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AN ITERATIVE ALGORITHM FOR SYSTEM OF MIXED VARIATIONAL–LIKE INEQUALITIES

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Abstract. In this paper we consider a system of mixed variational-like inequalities (for short, SMVLI) involving nondifferentiable terms, and its related system of auxiliary problems in the setting of real Hilbert spaces. An existence theorem for the system of auxiliary problems is established. By exploiting this theorem, an iterative algorithm for SMVLI is constructed. We derive the existence of a unique solution of SMVLI and discuss the convergence analysis of the proposed iterative algorithm. Our results represent the generalization, improvement and development of the previously known results in the literature.

1. Introduction

In 1980, Aubin [1] has pointed out that the Nash equilibrium problem [18] for differentiable functions can be formulated in the form of a variational inequality problem defined over the product of sets (for short, VIPPS). Further, Pang [19] showed that not only Nash equilibrium problem but also various equilibrium-type problems, like, traffic equilibrium, spatial equilibrium, and general equilibrium programming problems from operations research, economics, game theory, mathematical physics and other areas, can also be uniformly modeled as a VIPPS. Later, it is found that VIPPS is equivalent to the problem of system of variational inequalities (for short, SVI), see for example [12, 16] and references therein. Pang [19] suggested parallel and serial algorithms to compute the approximate solutions of a VIPPS. He also studied the convergence of the approximate solutions obtained by his algorithms to the exact solution of a VIPPS. The approximation methods for solving a VIPPS are also studied by Cohen and Chaplais [7], Ferris and Pang [8], Konnov [14, 15] and Makler-Sceimberg et al. [17]. In 1999, Ansari and Yao [4] used a fixed point theorem for a family of multivalued maps to prove the existence of a solution of SVI. Since than several authors, see for instance [2, 3, 4, 5, 12, 11, 16], studied the existence theory of various classes of systems of variational(-like) inequalities by exploiting fixed point theorems and maximal element

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theorems for a family of multivalued maps. In the recent past, system of variational(-like) inequalities and system of generalized variational(-like) inequalities emerged as tools to prove the existence of a solution of Nash equilibrium problem [18] for differentiable and non-differentiable functions, respectively. See for example [2, 3, 4, 5, 12] and references therein. On the other hand, only a few iterative algorithms have been constructed for finding the approximate solutions of systems of variational inequalities, see for example [7, 8, 14, 15, 17, 19] and references therein.

In this paper, we consider a system of mixed variational-like inequalities which includes system of variational(-like) inequalities as a special case. We propose a system of auxiliary problems and prove the existence of its unique solution. Further, by exploiting our existence result for a unique solution of our auxiliary problem, we propose an iterative algorithm for computing the approximate solutions of our system of mixed variational-like inequalities. Finally, we prove the existence of a unique solution of our system of mixed variational-like inequalities and discuss the convergence of the approximate solutions obtained by proposed algorithm to the solution of system of mixed variational-like inequalities. The results of this paper generalize and improve several results appeared in the literature.

2. Formulations and preliminaries

Throughout the paper, unless otherwise stated, we assume that $I = \{1,2,...,N\}$ is an index set and for each $i \in I$, H_i is a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle_i$ and $\|\cdot\|_i$, respectively. We set $H = \prod_{i\in I} H_i$. For each $i \in I$, let $F_i : H \to H_i$ and $\eta_i : H_i \times H_i \to H_i$ be nonlinear mappings. For each $i \in I$, let $b_i : H_i \times H_i \to \mathbb{R}$ be a bifunction which is not necessarily differentiable and satisfies the following properties:

(a) b_i is linear in the first argument;

(b) b_i is bounded, that is, there exists a constant $\gamma_i > 0$ such that

$$b_i(u_i, v_i) \leqslant \gamma_i \|u_i\|_i \|v_i\|_i, \quad \forall u_i, v_i \in H_i;$$

(c) $b_i(u_i, v_i) - b_i(u_i, w_i) \leqslant b_i(u_i, v_i - w_i), \quad \forall u_i, v_i, w_i \in H_i$;

(d) b_i is convex in the second argument.

REMARK 2.1. As in [[10], Remark 2.1], we have

(i) For each $i \in I$ and for arbitrary $u_i, v_i \in H_i$, property (a) implies that $-b_i(u_i, v_i) = b_i(-u_i, v_i)$ and property (b) implies that $b_i(-u_i, v_i) \leqslant \gamma_i \|u_i\|_i \|v_i\|_i$. Hence, we have

$$|b_i(u_i, v_i)| \leqslant \gamma_i \|u_i\|_i \|v_i\|_i, \quad \forall u_i, v_i \in H_i, i \in I,$$

$$b_i(u_i, 0) = b_i(0, v_i) = 0, \quad \forall u_i, v_i \in H_i, i \in I.$$

(ii) For each $i \in I$, it follows from properties (b) and (c) that, for all $u_i, v_i, w_i \in H_i$,

$$b_i(u_i, v_i) - b_i(u_i, w_i) \leqslant \gamma_i \|u_i\|_i \|v_i - w_i\|_i,$$

$$b_i(u_i,w_i)-b_i(u_i,v_i)\leqslant \gamma_i\|u_i\|_i\|w_i-v_i\|_i.$$

Therefore

$$|b_i(u_i,v_i)-b_i(u_i,w_i)|\leqslant \gamma_i\|u_i\|_i\|v_i-w_i\|_i,\quad \forall u_i,v_i,w_i\in H_i.$$

This implies that for each $i\in I$, b_i is continuous with respect to the second argument.

