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ON EQUIVALENCE BETWEEN CONVERGENCE OF ISHIKAWA—MANN AND NOOR ITERATIONS

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Abstract. In this paper, we prove the equivalence of convergence between the Mann–Ishikawa–Noor and multistep iterations for $\Phi-$ strongly pseudocontractive and $\Phi-$ strongly accretive type operators in an arbitrary Banach spaces. Results proved in this paper represent an extension and refinement of the previously known results in this area.

1. Introduction

Let E denote an arbitrary real Banach space and E^* denote the dual space of E. The duality map $J : E \to 2^{E^*}$ is defined by

$$Jx := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|^2; \|u^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of E and E^*. First of all, we recall and define the concepts as follows:

DEFINITION 1.1. ([1]) Let K be a nonempty subset of E and let $T : K \to K$ be an operator.

1) T is said to be strongly accretive if, for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geqslant k\|x-y\|^2, \tag{1.1}$$

where $k > 0$ is a constant. Without loss of generality, we can assume that $k \in (0,1)$. If $k = 0$ in (1.1) the T is said to be accretive operator.

2) T is said to be $\Phi-$strongly accretive if for all $x, y \in K$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geqslant \Phi(\|x-y\|)\|x-y\|. \tag{1.2}$$

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If I denotes the identity operator, it follows from inequalities (1.1) to (1.2) that T is pseudocontractive (respectively, strongly pseudocontractive, $\Phi-$ strongly pseudocontractive) if and only if $(I-T)$ is an accretive (respectively, strongly accretive, $\Phi-$ strongly accretive). It is shown in [2] that the class of single-valued strongly pseudocontractive operators is a proper subclass of the class of single-valued $\Phi-$ strongly pseudocontractive operators. The classes of single-valued operators have been studied by various authors (see, for example [2, 3, 4, 5, 6, 7, 8]).

Now, we state concepts and lemmas which will be needed in the sequel.

(1) The following iteration (see [9]):

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n+\alpha_n Ty_n^1+\lambda_n\xi_n, \\ y_n^1 = (1-\beta_n^1-c_n^1)x_n+\beta_n^1 Tx_n+c_n^1\eta_n^1, \quad n=0,1,2,\dots, \end{cases} \tag{1.3}$$

is called the Ishikawa iteration sequence with errors, where $\{\alpha_n\},\{\beta_n^1\},\{\lambda_n\},\{c_n^1\}$ are real sequences in $[0,1]$ and $\{\xi_n\},\{\eta_n^1\}$ are sequences in K satisfying appropriate conditions.

(2) In particular, if $\beta_n^1=c_n^1=0$ for $n\geqslant 0$ in (1.3) the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n-\lambda_n)x_n+\alpha_n Tx_n+\lambda_n\xi_n, \quad n=0,1,2,\dots, \tag{1.4}$$

is called the Mann iteration with errors (see [10]).

(3) In [11], Noor introduced the three-step procedure (Noor procedure). Now, we define the three step iterative sequence with errors as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n+\alpha_n Ty_n^1+\lambda_n\xi_n, \\ y_n^1 = (1-\beta_n^1-c_n^1)x_n+\beta_n^1 Ty_n^2+c_n^1\eta_n^1, \\ y_n^2 = (1-\beta_n^2-c_n^2)x_n+\beta_n^2 Tx_n+c_n^2\eta_n^2, \quad n=0,1,2,\dots, \end{cases} \tag{1.5}$$

where $\{\alpha_n\},\{\lambda_n\},\{\beta_n^i\},\{c_n^i\}$ are real sequences in $[0,1]$ and $\{\xi_n\},\{\eta_n^i\}$ are sequences in K satisfying appropriate conditions for $i=1,2$.

(4) In year 2004, Rhoades and Soltuz in [13] introduced the multi-step procedure. We generalize this to the multi-step iterative process with errors as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n+\alpha_n Ty_n^1+\lambda_n\xi_n, \\ y_n^i = (1-\beta_n^i-c_n^i)x_n+\beta_n^i Ty_n^{i+1}+c_n^i\eta_n^i, \quad i=1,\dots,p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1}-c_n^{p-1})x_n+\beta_n^{p-1} Tx_n+c_n^{p-1}\eta_n^{p-1}, \quad n=0,1,2,\dots, \end{cases} \tag{1.6}$$

where $p\geqslant 2$ is fixed order, $\{\alpha_n\},\{\lambda_n\},\{\beta_n^i\},\{c_n^i\}$ are sequences in $[0,1]$ and $\{\xi_n\},\{\eta_n^i\}$ are sequences in K for $i=1,2,\dots,p-1$.

