{(\lambda,p)} f(z)} \prec \frac{p + [pB + (A - B)(p - \alpha)]z}{1 + Bz} \equiv k(z).
$$

*(4.2)*

It is easy to check that the function $k$ defined by (4.2) and the function $h$ given by

$$
h(z) = H(z; s, t) = \int_{0}^{z} \left( \frac{s}{1 - su} - \frac{t}{1 - tu} \right) du
$$

are convex in $U$, and by combining a general subordination theorem [16, Theorem 4.1] with (4.2) we deduce

$$
\left( \frac{z(\Omega_{\lambda}^{(\lambda,p)} f(z))'}{\Omega_{\lambda}^{(\lambda,p)} f(z)} - p \right) * h(z) \prec \frac{(p - \alpha)(A - B)z}{1 + Bz} * h(z).
$$

*(4.3)*

A simple computation shows that, if $\varphi(z)$ is an analytic function in the unit disc $U$ with $\varphi(0) = 0$, then

$$
\varphi(z) * h(z) = \int_{iz}^{sz} \frac{\varphi(u)}{u} du
$$

*(4.4)*

and thus, from (4.3) and (4.4) we have

$$
\int_{iz}^{sz} \left( \frac{\left( \Omega_{\lambda}^{(\lambda,p)} f(u) \right)'}{\Omega_{\lambda}^{(\lambda,p)} f(u)} - \frac{p}{u} \right) du \prec (p - \alpha)(A - B) \int_{iz}^{sz} \frac{du}{1 + Bu}.
$$

This last subordination implies

$$
\exp \left[ \int_{iz}^{sz} \left( \frac{\left( \Omega_{\lambda}^{(\lambda,p)} f(u) \right)'}{\Omega_{\lambda}^{(\lambda,p)} f(u)} - \frac{p}{u} \right) du \right] \prec \exp \left[ (p - \alpha)(A - B) \int_{iz}^{sz} \frac{du}{1 + Bu} \right],
$$

which represents the conclusion (4.1).
Corollary 6. If $f \in \mathcal{L}_{p,n}^\lambda(A,B)$, then for all $|z| = r < 1$, the next inequalities hold:

$$
\left| \Omega_z^{(\lambda,p)} f(z) \right| \leq \begin{cases} 
  r^p (1 + Br)^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\
  r^p \exp \left[ (p-\alpha)Ar \right], & \text{if } B = 0,
\end{cases}
$$

(4.5)

$$
\left| \Omega_z^{(\lambda,p)} f(z) \right| \geq \begin{cases} 
  r^p (1 - Br)^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\
  r^p \exp \left[ -(p-\alpha)Ar \right], & \text{if } B = 0,
\end{cases}
$$

(4.6)

and

$$
\left| \arg \left( \frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} \right) \right| \leq \begin{cases} 
  \frac{(p-\alpha)(A-B)}{|B|} \sin^{-1}(|B|r), & \text{if } B \neq 0, \\
  (p-\alpha)Ar, & \text{if } B = 0.
\end{cases}
$$

(4.7)

All of the estimates asserted here are sharp.

**Proof.** Taking $s = 1$ and $t = 0$ in (4.1), and using the definition of subordination, we obtain

$$
\left( \frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} \right) = \begin{cases} 
  (1 + Bw(z))^{\frac{(p-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\
  \exp \left[ (p-\alpha)Aw(z) \right], & \text{if } B = 0,
\end{cases}
$$

(4.8)

where $w$ is an analytic function in $U$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$. According to the well-known Schwarz’s Theorem, we have $|w(z)| \leq |z|$ for all $z \in U$.

(i) If $B > 0$, then from (4.8) are find that

$$
\left| \Omega_z^{(\lambda,p)} f(z) \right| = \exp \left[ \frac{(p-\alpha)(A-B)}{B} \log |1 + Bw(z)| \right] = |1 + Bw(z)|^{\frac{(p-\alpha)(A-B)}{B}} \leq (1 + Br)^{\frac{(p-\alpha)(A-B)}{B}}.
$$

(ii) If $B < 0$, we easily obtain that

$$
\left| \Omega_z^{(\lambda,p)} f(z) \right| = \left| (1 + Bw(z))^{-1} \right|^{\frac{(p-\alpha)(A-B)}{-B}} \leq \left[ (1 + Br)^{-1} \right]^{\frac{(p-\alpha)(A-B)}{-B}} = (1 + Br)^{\frac{(p-\alpha)(A-B)}{B}}.
$$

This proves the inequality (4.5) for $B \neq 0$, and similarly we may prove the other inequalities in (4.6) and (4.7).