We consider the following *system of mixed variational-like inequalities* (for short, SMVLI):

$$\begin{cases}\text{Find } (x_1,x_2,\ldots,x_N)\in H \text{ such that}\\ \langle F_1(x_1,x_2,\ldots,x_N),\eta_1(y_1,x_1)\rangle_1+b_1(x_1,y_1)-b_1(x_1,x_1)\geqslant 0,\quad \forall y_1\in H_1,\\ \langle F_2(x_1,x_2,\ldots,x_N),\eta_2(y_2,x_2)\rangle_2+b_2(x_2,y_2)-b_2(x_2,x_2)\geqslant 0,\quad \forall y_2\in H_2,\\ \vdots\\ \langle F_N(x_1,x_2,\ldots,x_N),\eta_N(y_N,x_N)\rangle_N+b_N(x_N,y_N)-b_N(x_N,x_N)\geqslant 0,\quad \forall y_N\in H_N.\end{cases} \tag{2.1}$$

The following definitions, assumptions and results will be used in the sequel.

DEFINITION 2.1. For each $i\in I$, let $\eta_i:H_i\times H_i\to H_i$ is a mapping. A mapping $F_i:H\to H_i$ is said to be

(i) $\alpha_i-\eta_i$*-strongly monotone in the ith argument* if there exists a constant $\alpha_1>0$ such that

$$\langle F_i(u_1,u_2,\ldots,u_N)-F_i(v_1,u_2,\ldots,u_N),\eta_i(u_i,v_i)\rangle_i \geqslant \alpha_i\|u_i-v_i\|_i^2,\quad \forall u_i,v_i\in H_i,\ (u_1,\ldots u_N)\in H;$$

(ii) $(\beta_{i1},\beta_{i2},\ldots,\beta_{iN})$*-Lipschitz continuous* if there exist constants $\beta_{i1},\beta_{i2},\ldots,\beta_{iN}>0$ such that

$$\|F_i(u_1,u_2,\ldots,u_N)-F_i(v_1,v_2,\ldots,v_N)\|_i \leqslant \beta_{i1}\|u_1-v_1\|_1+\beta_{i2}\|u_2-v_2\|_2+\cdots+\beta_{iN}\|u_N-v_N\|_N$$

for all $(u_1,u_2,\ldots,u_N),(v_1,v_2,\ldots,v_N)\in H$.

ASSUMPTION 2.1. For each $i\in I$, $F_i:H\to H_i$ and $\eta_i:H_i\times H_i\to H_i$ satisfy the following conditions:

(i) For all $(x_1,x_2,\ldots,x_N)\in H$, there exists a constant $\tau_i>0$ such that

$$\|F_i(x_1,x_2,\ldots,x_N)\|_i\leqslant \tau_i\left(\|x_1\|_1+\|x_2\|_2+\cdots+\|x_N\|_N\right);$$

(ii) $\eta_i(x_i,z_i)=\eta_i(x_i,y_i)+\eta_i(y_i,z_i)$, $\forall x_i,y_i,z_i\in H_i$;

(iii) η_i is affine in the second argument;

(iv) For each fixed $u_i\in H_i$, the mapping $v_i\mapsto \eta_i(u_i,v_i)$ is continuous from the weak topology to the weak topology.

REMARK 2.2. It is easy to see that condition (ii) in Assumption 2.1 implies the following conclusions: For all $u_i, v_i \in H_i$

(a) $\eta_i(u_i, u_i) = 0$;

(b) $\eta_i(v_i, u_i) = -\eta_i(u_i, v_i)$.

LEMMA 2.1. [6] *Let X be a nonempty, closed and convex subset of a real Hausdorff topological vector space E and let $\phi, \psi : X \times X \to \mathbb{R}$ be mappings satisfying the following conditions:*

(i) *$\psi(x,y) \leqslant \phi(x,y)$, for all $x, y \in X$ and $\psi(x,x) \geqslant 0$, for all $x \in X$;*

(ii) *For each fixed $x \in X$, the mapping $y \mapsto \phi(x,y)$ is upper semicontinuous;*

(iii) *For each $y \in X$, the set $\{x \in X : \psi(x,y) < 0\}$ is convex;*

(iv) *There exists a nonempty compact set $K \subseteq X$ and $x_0 \in K$ such that*

$$\psi(x_0, y) < 0, \quad \forall y \in X \setminus K.$$

Then, there exists $y^* \in K$ such that $\phi(x, y^*) \geqslant 0$, for all $x \in X$.

LEMMA 2.2. [20] *Let K be a nonempty and convex subset of a real Hilbert space H and $f : K \to \mathbb{R}$ be a lower semicontinuous and convex functional. Then, f is weakly lower semicontinuous.*

REMARK 2.3. [20] By using the same argument as in the proof of Lemma 2.2, it is easy to see that $f : K \to \mathbb{R}$ is upper semicontinuous and concave if f is weakly upper semicontinuous.