(5) In 2006, Huang et al. in [14] introduced the multi-step iterations with errors as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n+Ty_n^1+\xi_n, \\ y_n^i = (1-\beta_n^i)x_n+\beta_n^i Ty_n^{i+1}+\eta_n^i, \quad i=1,2,\dots,p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1})x_n+\beta_n^{p-1} Tx_n+\eta_n^{p-1}, \ p\geqslant 2, \quad n=0,1,2,\cdots, \end{cases} \tag{1.7}$$

It is clear that if K is a nonempty convex subset of K and $\{\xi_n\}$, $\{\eta_n^i\} \subset K$ such that $\sum_{n=1}^{\infty}\|\xi_n\| < \infty$, $\lim_{n\to\infty}\|\eta_n^1\| = 0$, then the multi-step iterations with errors in the sense of [14] need not be well defined, i.e., $\{x_n\}$ may fail to be in K. More precisely, the conditions imposed on the error terms are not compatible with the randomness of the occurrence of errors. Unlike iteration methods with errors (1.7) of [14], our iteration method with random errors (1.6) is always well defined, that is $\{x_n\}$ is always in K. We would like to emphasize that the multi-step iterations can be viewed as the predictor—corrector methods for solving the nonlinear equations in Banach spaces. For the convergence analysis of the predictor—corrector and multi-step iterative methods for solving the variational inequalities and optimization problems, see Noor [23] and the references therein.

Taking $p = 3$ in (1.6), we obtain that the three-step iteration with errors (1.5). Taking $p = 2$ in (1.6), we obtain the Ishikawa iteration with errors (1.3). Thus, our iteration scheme (1.6) generalizes the Mann, Ishikawa and three-step iteration schemes with errors. It is worth mentioning that other important iteration schemes introduced recently by Das and Debata [21] and Kim et al. [22] are all special cases of our iteration scheme. Iterative methods for approximating fixed points of strongly ($\Phi-$strongly) accretive operator have been studied by some authors (see, e.g., [2, 5, 6, 7, 24, 1]), using the Mann iteration process or the Ishikawa iteration process. Then we have a question: are there any differences of the convergence between these two kinds of sequences? Can we prove the equivalence of the convergence between these two kinds of sequences? B.E. Rhoades and S.M. Soltuz in [13] show that the convergence of the Mann, Ishikawa iterations are equivalent to the multi-step iteration for strongly pseudocontractive operator and strongly accretive operator in uniformly convex Banach space. Z. Huang et al in [14] shows the equivalence of the convergence between the modified Mann–Ishikawa and multi-step Noor iterations with errors for the successively strongly pseudo-contractive operators and the strongly pseudo-contractive operators in uniformly smooth Banach spaces. Recently, the study of equivalence for Mann and Ishikawa iterations has been investigated extensively by many authors (see, e.g., [16, 17, 18, 19, 20, 15] and the references therein). Motivated and inspired by the results of [13] and [14], we prove in this paper that the convergence of the Mann, Ishikawa iterations with errors are equivalent to the multi-step iteration with errors for $\Phi-$strongly pseudocontractive type operator and $\Phi-$strongly accretive type operator in an arbitrary Banach space. Moreover, we will modify some gaps in [15]. Indeed, we discovered that there are some gaps in the proof of [15, Theorem 2.1] (see Page 1265). In [15], the author used $b_{n_{j_o}+i} \leqslant \varepsilon$ for all $i \geqslant 1$ to deduce $b_n \to 0$ $(n \to \infty)$. But it is known that this is not always true. For example,

$$\{b_n\} = \{1,0,2,0,0,3,0,0,0,4,0,0,0,0,5,0,0,0,0,0,6,0,0,\ldots\}. \qquad (*)$$

If we take the second term, the fourth term, the seventh term, the eleventh term, the sixteenth term, $\ldots$, we obtain that the subsequence $\{b_{n_j}\} = \{0\}$. Obviously, $b_{n_j} \to 0$ $(j \to \infty)$. For all $i \in \mathbb{N}$ (a positive integer set), apart from $b_{n_j+i} \neq 0$, the other terms are zero, that is, $b_{n_j+1} = b_{n_j+2} = \cdots = b_{n_j+i-1} = b_{n_j+i+1} = \cdots = 0$. Therefore, $b_{n_j+i} \to 0$ $(n_j \to \infty)$. But $\lim_{n\to\infty} b_n \neq 0$. In fact, the sequence $\{b_n\}$ defined by $(*)$ is divergent.

In the sequel, we shall need the following results.

LEMMA 1.2. ([8]) *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leqslant (1-t_n)a_n + b_n + c_n, \quad n \geqslant 1,$$

where $\{t_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} t_n = \infty$. *Suppose*

(a) $b_n = O(t_n)$,

(b) $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\{a_n\}$ *is bounded.*

The next lemma plays a crucial role for proving the main theorem.