Now, for $|z| = r$ and $B \neq 0$, from (4.8) we see that

$$
\left| \arg \left( \frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} \right) \right| = \frac{(p-\alpha)(A-B)}{|B|} |\arg(1 + Bw(z))| \leq \frac{(p-\alpha)(A-B)}{|B|} \sin^{-1}(|B|r),
$$

and for $B = 0$, (4.7) is a direct consequence of (4.8).
It is easy to verify that all of the estimates from this corollary are sharp, being attained by the function \( f_0 \) defined in \( U \) by
\[
\Omega^{(\lambda,p)}_{\alpha}(z) = \begin{cases} z^p (1 + Bz)^{(p-\alpha)(A-B)} \exp[(p-\alpha)Az], & \text{if } B \neq 0, \\ \exp[(p-\alpha)Az], & \text{if } B = 0. \end{cases}
\] (4.9)

**COROLLARY 7.** If \( f \in V_{p,n}^{(\lambda)}(\alpha;A,B) \), then for all \( |z| = r < 1 \), the next inequalities hold:
\[
\left| (\Omega^{(\lambda,p)}_{\alpha} f(z))' \right| \leq \begin{cases} r^{p-1} \left\{ p + [\alpha B + (p - \alpha)A] r \right\} (1 + Br)^{(p-\alpha)(A-B)} \exp[(p-\alpha)Ar], & \text{if } B \neq 0, \\ r^{p-1} \left\{ p - (p - \alpha) |A| r \right\} \exp[-(p-\alpha)Ar], & \text{if } B = 0, \end{cases}
\] (4.10)
\[
\left| (\Omega^{(\lambda,p)}_{\alpha} f(z))' \right| \geq \begin{cases} r^{p-1} \left\{ p - [\alpha B + (p - \alpha)A] r \right\} (1 - Br)^{(p-\alpha)(A-B)} \exp[-(p-\alpha)Ar], & \text{if } B \neq 0, \\ r^{p-1} \left\{ p - (p - \alpha) |A| r \right\} \exp[-(p-\alpha)Ar], & \text{if } B = 0, \end{cases}
\] (4.11)
and
\[
\left| \arg \left( \frac{(\Omega^{(\lambda,p)}_{\alpha} f(z))'}{z^{p-1}} \right) \right| \leq \begin{cases} \frac{(p-\alpha)(A-B)}{|B|} \sin^{-1}(|B| r) + \sin^{-1} \left[ \frac{(p-\alpha)(A-B)r}{p - [\alpha B + (p - \alpha)A]Br^2} \right], & \text{if } B \neq 0, \\ (p - \alpha)Ar + \sin^{-1} \left[ \frac{A(p-\alpha)r}{p} \right], & \text{if } B = 0. \end{cases}
\] (4.12)

All of the estimates asserted here are sharp.

**Proof.** If we define the function \( g \) by
\[
g(z) = \frac{z(\Omega^{(\lambda,p)}_{\alpha} f(z))'}{(\Omega^{(\lambda,p)}_{\alpha} f(z))} \quad (z \in U),
\] (4.13)
then \( g \) is analytic in \( U \) with \( g(0) = p \) and
\[
g(z) \leq \frac{p + [pB + (A-B)(p-\alpha)]z}{1 + Bz},
\]
It is known from [1] that the function \( g \) satisfies the following sharp inequalities:
\[
\frac{p - [\alpha B + (p - \alpha)A]r}{1 - Br} \leq |g(z)| \leq \frac{p + [\alpha B + (p - \alpha)A]r}{1 + Br}, \quad |z| = r < 1,
\] (4.14)
\[
\left| g(z) - \frac{p - [\alpha B + (p - \alpha)A]Br^2}{1 - B^2r^2} \right| \leq \frac{(A-B)(p-\alpha)r}{1 - B^2r^2}, \quad |z| = r < 1,
\] (4.15)
and
\[
|\arg g(z)| \leq \sin^{-1} \left[ \frac{(A-B)(p-\alpha)r}{p - [\alpha B + (p - \alpha)A]Br^2} \right], \quad |z| = r < 1.
\] (4.16)
Applying to the function \( g \) given by (4.13) the inequalities (4.14), (4.15) and (4.16), in conjunction with the estimates given by Corollary 6, we deduce the relations (4.10), (4.11) and (4.12). All of the estimates are sharp for the function \( f_0 \) defined by (4.9).

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