3. System of auxiliary problems and an iterative algorithm

Related to SMVLIP (2.1), we introduce the following *system of auxiliary problems* (for short, SAP):

Given $(x_1, x_2, ..., x_N) \in H$, find $(z_1, z_2, ..., z_N) \in H$ such that

$$\begin{aligned}
&\langle \eta_1(z_1,x_1) + \rho F_1(x_1,x_2,...,x_N), \eta_1(y_1,z_1)\rangle_1 + \rho\,[b_1(x_1,y_1) - b_1(x_1,x_1)] \geqslant 0, \\
&\hspace{20em} \forall y_1 \in H_1, \\
&\langle \eta_2(z_2,x_2) + \rho F_2(x_1,x_2,...,x_N), \eta_2(y_2,z_2)\rangle_2 + \rho\,[b_2(x_2,y_2) - b_2(x_2,x_2)] \geqslant 0, \\
&\hspace{20em} \forall y_2 \in H_2, \\
&\qquad\vdots \\
&\langle \eta_N(z_N,x_N) + \rho F_N(x_1,x_2,...,x_N), \eta_N(y_N,z_N)\rangle_N + \rho\,[b_N(x_N,y_N) - b_N(x_N,x_N)] \geqslant 0, \\
&\hspace{20em} \forall y_N \in H_N,
\end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant.

We establish the following existence result for a solution of SAP.

THEOREM 3.1. *For each $i \in I$, let the mapping $\eta_i : H_i \times H_i \to H_i$ be σ_i-strongly monotone and δ_i-Lipschitz continuous and the bifunction $b_i : H_i \times H_i \to \mathbb{R}$ satisfy the conditions (a)-(d). If the Assumption 2.1 holds, then SAP has a solution.*

Proof. For each $i \in I$, define the mappings $\phi_i, \psi_i : H_i \times H_i \to \mathbb{R}$ by

$$\phi_i(y_i,z_i) = \langle \eta_i(y_i,x_i) + \rho F_i(x_1,x_2,\dots,x_N), \eta_i(y_i,z_i)\rangle_i + \rho\,[b_i(x_i,y_i) - b_i(x_i,z_i)]$$

and

$$\psi_i(y_i,z_i) = \langle \eta_i(z_i,x_i) + \rho F_i(x_1,x_2,\dots,x_N), \eta_i(y_i,z_i)\rangle_i + \rho\,[b_i(x_i,y_i) - b_i(x_i,z_i)]$$

for all $y_i, z_i \in H_i$ and for a given $(x_1,x_2,\dots,x_N) \in H$. For each $i \in I$, we show that the mappings ϕ_i and ψ_i satisfy all the conditions of Lemma 2.1 in the weak topology. Indeed, since condition (ii) in Assumption 2.1 implies (b) in Remark 2.2, we have

$$\begin{aligned} 0 \leqslant \|\eta_i(y_i,z_i)\|_i^2 &= \langle \eta_i(y_i,z_i), \eta_i(y_i,z_i)\rangle_i \\ &= \langle \eta_i(y_i,x_i) + \eta_i(x_i,z_i), \eta_i(y_i,z_i)\rangle_i \\ &= \langle \eta_i(y_i,x_i) - \eta_i(z_i,x_i), \eta_i(y_i,z_i)\rangle_i, \end{aligned}$$

and hence

$$\langle \eta_i(z_i,x_i), \eta_i(y_i,z_i)\rangle_i \leqslant \langle \eta_i(y_i,x_i), \eta_i(y_i,z_i)\rangle_i.$$

Thus, it follows that ϕ_i and ψ_i satisfy condition (i) of Lemma 2.1. Since the bifunction b_i is convex in the second argument and η_i is affine in the second argument, it follows from Remark 2.1 (ii) and Assumption 2.1 (iv) that $z_i \mapsto \phi_i(y_i,z_i)$ is weakly upper semicontinuous. Further, it is easy to show that for each fixed $z_i \in H_i$, the set $\{y_i \in H_i : \psi_i(y_i,z_i) < 0\}$ is convex, and so the conditions (ii) and (iii) of Lemma 2.1 hold.

For each $i \in I$, let

$$\omega_i = \sigma_i^{-2}\left(\delta_i^2\|x_i\|_i + \rho\tau_i\delta_i\,(\|x_1\|_1 + \cdots + \|x_N\|_N) + \rho\gamma_i\|x_i\|_i\right)$$

and

$$K_i = \{z_i \in H_i : \|z_i\|_i \leqslant \omega_i\}.$$

Then, for each $i \in I$, K_i is a weakly compact subset of H_i. For any fixed $z_i \in H_i \setminus K_i$, take $v_{0_i} = 0 \in K_i$. From Assumption 2.1, Lipschitz continuity and strong monotonicity of η_i, and Remark 2.1, we have

$$\begin{aligned} \psi_i(v_{0_i},z_i) &= \psi_i(0,z_i) \\ &= \langle \eta_i(z_i,x_i) + \rho F_i(x_1,x_2,\dots,x_N), \eta_i(0,z_i)\rangle + \rho[b_i(x_i,0) - b_i(x_i,z_i)] \\ &= -\langle \eta_i(0,z_i), \eta_i(0,z_i)\rangle_i + \langle \eta_i(0,x_i), \eta_i(0,z_i)\rangle_i \\ &\quad + \rho\langle F_i(x_1,x_2,\dots,x_N), \eta_i(0,z_i)\rangle_i \quad + \rho[b_i(x_i,0) - b_i(x_i,z_i)] \\ &\leqslant -\sigma_i^2\|z_i\|_i^2 + \delta_i^2\|x_i\|_i\|z_i\|_i + \rho\tau_i\delta_i(\|x_1\|_1 + \cdots + \|x_N\|_N)\|z_i\|_i + \rho\gamma_i\|x_i\|_i\|z_i\|_i \\ &= -\sigma_i^2\|z_i\|_i\{\|z_i\|_i - \sigma_i^{-2}(\delta_i^2\|x_i\|_i + \rho\tau_i\delta_i(\|x_1\|_1 + \cdots + \|x_N\|_N) + \rho\gamma_i\|x_i\|_i)\} \\ &< 0. \end{aligned}$$