LEMMA 1.3. *Let* $\{a_n\}$, $\{\rho_n\}$, $\{\mu_n\}$, $\{c_n\}$ *be four sequences of non-negative real numbers such that*

$$a_{n+1}^2 \leqslant a_n^2 - \rho_n \phi(a_{n+1}) + \rho_n \mu_n + c_n, \tag{1.8}$$

where $\phi : [0,\infty) \to [0,\infty)$ *is a non-decreasing real function such that* $\phi(t) = 0$ *if and only if* $t = 0$. *Suppose that* (i) $\sum_{n=0}^{\infty} \rho_n = \infty$; (ii) $\lim_{n\to\infty} \mu_n = 0$; (iii) $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Proof. First we show that

$$\liminf_{n\to\infty} a_n = \delta = 0.$$

Suppose, to the contrary that $\delta > 0$ or $\delta = +\infty$, for arbitrary $r \in (0,\delta)$, there exists some positive integer n_0 such that $a_n \geqslant r > 0$ for all $n \geqslant n_0$. Since $\phi(t)$ is non-decreasing, then $\phi(a_{n+1}) \geqslant \phi(r) > 0$ for all $n \geqslant n_0$. As $\lim_{n\to\infty} \mu_n = 0$, there exists a positive integer $n_1 \geqslant n_0$ such that

$$\mu_n \leqslant \frac{\phi(r)}{2}$$

for all $n \geqslant n_1$. Thus, from (1.8) and using the fact that $\phi(t)$ is non-decreasing, for all $n \geqslant n_1$, we have

$$a_{n+1}^2 \leqslant a_n^2 - \rho_n \phi(r) + \frac{1}{2}\rho_n \phi(r) + c_n,$$

or rewritten,

$$\frac{1}{2}\phi(r)\rho_n \leqslant a_n^2 - a_{n+1}^2 + c_n.$$

Hence, for any $n > n_1$,

$$\frac{1}{2}\phi(r)\sum_{j=n_1}^{n}\rho_j \leqslant a_{n_1}^2 - a_{n+1}^2 + \sum_{j=n_1}^{n} c_j \leqslant a_{n_1}^2 + \sum_{j=n_1}^{n} c_j.$$

This implies

$$\phi(r)\sum_{j=n_1}^{\infty}\rho_j \leqslant 2a_{n_1}^2 + \sum_{j=n_1}^{\infty} c_j,$$

which is contradiction with (i). Then $\delta = 0$, i.e., $\liminf_{n\to\infty} a_n = 0$. Now we show that $\limsup_{n\to\infty} a_n = 0$. Indeed, for arbitrary $\varepsilon > 0$, From conditions (ii), (iii) there exists a positive integer n_2 such that

$$\mu_n < \phi(\varepsilon), \quad \sum_{n=n_2}^{\infty} c_n < \varepsilon^2 \tag{1.9}$$

for all $n \geqslant n_2$. Since $\liminf_{n\to\infty} a_n = 0$, there exists a positive integer N such that $a_N < \varepsilon$ for all $n \geqslant N$. Next we shall prove

$$a_k^2 \leqslant \varepsilon^2 + \sum_{n=N}^{k-1} c_n, \quad \forall k \geqslant N. \tag{1.10}$$

The proof of this is by mathematical induction. Clearly, (1.10) holds for $k = N$. Assume now it holds for $n \geqslant N$. We prove that $a_{k+1}^2 \leqslant \varepsilon^2 + \sum_{n=N}^{k} c_n$. Suppose this not the case. Then $a_{k+1}^2 > \varepsilon^2 + \sum_{n=N}^{k} c_n$. This implies that $a_{k+1}^2 > \varepsilon^2$, and $a_{k+1} > \varepsilon$. Since $\phi(t)$ is non-decreasing and $\varepsilon > 0$, then $\phi(a_{k+1}) \geqslant \phi(\varepsilon)$. It follows from (1.8) and (1.9) that

$$\begin{aligned} a_{k+1}^2 &\leqslant a_k^2 - \rho_k\phi(a_{k+1}) + \rho_k\mu_k + c_k \leqslant a_k^2 - \rho_k\phi(\varepsilon) + \rho_k\phi(\varepsilon) + c_k \\ &= a_k^2 + c_k \leqslant \varepsilon^2 + \sum_{n=N}^{k} c_n, \end{aligned}$$

a contradiction. Hence (1.10) holds. Therefore, it follows from (1.9) and (1.10) that

$$\limsup_{k\to\infty} a_k \leqslant \sqrt{\varepsilon^2 + \sum_{n=N}^{\infty} c_n} \leqslant \sqrt{2}\,\varepsilon,$$

then $\limsup_{n\to\infty} a_n = 0$, So $\lim_{n\to\infty} a_n = 0$. This completes the proof. □

LEMMA 1.4. (see [24, Lemma 2.3]) *Let E be a real Banach space and $T : E \to E$ be a continuous $\Phi-$strongly accretive operator. Then the equation $Tx = f$ has a unique solution for any $f \in E$.*

2. Main results

Let S, $T : E \to E, f \in E$ be given. It is well known that T is a $\Phi-$strongly accretive type operator if and only if $(I-T)$ is $\Phi-$strongly pseudocontractive. Moreover, x^* is the solution of the equation $Tx = f$ if and only if x^* is the fixed point for the operator $Sx = f + (I-T)x$.

Replacing T by $f + (I-T)$ in (1.4), (1.6), we obtain the following ordinary Mann and multi-step iteration with errors, respectively:

$$u_{n+1} = (1-\alpha_n-\lambda_n)u_n + \alpha_n(f+u_n-Tu_n) + \lambda_n\omega_n,\ n=0,1,\ldots, \tag{2.1}$$

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n(f+y_n^1-Ty_n^1) + \lambda_n\xi_n, \\ y_n^i = (1-\beta_n^i-c_n^i)x_n + \beta_n^i(f+y_n^{i+1}-Ty_n^{i+1}) + c_n^i\eta_n^i, \quad i=1,\ldots,p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1}-c_n^{p-1})x_n + \beta_n^{p-1}(f+x_n-Tx_n) + c_n^{p-1}\eta_n^{p-1}, \quad n=0,1,2,\ldots \end{cases} \tag{2.2}$$

THEOREM 2.1. *Let K be a nonempty closed convex subset of an arbitrary real Banach space and $T : K \to K$ be a uniformly continuous $\Phi-$strongly accretive operator. Let $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n^i\}$ and $\{c_n^i\}$ be sequences in $[0,1]$ for $i = 1,2,\ldots,p-1$ satisfying the following conditions:*

(a) $\alpha_n + \lambda_n \in [0,1]$ *and* $\beta_n^i + c_n^i \in [0,1]$, $n \geqslant 1$, $i = 1,2,p-1$*;*

(b) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n^1 = 0$ *and* $\lim_{n\to\infty}c_n^1 = 0$*;*

(c) $0 < \alpha_n < 1$, $n \geqslant 1$*;*

(d) $\lim_{n\to\infty}\frac{\lambda_n}{\alpha_n+\lambda_n} = 0$*;*

(e) $\sum_{n=1}^{\infty}(\alpha_n+\lambda_n) = \infty$.

Assume that the sequences $\{Tu_n\}$, $\{Ty_n^j\}$ or $\{(I-T)u_n\}$, $\{(I-T)y_n^j\}$ are bounded for $j = 1,2$. For error term sequences $\{\omega_n\}$, $\{\xi_n\}$, $\{\eta_n^1\}$ bounded in K. If $u_0 = x_0 \in K$, then the following are equivalent:

(i) *the Mann iterative sequence with errors* (2.1) *converges strongly to the solution of the equation $Tx = f$ for any given $f \in K$.*

(ii) *the multi-step iterative sequence with errors* (2.2) *converges strongly to the solution of the equation $Tx = f$ for any given $f \in K$.*

Proof. It follows from Lemma 1.4 that the equation $Tx = f$ has a unique solution $q \in K$. Define $S : K \to K$ by $Sx = f + (I-T)x$ for all $x \in K$, we know that S is uniformly continuous and q is a unique fixed point of S. And (2.1), (2.2) become respectively,

$$u_{n+1} = (1-\alpha_n-\lambda_n)u_n + \alpha_n Su_n + \lambda_n\omega_n,\ n=0,1,\ldots, \tag{2.3}$$

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n S y_n^1 + \lambda_n \xi_n, \\ y_n^i = (1-\beta_n^i - c_n^i)x_n + \beta_n^i S y_n^{i+1} + c_n^i \eta_n^i, \quad i = 1,\dots,p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1} - c_n^{p-1})x_n + \beta_n^{p-1} S x_n + c_n^{p-1}\eta_n^{p-1}, \quad n = 0,1,2,\dots \end{cases} \tag{2.4}$$

If the multi-step iterative sequence with errors (2.4) converges strongly to a point x^*, analogy to [12], we can prove that x^* is a fixed point. Setting $\beta_n^i = 0$ $(i = 1,2,\dots,p-1)$ in (2.4), we get the convergence of the Mann iteration with errors. Conversely, we will prove that the convergence of the Mann iteration with errors implies the convergence of multi-step iteration with errors. Since T is Φ-strongly accretive type operator, operator S is Φ-strongly pseudocontractive type and for all $x,y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geqslant \Phi(\|x-y\|)\|x-y\|,$$

which implies that

$$\Phi(\|x-y\|) \leqslant \|Tx-Ty\|.$$

Then, for all $x,y \in K$, we have

$$\|Sx-Sy\| \leqslant \|x-y\| + \|Tx-Ty\| \leqslant \Phi^{-1}(\|Tx-Ty\|) + \|Tx-Ty\| \tag{2.5}$$