Hence, condition (iv) of Lemma 2.1 holds. Thus, by Lemma 2.1, for each $i \in I$, there exists $z_i^* \in H_i$ such that

$$\phi_i(y_i, z_i^*) \geqslant 0, \quad \forall y_i \in H_i,$$

that is,

$$\langle \eta_i(y_i, x_i) + \rho F_i(x_1, x_2, \ldots, x_N), \eta_i(y_i, z_i^*) \rangle_i + \rho\, [b_i(x_i, y_i) - b_i(x_i, z_i^*)] \geqslant 0, \quad \forall y_i \in H_i. \tag{3.2}$$

For arbitrary $t \in (0,1]$ and $y_i \in H_i$, let $y_{i,t} := t y_i + (1-t) z_i^*$. By replacing y_i by $y_{i,t}$ in (3.2), we obtain

$$\begin{aligned}
0 &\leqslant \langle \eta_i(y_{i,t}, x_i) + \rho F_i(x_1, x_2, \ldots, x_N), \eta_i(y_{i,t}, z_i^*) \rangle_i + \rho [b_i(x_i, y_{i,t}) - b_i(x_i, z_i^*)] \\
&= \langle \eta_i(x_i, y_{i,t}) - \rho F_i(x_1, x_2, \ldots, x_N), \eta_i(z_i^*, y_{i,t}) \rangle_i + \rho [b_i(x_i, y_{i,t}) - b_i(x_i, z_i^*)] \\
&\leqslant \langle t \eta_i(x_i, y_i) + (1-t) \eta_i(x_i, z_i^*) - \rho F_i(x_1, x_2, \ldots, x_N), t \eta_i(z_i^*, y_i) \\
&\quad + (1-t) \eta_i(z_i^*, z_i^*) \rangle_i + \rho [t b_i(x_i, y_i) + (1-t) b_i(x_i, z_i^*) - b_i(x_i, z_i^*)] \\
&\leqslant t \langle t \eta_i(x_i, y_i) + (1-t) \eta_i(x_i, z_i^*) - \rho F_i(x_1, x_2, \ldots, x_N), \eta_i(z_i^*, y_i) \rangle_i \\
&\quad + t \rho [b_i(x_i, y_i) - b_i(x_i, z_i^*)].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\langle t \eta_i(x_i, y_i) + (1-t) \eta_i(x_i, z_i^*) - \rho F_i(x_1, x_2, \ldots, x_N), \eta_i(z_i^*, y_i) \rangle_i \\
&\qquad + \rho\, [b_i(x_i, y_i) - b_i(x_i, z_i^*)] \geqslant 0.
\end{aligned}$$

Letting $t \to 0^+$, we have

$$\langle \eta_i(x_i, z_i^*) - \rho F_i(x_1, x_2, \ldots, x_N), \eta_i(z_i^*, y_i) \rangle_i + \rho\, [b_i(x_i, y_i) - b_i(x_i, z_i^*)] \geqslant 0, \quad \forall y_i \in H_i,$$

which implies that

$$\langle \eta_i(z_i^*, x_i) + \rho F_i(x_1, x_2, \ldots, x_N), \eta_i(y_i, z_i^*) \rangle_i + \rho\, [b_i(x_i, y_i) - b_i(x_i, z_i^*)] \geqslant 0, \quad \forall y_i \in H_i.$$

Therefore, $(z_1^*, z_2^*, \ldots, z_N^*) \in H$ is the solution of SAP. $\square$

Based on Theorem 3.1, we construct an iterative algorithm for computing the approximate solutions of SMVLIP (2.1) in the following way.

For given $(x_1^0, x_2^0, \ldots, x_N^0) \in H$, from Theorem 3.1, SAP has a solution, say, $(x_1^1, x_2^1, \ldots, x_N^1) \in H$, that is,

$$\begin{aligned}
&\left\langle \eta_1(x_1^1, x_1^0) + \rho F_1(x_1^0, x_2^0, \ldots, x_N^0), \eta_1(y_1, x_1^1) \right\rangle_1 + \rho \left[b_1(x_1^0, y_1) - b_1(x_1^0, x_1^1) \right] \geqslant 0, \\
&\hspace{30em} \forall y_1 \in H_1, \\
&\left\langle \eta_2(x_2^1, x_2^0) + \rho F_2(x_1^0, x_2^0, \ldots, x_N^0), \eta_2(y_2, x_2^1) \right\rangle_2 + \rho \left[b_2(x_2^0, y_2) - b_2(x_2^0, x_2^1) \right] \geqslant 0, \\
&\hspace{30em} \forall y_2 \in H_2, \\
&\qquad\qquad\qquad \vdots \\
&\left\langle \eta_N(x_N^1, x_N^0) + \rho F_N(x_1^0, x_2^0, \ldots, x_N^0), \eta_N(y_N, x_N^1) \right\rangle_N + \rho \left[b_N(x_N^0, y_N) - b_N(x_N^0, x_N^1) \right] \geqslant 0, \\
&\hspace{30em} \forall y_N \in H_N.
\end{aligned}$$