and

$$\|Sx-Sy\| \leqslant \|x-Tx\| + \|y-Ty\|. \tag{2.6}$$

Since the sequences $\{Tu_n\}$, $\{Ty_n^j\}$ or $\{(I-T)u_n\}$, $\{(I-T)y_n^j\}$ are bounded for $j = 1,2$, (2.5) and (2.6), we have that the sequences $\{Su_n\}, \{Sy_n^i, i = 1,2\}$ are bounded. Since $\{\omega_n\}$, $\{\xi_n\}$, $\{\eta_n^1\}$ are bounded in K, then for $q \in F(S)$, the set of fixed points of S, we can find a constant D' such that

$$D' = \max\Big\{\sup_{n\in\mathbb{N}}\{\|Su_n - q\|\}, \sup_{n\in\mathbb{N}}\{\|Sy_n^i - q\| : i = 1,2\},\ \sup_{n\in\mathbb{N}}\{\|\omega_n - q\|\},$$
$$\sup_{n\in\mathbb{N}}\{\|\xi_n - q\|\},\ \sup_{n\in\mathbb{N}}\{\|\eta_n^1 - q\|\}\Big\} + \|x_0 - q\|,$$

then $D' < \infty$. It follows from (2.4) that

$$\begin{aligned} \|x_{n+1} - q\| &\leqslant (1-\delta_n)\|x_n - q\| + \alpha_n\|Sy_n^1 - q\| + \lambda_n\|\omega_n - q\| \\ &\leqslant (1-\delta_n)\|x_n - q\| + \delta_n D', \end{aligned}$$

where $\delta_n = \alpha_n + \lambda_n$. Since $\sum_{n=1}^{\infty}\delta_n = \infty$, Lemma 1.2 guarantees that $\{\|x_{n+1} - q\|\}$ is bounded. As a result, $\{x_{n+1}\}$ is bounded. Similarity, we can prove the sequences $\{y_n^1\}$, $\{u_n\}$ are bounded. So we can find D'' such that

$$\|x_{n+1} - q\| \leqslant D'',\ \|y_n^1 - q\| \leqslant D'',\ \|u_n - q\| \leqslant D'',$$

for all n. Let $D=\max\{D',D''\}$. It follows from (2.3) and (2.4) that

$$\begin{aligned}
u_{n+1} &= (1-\delta_n)u_n+\delta_n Su_n+\lambda_n(\omega_n-Su_n),\\
x_{n+1} &= (1-\delta_n)x_n+\delta_n Sy_n^1+\lambda_n(\xi_n-Sy_n^1).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\|^2 &\leqslant (1-\delta_n)\|x_n-u_n\|\cdot\|x_{n+1}-u_{n+1}\|\\
&\quad +\delta_n\langle Sy_n^1-Su_n, j(x_{n+1}-u_{n+1})\rangle\\
&\quad +\lambda_n\langle(\xi_n-Sy_n^1)-(\omega_n-Su_n), j(x_{n+1}-u_{n+1})\rangle\\
&= (1-\delta_n)\|x_n-u_n\|\cdot\|x_{n+1}-u_{n+1}\|\\
&\quad +\delta_n\langle Sx_{n+1}-Su_{n+1}, j(x_{n+1}-u_{n+1})\rangle\\
&\quad +\delta_n\langle Sy_n^1-Su_n-(Sx_{n+1}-Su_{n+1}), j(x_{n+1}-u_{n+1})\rangle\\
&\quad +\lambda_n\langle(\xi_n-Sy_n^1)-(\omega_n-Su_n), j(x_{n+1}-u_{n+1})\rangle\\
&\leqslant (1-\delta_n)\|x_n-u_n\|\cdot\|x_{n+1}-u_{n+1}\|\\
&\quad +\delta_n\left[\|x_{n+1}-u_{n+1}\|^2-\Phi(\|x_{n+1}-u_{n+1}\|)\|x_{n+1}-u_{n+1}\|\right]\\
&\quad +(\delta_n\theta_n+M\lambda_n)\|x_{n+1}-u_{n+1}\|\\
&\leqslant (1-\delta_n)\|x_n-u_n\|\cdot\|x_{n+1}-u_{n+1}\|\\
&\quad +\delta_n\left[\|x_{n+1}-u_{n+1}\|^2-\Phi(\|x_{n+1}-u_{n+1}\|)\|x_{n+1}-u_{n+1}\|\right]\\
&\quad +\delta_n q_n\|x_{n+1}-u_{n+1}\|, \qquad (2.7)
\end{aligned}$$

where $\theta_n=\|Sy_n^1-Su_n-(Sx_{n+1}-Su_{n+1})\|$, $M=\|(\xi_n-Sy_n^1)-(\omega_n-Su_n)\|<\infty$, $q_n=\theta_n+\frac{M\lambda_n}{\delta_n}$. Notice

$$\begin{aligned}
\|u_{n+1}-u_n\| &\leqslant \delta_n\|Su_n-u_n\|+\lambda_n\|\omega_n-Su_n\|\\
&\leqslant 2D(\delta_n+\lambda_n),\\
\|x_{n+1}-y_n^1\| &\leqslant \delta_n\|Sy_n^1-x_n\|+\lambda_n\|\xi_n-Sy_n^1\|\\
&\quad +\delta_n^1\|Sy_n^2-x_n\|+c_n^1\|\eta_n^1-Sy_n^2\|\\
&\leqslant 2D(\delta_n+\lambda_n)+2D(\delta_n^1+c_n^1),
\end{aligned}$$

where $\delta_n^1=\beta_n^1+c_n^1$. This implies that $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$, $\lim_{n\to\infty}\|x_{n+1}-y_n^1\|=0$ since $\lim_{n\to\infty}\lambda_n=0$, $\lim_{n\to\infty}\delta_n=0$, $\lim_{n\to\infty}\delta_n^1=0$, $\lim_{n\to\infty}c_n^1=0$. Since S is uniformly continuous, we have

$$\theta_n\leqslant\|Sx_{n+1}-Sy_n^1\|+\|Su_{n+1}-Su_n\|\to 0, (n\to\infty).$$

Note that $\lim_{n\to\infty}\frac{\lambda_n}{\delta_n}=0$, we have $q_n\to 0$ as $n\to\infty$. Notice that

$$\begin{aligned}
(1-\delta_n)\|x_n-u_n\|\cdot\|x_{n+1}-u_{n+1}\| &\leqslant \frac{1}{2}\Big((1-\delta_n)^2\|x_n-u_n\|^2+\|x_{n+1}-u_{n+1}\|^2\Big),\\
\|x_{n+1}-u_{n+1}\| &\leqslant \frac{1}{2}(1+\|x_{n+1}-u_{n+1}\|^2).
\end{aligned}$$

If we set $a_n = \|x_n - u_n\|$, it follows from (2.7) that we have

$$[1-\delta_n(2+q_n)]a_{n+1}^2 \leqslant (1-\delta_n)^2 a_n^2 - 2\delta_n \Phi(a_{n+1})a_{n+1} + \delta_n q_n. \tag{2.8}$$

Owing to $\lim_{n\to\infty}[1-\delta_n(2+q_n)] = 1 > 0$, then there exists a positive integer N_0 such that $1-\delta_n(2+q_n) > 0$ for $n \geqslant N_0$. Without loss of generality, let $1-\delta_n(2+q_n) > 0$ for all $n > 0$. Thus, for all $n > 0$, it follows from (2.8) that we have

$$a_{n+1}^2 \leqslant \frac{(1-\delta_n)^2}{1-\delta_n(2+q_n)} a_n^2 - \frac{2\delta_n}{1-\delta_n(2+q_n)} \Phi(a_{n+1})a_{n+1} + \frac{\delta_n q_n}{1-\delta_n(2+q_n)}. \tag{2.9}$$

Since $\lim_{n\to\infty}[1-\delta_n(2+q_n)] = 1 > \frac{1}{2}$, there exists a positive integer N_1 such that $1 > [1-\delta_n(2+q_n)] > \frac{1}{2}$ for all $n \geqslant N_1$. Notice that $a_n \leqslant \|x_n - q\| + \|u_n - q\| \leqslant 2D$. Thus, it follows from (2.9) that

$$a_{n+1}^2 \leqslant a_n^2 - 2\delta_n \Phi(a_{n+1})a_{n+1} + \delta_n(\delta_n + 2q_n + 8D^2 q_n), \quad \forall n \geqslant N_1. \tag{2.10}$$

Taking

$$\rho_n = 2\delta_n,\ \phi(t) = \Phi(t)t,\ \mu_n = \frac{1}{2}(\delta_n + 2q_n + 8D^2 q_n),$$

then (2.10) becomes

$$a_{n+1}^2 \leqslant a_n^2 - \rho_n \phi(a_{n+1}) + \rho_n \mu_n, \quad \forall n \geqslant N_1.$$

This with Lemma 1.3 we get $a_n \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.11}$$

Suppose that $\lim_{n\to\infty} u_n = q$. The inequality

$$0 \leqslant \|x_n - q\| \leqslant \|x_n - u_n\| + \|u_n - q\|$$

and (2.11) imply that $\lim_{n\to\infty} x_n = q$. This completes the proof. □

REMARK 2.2. Theorem 2.1 extends, improves theorem 2.1 of [13] and theorem 2 of [14] in some aspects.

(1). Abolish the condition that E^* is uniformly convex of [13] and E is uniformly smooth of [14].

(2). The hypotheses conditions that a bounded subset K of E in [13], [14] is replaced by the more general conditions $\{(I-T)u_n\}$, $\{(I-T)y_n^i\}_{i=1}^2$ or $\{Tu_n\}$, $\{Ty_n^i\}_{i=1}^2$ are bounded.