Again by Theorem 3.1, for given $(x_1^1, x_2^1, \ldots, x_N^1) \in H$, SAP has a solution $(x_1^2, x_2^2, \ldots, x_N^2) \in H$, that is,

$$\begin{aligned}
&\left\langle \eta_1(x_1^2, x_1^1) + \rho F_1(x_1^1, x_2^1, \ldots, x_N^1), \eta_1(y_1, x_1^2) \right\rangle_1 + \rho \left[b_1(x_1^1, y_1) - b_1(x_1^1, x_1^2) \right] \geqslant 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y_1 \in H_1, \\
&\left\langle \eta_2(x_2^2, x_2^1) + \rho F_2(x_1^1, x_2^1, \ldots, x_N^1), \eta_2(y_2, x_2^2) \right\rangle_2 + \rho \left[b_2(x_2^1, y_2) - b_2(x_2^1, x_2^2) \right] \geqslant 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y_2 \in H_2, \\
&\qquad\qquad\vdots \\
&\left\langle \eta_N(x_N^2, x_N^1) + \rho F_N(x_1^1, x_2^1, \ldots, x_N^1), \eta_N(y_N, x_N^2) \right\rangle_N + \rho \left[b_N(x_N^1, y_N) - b_N(x_N^1, x_N^2) \right] \geqslant 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y_N \in H_N.
\end{aligned}$$

By induction, we construct the following iterative algorithm for finding the approximate solutions of SMVLIP (2.1):

ALGORITHM 3.1. For given $(x_1^0,\ x_2^0,\ \ldots,\ x_N^0) \in H$, compute a sequence $\{(x_1^n, x_2^n, \ldots, x_N^n)\}$ of approximate solutions in H for SMVLIP (2.1) by the following iterative scheme

$$\begin{aligned}
&\left\langle \eta_1(x_1^{n+1}, x_1^n) + \rho F_1(x_1^n, x_2^n, \ldots, x_N^n), \eta_1(y_1, x_1^{n+1}) \right\rangle_1 + \rho \left[b_1(x_1^n, y_1) - b_1(x_1^n, x_1^{n+1}) \right] \geqslant 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y_1 \in H_1, \\
&\left\langle \eta_2(x_2^{n+1}, x_2^n) + \rho F_2(x_1^n, x_2^n, \ldots, x_N^n), \eta_2(y_2, x_2^{n+1}) \right\rangle_2 + \rho \left[b_2(x_2^n, y_2) - b_2(x_2^n, x_2^{n+1}) \right] \geqslant 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y_2 \in H_2, \\
&\qquad\qquad\vdots \\
&\left\langle \eta_N(x_N^{n+1}, x_N^n) + \rho F_N(x_1^n, x_2^n, \ldots, x_N^n), \eta_N(y_N, x_N^{n+1}) \right\rangle_N + \rho \left[b_N(x_N^n, y_N) - b_N(x_N^n, x_N^{n+1}) \right] \geqslant 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y_N \in H_N,
\end{aligned} \tag{3.3}$$

where $\rho > 0$ is a constant.

4. Existence of a unique solution and convergence analysis

We prove the existence of a unique solution of SMVLIP (2.1) and discuss the convergence analysis of the sequences generated by Algorithm 3.1.

THEOREM 4.1. *For each $i \in I$, let the mapping $F_i : H \to H_i$ be α_i-η_i-strongly monotone in the ith argument and $(\beta_{i1}, \beta_{i2}, \ldots, \beta_{iN})$-Lipschitz continuous, the mapping $\eta_i : H_i \times H_i \to H_i$ be σ_i-strongly monotone and δ_i-Lipschitz continuous and the bifunction $b_i : H_i \times H_i \to \mathbb{R}$ satisfy the properties (a)-(d). Assume that the Assumption 2.1 hold and the following conditions hold for $\rho > 0$*

$$\left| \rho - \frac{\alpha_i - t_i^2 \varepsilon_i}{\beta_{ii}^2 - t_i^2 \varepsilon_i^2} \right| < \frac{\sqrt{(\alpha_i - t_i^2 \varepsilon_i)^2 - (\delta_i^2 - t_i^2)(\beta_{ii}^2 - t_i^2 \varepsilon_i^2)}}{\beta_{ii}^2 - t_i^2 \varepsilon_i^2}, \tag{4.1}$$

$$\alpha_i > t_i^2 \varepsilon_i + \sqrt{(\delta_i^2 - t_i^2)(\beta_{ii}^2 - t_i^2 \varepsilon_i^2)} \quad \text{and} \quad \beta_{ii} > t_i \varepsilon_i,$$

where

$$t_i := \frac{\sigma_i^2}{\delta_i} \quad \text{and} \quad \varepsilon_i := \frac{\gamma_i}{\sigma_i^2} + \sum_{j=1, j\neq i}^{N} \frac{\beta_{ji}}{t_j}, \quad i = 1,2,...,N.$$

Then the approximate solution $(x_1^n, x_2^n, ..., x_N^n)$ *generated by Algorithm 3.1 strongly converges to a solution* $(x_1^*, x_2^*, ..., x_N^*)$ *of SMVLIP (2.1).*