(3). The strongly pseudocontractive operator in [13] and [14] is replaced by the $\Phi-$ strongly pseudocontractive operator.

(4). The assumption that $\{\omega_n\}$, $\{\xi_n\}$ be summable and $\lim_{n\to\infty}\|\eta_n^1\| = 0$ are replaced by the assumption that $\{\omega_n\}$, $\{\xi_n\}$, $\{\eta_n^1\}$ be bounded in K in [14].

(5). The domain of T need not be the whole of E.

(6). The iterative sequences in [13], [14] are replaced by the iterative sequence with random errors which appear to be more satisfactory in this paper.

Replacing T by $f+(I-T)$ in (1.3), (1.5), we obtain the following ordinary Ishikawa and three-step iterations, respectively:

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n(f+y_n^1 - Ty_n^1) + \lambda_n\xi_n, \\ y_n^1 = (1-\beta_n^1 - c_n^1)x_n + \beta_n^1(f+x_n-Tx_n) + c_n^1\eta_n^1, \quad n=0,1,2,\ldots \end{cases} \tag{2.12}$$

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n(f+y_n^1 - Ty_n^1) + \lambda_n\xi_n, \\ y_n^1 = (1-\beta_n^1 - c_n^1)x_n + \beta_n^1(f+y_n^2-Ty_n^2) + c_n^1\eta_n^1, \\ y_n^2 = (1-\beta_n^2 - c_n^2)x_n + \beta_n^2(f+x_n-Tx_n) + c_n^2\eta_n^2, \quad n=0,1,2,\cdots \end{cases} \tag{2.13}$$

Taking $p=2,3$ in (2.2), respectively, Theorem 2.1 leads to the following result.

COROLLARY 2.3. *Let K be a nonempty closed convex subset of an arbitrary real Banach space and $T: K \to K$ be a uniformly continuous $\Phi-$strongly accretive operator. Let $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n^i\}$ and $\{c_n^i\}$ be sequences in $[0,1]$ for $i=1,2,\ldots,p-1$ satisfying the following conditions:*

(a) $\alpha_n+\lambda_n \in [0,1]$ *and* $\beta_n^i + c_n^i \in [0,1]$, $n \geqslant 1$, $i=1,2,p-1$*;*

(b) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n^1 = 0$ *and* $\lim_{n\to\infty} c_n^1 = 0$*;*

(c) $0<\alpha_n<1$, $n \geqslant 1$*;*

(d) $\lim_{n\to\infty}\frac{\lambda_n}{\alpha_n+\lambda_n} = 0$*;*

(e) $\sum_{n=1}^{\infty}(\alpha_n+\lambda_n) = \infty$.

Assume that the sequences $\{Tu_n\}$, $\{Ty_n^j\}$ or $\{(I-T)u_n\}$, $\{(I-T)y_n^j\}$ are bounded for $j=1,2$. For error term sequences $\{\omega_n\}$, $\{\xi_n\}$, $\{\eta_n^1\}$ bounded in K. If $u_0=x_0\in K$, then the following are equivalent:

(i) *the Mann iterative sequence (2.1)converges strongly to the solution of the equation $Tx=f$ for any given $f\in K$;*

(ii) *the Ishikawa iterative sequence* (2.12) *converges strongly to the solution of the equation $Tx=f$ for any given $f\in K$;*

(iii) *the three-step iterative sequence* (2.13) *converges strongly to the solution of the equation $Tx=f$ for any given $f\in K$;*

(iv) *the multi-step iterative sequence (2.2) converges strongly to the solution of the equation $Tx=f$ for any given $f\in K$.*

If we put $S=I+T$ and $T:K\to K$ be a uniformly continuous $\Phi-$strongly accretive operator. It is easy to prove that S is a uniformly continuous $\Phi-$strongly accretive operator. For all $x\in K$, we have $f-Tx = f-(S-I)x = f+x-Sx$. Thus, the Mann iterative sequence with errors (2.1) becomes

$$u_{n+1} = (1-\alpha_n-\lambda_n)u_n + \alpha_n(f-Tu_n) + \lambda_n\omega_n, n=0,1,\ldots. \tag{2.14}$$

The Ishikawa iterative sequence with errors (2.12) becomes

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n(f - Ty_n^1) + \lambda_n\xi_n, \\ y_n^1 = (1-\beta_n^1 - c_n^1)x_n + \beta_n^1(f - Tx_n) + c_n^1\eta_n^1, \quad n = 0,1,2,\dots. \end{cases} \tag{2.15}$$