Proof. For any $(y_1, y_2, ..., y_N) \in H$, it follows from (3.3) that

$$\begin{aligned}
&\langle \eta_1(x_1^n, x_1^{n-1}) + \rho F_1(x_1^{n-1}, x_2^{n-1}, ..., x_N^{n-1}), \eta_1(y_1, x_1^n)\rangle_1 \\
&\qquad\qquad + \rho\left[b_1(x_1^{n-1}, y_1) - b_1(x_1^{n-1}, x_1^n)\right] \geqslant 0, \\
&\langle \eta_2(x_2^n, x_2^{n-1}) + \rho F_2(x_1^{n-1}, x_2^{n-1}, ..., x_N^{n-1}), \eta_2(y_2, x_2^n)\rangle_2 \\
&\qquad\qquad + \rho\left[b_2(x_2^{n-1}, y_2) - b_2(x_2^{n-1}, x_2^n)\right] \geqslant 0, \\
&\qquad\vdots \\
&\langle \eta_N(x_N^n, x_N^{n-1}) + \rho F_N(x_1^{n-1}, x_2^{n-1}, ..., x_N^{n-1}), \eta_N(y_N, x_N^n)\rangle_N \\
&\qquad\qquad + \rho\left[b_N(x_N^{n-1}, y_N) - b_N(x_N^{n-1}, x_N^n)\right] \geqslant 0
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\langle \eta_1(x_1^{n+1}, x_1^n) + \rho F_1(x_1^n, x_2^n, ..., x_N^n), \eta_1(y_1, x_1^{n+1})\rangle_1 \\
&\qquad\qquad + \rho\left[b_1(x_1^n, y_1) - b_1(x_1^n, x_1^{n+1})\right] \geqslant 0, \\
&\langle \eta_2(x_2^{n+1}, x_2^n) + \rho F_2(x_1^n, x_2^n, ..., x_N^n), \eta_2(y_2, x_2^{n+1})\rangle_2 \\
&\qquad\qquad + \rho\left[b_2(x_2^n, y_2) - b_2(x_2^n, x_2^{n+1})\right] \geqslant 0, \\
&\qquad\vdots \\
&\langle \eta_N(x_N^{n+1}, x_N^n) + \rho F_N(x_1^n, x_2^n, ..., x_N^n), \eta_N(y_N, x_N^{n+1})\rangle_N \\
&\qquad\qquad + \rho\left[b_N(x_N^n, y_N) - b_N(x_N^n, x_N^{n+1})\right] \geqslant 0.
\end{aligned} \tag{4.3}$$

Taking $y_i = x_i^{n+1}$ in the ith inequality of (4.2) and $y_i = x_i^n$ in the ith inequality of (4.3), we obtain

$$\begin{aligned}
&\langle \eta_i(x_i^n, x_i^{n-1}) + \rho F_i(x_1^{n-1}, x_2^{n-1}, ..., x_N^{n-1}), \eta_i(x_i^{n+1}, x_i^n)\rangle_i \\
&\qquad\qquad + \rho\left[b_i(x_i^{n-1}, x_i^{n+1}) - b_i(x_i^{n-1}, x_i^n)\right] \geqslant 0,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\langle \eta_i(x_i^{n+1}, x_i^n) + \rho F_i(x_1^n, x_2^n, ..., x_N^n), \eta_i(x_i^n, x_i^{n+1})\rangle_i \\
&\qquad\qquad + \rho\left[b_i(x_i^n, x_i^n) - b_i(x_i^n, x_i^{n+1})\right] \geqslant 0.
\end{aligned} \tag{4.5}$$

Adding (4.4) and (4.5), we get

$$
\begin{aligned}
&\left\|\eta_i(x_i^n,x_i^{n+1})\right\|_i^2 \\
&\quad = \left\langle \eta_i(x_i^n,x_i^{n+1}), \eta_i(x_i^n,x_i^{n+1})\right\rangle_i \\
&\quad \leqslant \left\langle \eta_i(x_i^{n-1},x_i^n) - \rho(F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^n,x_2^n,\dots,x_N^n)), \eta_i(x_i^n,x_i^{n+1})\right\rangle_i \\
&\qquad +\rho\left[b_i(x_i^{n-1}-x_i^n,x_i^{n+1}) + b_i(x_i^n-x_i^{n-1},x_i^n)\right] \\
&\quad \leqslant \left\langle \eta_i(x_i^{n-1},x_i^n) - \rho(F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^n,x_2^n,\dots,x_N^n)), \eta_i(x_i^n,x_i^{n+1})\right\rangle_i \\
&\qquad +\rho b_i\left(x_i^n - x_i^{n-1}, x_i^n - x_i^{n+1}\right).
\end{aligned} \tag{4.6}
$$

Since η_i is σ_i-strongly monotone and δ_i-Lipschitz continuous, from (4.6) we obtain

$$
\begin{aligned}
&\sigma_i^2\left\|x_i^n - x_i^{n+1}\right\|_i^2 \\
&\quad \leqslant \left\|\eta_i(x_i^{n-1},x_i^n) - \rho(F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^n,x_2^n,\dots,x_N^n))\right\|_i \; \left\|\eta_i(x_i^n,x_i^{n+1})\right\|_i \\
&\qquad +\rho\gamma_i\left\|x_i^n - x_i^{n-1}\right\|_i \; \left\|x_i^n - x_i^{n+1}\right\|_i \\
&\quad \leqslant \delta_i\left\|\eta_i(x_i^{n-1},x_i^n) - \rho(F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^n,x_2^n,\dots,x_N^n))\right\|_i \; \left\|x_i^n - x_i^{n+1}\right\|_i \\
&\qquad +\rho\gamma_i\left\|x_i^n - x_i^{n-1}\right\|_i \; \left\|x_i^n - x_i^{n+1}\right\|_i,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left\|x_i^n - x_i^{n+1}\right\|_i \\
&\quad \leqslant \tfrac{1}{\sigma_i^2}[\delta_i\left\|\eta_i(x_i^{n-1},x_i^n) - \rho(F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^n,x_2^n,\dots,x_N^n))\right\|_i \\
&\qquad\quad +\rho\gamma_i\left\|x_i^n - x_i^{n-1}\right\|_i] \\
&\quad \leqslant \tfrac{1}{\sigma_i^2}[\delta_i(\left\|\eta_i(x_i^{n-1},x_i^n) - \rho(F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1})\right. \\
&\qquad\qquad\quad \left.- F_i(x_1^{n-1},x_2^{n-1},\dots,x_{i-1}^{n-1},x_i^n,x_{i+1}^{n-1},\dots,x_N^{n-1}))\right\|_i \\
&\qquad\quad +\rho\left\|F_i(x_1^{n-1},x_2^{n-1},\dots,x_{i-1}^{n-1},x_i^n,x_{i+1}^{n-1},\dots,x_N^{n-1}) - F_i(x_1^n,x_2^n,\dots,x_N^n)\right\|_i) \\
&\qquad\quad +\rho\gamma_i\|x_i^n - x_i^{n-1}\|_i].
\end{aligned} \tag{4.7}
$$

Since F_i is α_i-η_i-strongly monotone in the ith argument and $(\beta_{i1},\beta_{i2},\dots,\beta_{iN})$-Lipschitz continuous, we deduce that

$$
\begin{aligned}
&\left\|\eta_i(x_i^{n-1},x_i^n) - \rho(F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^{n-1},x_2^{n-1},\dots,x_{i-1}^{n-1},x_i^n,x_{i+1}^{n-1},\dots,x_N^{n-1}))\right\|_i^2 \\
&= \left\|\eta_i(x_i^{n-1},x_i^n)\right\|^2 \\
&\ -2\rho\langle F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^{n-1},x_2^{n-1},\dots,x_{i-1}^{n-1},x_i^n,x_{i+1}^{n-1},\dots,x_N^{n-1}), \eta_i(x_i^{n-1},x_i^n)\rangle \\
&\ +\rho^2\left\|F_i(x_1^{n-1},x_2^{n-1},\dots,x_N^{n-1}) - F_i(x_1^{n-1},x_2^{n-1},\dots,x_{i-1}^{n-1},x_i^n,x_{i+1}^{n-1},\dots,x_N^{n-1})\right\|_i^2 \\
&\leqslant \delta_i^2\left\|x_i^{n-1} - x_i^n\right\|^2 - 2\rho\alpha_i\left\|x_i^{n-1} - x_i^n\right\|^2 + \rho^2\beta_{ii}^2\left\|x_i^{n-1} - x_i^n\right\|_i^2 \\
&= \left(\delta_i^2 - 2\rho\alpha_i + \rho^2\beta_{ii}^2\right)\left\|x_i^{n-1} - x_i^n\right\|_i^2,
\end{aligned} \tag{4.8}
$$

and

$$
\begin{aligned}
&\left\|F_i(x_1^{n-1},x_2^{n-1},\dots,x_{i-1}^{n-1},x_i^n,x_{i+1}^{n-1},\dots,x_N^{n-1}) - F_i(x_1^n,x_2^n,\dots,x_N^n)\right\|_i \\
&\qquad \leqslant \sum_{j=1,j\neq i}^{N} \beta_{ij}\left\|x_j^{n-1} - x_j^n\right\|_j.
\end{aligned} \tag{4.9}
$$

It follows from (4.7), (4.8) and (4.9) that

$$\left\|x_i^n - x_i^{n+1}\right\|_i \leqslant \frac{1}{\sigma_i^2}\left[\delta_i\sqrt{\delta_i^2 - 2\rho\alpha_i + \rho^2\beta_{ii}^2} + \rho\gamma_i\right]\left\|x_i^{n-1} - x_i^n\right\|_i + \frac{\rho\delta_i}{\sigma_i^2}\sum_{j=1, j\neq i}^{N}\beta_{ij}\left\|x_j^{n-1} - x_j^n\right\|_j. \tag{4.10}$$

Taking $i = 1,2,...,N$ in (4.10) and summing the resultant inequalities, we obtain

$$\begin{aligned}
&\sum_{j=1}^{N}\left\|x_j^n - x_j^{n+1}\right\|_j \\
&\leqslant \left\{\tfrac{1}{\sigma_1^2}\left[\delta_1\sqrt{\delta_1^2 - 2\rho\alpha_1 + \rho^2\beta_{11}^2} + \rho\gamma_1\right] + \tfrac{\rho\delta_2\beta_{21}}{\sigma_2^2} + \cdots + \tfrac{\rho\delta_N\beta_{N1}}{\sigma_N^2}\right\}\left\|x_1^{n-1} - x_1^n\right\|_1 \\
&\quad + \left\{\tfrac{\rho\delta_1\beta_{12}}{\sigma_1^2} + \tfrac{1}{\sigma_2^2}\left[\delta_2\sqrt{\delta_2^2 - 2\rho\alpha_2 + \rho^2\beta_{22}^2} + \rho\gamma_2\right] + \tfrac{\rho\delta_3\beta_{32}}{\sigma_3^2} + \cdots + \tfrac{\rho\delta_N\beta_{N2}}{\sigma_N^2}\right\}\left\|x_2^{n-1} - x_2^n\right\|_2 \\
&\quad \vdots \\
&\quad + \left\{\tfrac{\rho\delta_1\beta_{1N}}{\sigma_1^2} + \cdots + \tfrac{\rho\delta_{N-1}\beta_{N-1\ N}}{\sigma_{N-1}^2} + \tfrac{1}{\sigma_N^2}\left[\delta_N\sqrt{\delta_N^2 - 2\rho\alpha_N + \rho^2\beta_{NN}^2} + \rho\gamma_N\right]\right\}\left\|x_N^{n-1} - x_N^n\right\|_N \\
&\leqslant \max\{\theta_i : 1 \leqslant i \leqslant N\}\sum_{j=1}^{N}\left\|x_j^{n-1} - x_j^n\right\|_j,
\end{aligned} \tag{4.11}$$

where for each $i \in I$

$$\theta_i := \frac{1}{\sigma_i^2}\left[\delta_i\sqrt{\delta_i^2 - 2\rho\alpha_i + \rho^2\beta_{ii}^2} + \rho\gamma_i\right] + \sum_{j=1, j\neq i}^{N}\frac{\rho\delta_j\beta_{ji}}{\sigma_j^2}. \tag{4.12}$$

Now, define the norm $\|\cdot\|_*$ on $H = \prod_{j=1}^N H_j$ by

$$\|(y_1,y_2,...,y_N)\|_* = \|y_1\|_1 + \|y_2\|_2 + \cdots + \|y_N\|_N, \quad \forall (y_1,y_2,...,y_N) \in H. \tag{4.13}$$

We observe that $(H, \|\cdot\|_*)$ is a Banach space. Hence, (4.12) implies that

$$\begin{aligned}
&\left\|(x_1^n, x_2^n, ..., x_N^n) - (x_1^{n+1}, x_2^{n+1}, ..., x_N^{n+1})\right\|_* \\
&\qquad \leqslant \max\{\theta_i : 1 \leqslant i \leqslant N\}\left\|(x_1^{n-1}, x_2^{n-1}, ..., x_N^{n-1}) - (x_1^n, x_2^n, ..., x_N^n)\right\|_*.
\end{aligned} \tag{4.14}$$

From condition (4.1), it follows that $\max\{\theta_i : 1 \leqslant i \leqslant N\} \in (0,1)$ and hence (4.14) implies that $\{(x_1^n, x_2^n, ..., x_N^n)\}$ is a Cauchy sequence in H. Let $(x_1^n, x_2^n, ..., x_N^n) \to (x_1^*, x_2^*, ..., x_N^*)$ in H as $n \to \infty$. From (3.3), and from the fact that $F_1, F_2, ..., F_N$, $\eta_1, \eta_2, ..., \eta_N$, $b_1, b_2, ..., b_N$ are continuous, we have for each $(y_1, y_2, ..., y_N) \in H$,

$$\begin{aligned}
&\langle \eta_1(x_1^*, x_1^*) + \rho F_1(x_1^*, x_2^*, ..., x_N^*), \eta_1(y_1, x_1^*)\rangle_1 + \rho[b_1(x_1^*, y_1) - b_1(x_1^*, x_1^*)] \geqslant 0, \\
&\langle \eta_2(x_2^*, x_2^*) + \rho F_2(x_1^*, x_2^*, ..., x_N^*), \eta_2(y_2, x_2^*)\rangle_2 + \rho[b_2(x_2^*, y_2) - b_2(x_2^*, x_2^*)] \geqslant 0, \\
&\qquad \vdots \\
&\langle \eta_N(x_N^*, x_N^*) + \rho F_N(x_1^*, x_2^*, ..., x_N^*), \eta_N(y_N, x_N^*)\rangle_N + \rho[b_N(x_N^*, y_N) - b_N(x_N^*, x_N^*)] \geqslant 0,
\end{aligned}$$

that is,

$$\begin{gathered}\langle F_1(x_1^*,x_2^*,...,x_N^*),\eta_1(y_1,x_1^*)\rangle_1+b_1(x_1^*,y_1)-b_1(x_1^*,x_1^*)\geqslant 0,\\ \langle F_2(x_1^*,x_2^*,...,x_N^*),\eta_2(y_2,x_2^*)\rangle_2+b_2(x_2^*,y_2)-b_2(x_2^*,x_2^*)\geqslant 0,\\ \vdots\\ \langle F_N(x_1^*,x_2^*,...,x_N^*),\eta_N(y_N,x_N^*)\rangle_N+b_N(x_N^*,y_N)-b_N(x_N^*,x_N^*)\geqslant 0.\end{gathered}$$

Therefore, $(x_1^*,x_2^*,...,x_N^*)$ is a solution of SMVLIP (2.1). □

REMARK 4.1. Zeng et al. [20] considered the single mixed variational-like inequality problem involving set-valued mappings. There is no doubt that it is of further research interest to extend the method presented in this paper for iterative approximations of solutions of the SMVLIP involving set-valued mappings, that is, utilizing the method presented in this paper, one can extend Theorems 3.1 and 4.1 to the system of N-mixed variational-like inequality problems involving set-valued mappings where N is any given positive integer.

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