The three-step iterative sequence with errors (2.13) becomes

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n(f - Ty_n^1) + \lambda_n\xi_n, \\ y_n^1 = (1-\beta_n^1 - c_n^1)x_n + \beta_n^1(f - Ty_n^2) + c_n^1\eta_n^1, \\ y_n^2 = (1-\beta_n^2 - c_n^2)x_n + \beta_n^2(f - Tx_n) + c_n^2, \ n = 0,1,2,\dots. \end{cases} \tag{2.16}$$

The multi-step iterative sequence with errors (2.2) becomes

$$\begin{cases} x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n(f - Ty_n^1) + \lambda_n\xi_n, \\ y_n^i = (1-\beta_n^i - c_n^i)x_n + \beta_n^i(f - Ty_n^{i+1}) + c_n^i\eta_n^i, \quad i = 1,\dots,p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1} - c_n^{p-1})x_n + \beta_n^{p-1}(f - Tx_n) + c_n^{p-1}\eta_n^{p-1}, \quad n = 0,1,2,\dots. \end{cases} \tag{2.17}$$

It follows from Corollary 2.3 that we have

COROLLARY 2.4. *Let K be a nonempty closed convex subset of an arbitrary real Banach space and $T : K \to K$ be a uniformly continuous $\Phi-$strongly accretive operator. Let $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n^i\}$ and $\{c_n^i\}$ be sequences in $[0,1]$ for $i = 1,2,\dots,p-1$ satisfying the following conditions:*

(a) $\alpha_n + \lambda_n \in [0,1]$ *and* $\beta_n^i + c_n^i \in [0,1]$, $n \geqslant 1$, $i = 1,2,p-1$*;*

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n^1 = 0$ *and* $\lim_{n\to\infty} c_n^1 = 0$*;*

(c) $0 < \alpha_n < 1$, $n \geqslant 1$*;*

(d) $\lim_{n\to\infty} \frac{\lambda_n}{\alpha_n+\lambda_n} = 0$*;*

(e) $\sum_{n=1}^{\infty}(\alpha_n + \lambda_n) = \infty$.

Assume that $\{u_n + Tu_n\}$, $\{y_n^i + Ty_n^i\}_{i=1}^2$ or the sequences $\{Tu_n\}$, $\{Ty_n^i\}_{i=1}^2$ are bounded and $u_0 = x_0 \in K$, then the following are equivalent:

(i) *the Mann iterative sequence with errors* (2.14) *converges strongly to the solution of the equation $x + Tx = f$ for any given $f \in K$.*

(ii) *the Ishikawa iterative sequence with errors* (2.15) *converges strongly to the solution of the equation $x + Tx = f$ for any given $f \in K$.*

(iii) *the three-step iterative sequence with errors* (2.16) *converges strongly to the solution of the equation $x + Tx = f$ for any given $f \in K$.*

(iv) *the multi-step iterative sequence with errors* (2.17) *converges strongly to the solution of the equation $x + Tx = f$ for any given $f \in K$.*

Let $S = I - T$ and $f = 0$. Suppose that T is a uniformly continuous $\Phi-$ strongly pseudocontractive operator, then S is a uniformly continuous $\Phi-$ strongly accretive type operator. It follows from Lemma 1.4 that equation $Sx = 0$ has a unique solution $q \in K$ if and only if operator T has a unique fixed point $q \in K$. On the other hand, for all $x \in K$, we have $Tx = f + (I - S)x = (I - S)x$. Thus, the Mann iterative sequence with errors (2.1) becomes

$$u_{n+1} = (1 - \alpha_n - \lambda_n)u_n + \alpha_n T u_n + \lambda_n \omega_n, n = 0, 1, \dots. \tag{2.18}$$

The Ishikawa iterative sequence with errors (2.12) becomes

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \lambda_n)x_n + \alpha_n T y_n^1 + \lambda_n \xi_n, \\ y_n^1 \quad = (1 - \beta_n^1 - c_n^1)x_n + \beta_n^1 T x_n + c_n^1, \quad n = 0, 1, 2, \dots. \end{cases} \tag{2.19}$$

The multi-step iterative sequence with errors (2.2) becomes

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \lambda_n)x_n + \alpha_n T y_n^1 + \lambda_n \xi_n, \\ y_n^i \quad = (1 - \beta_n^i - c_n^i)x_n + \beta_n^i T y_n^{i+1} + c_n^i \eta_n^i, \quad i = 1, \dots, p-2, \\ y_n^{p-1} = (1 - \beta_n^{p-1} - c_n^{p-1})x_n + \beta_n^{p-1} T x_n + c_n^{p-1} \eta_n^{p-1}, \quad n = 0, 1, 2, \dots. \end{cases} \tag{2.20}$$

It follows from Theorem 2.1 that we get the following result.

COROLLARY 2.5. *Let K be a nonempty closed convex subset of an arbitrary real Banach space and $T : K \to K$ be a uniformly continuous $\Phi-$ strongly pseudo-contractive operator. Let $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n^i\}$ and $\{c_n^i\}$ be sequences in $[0,1]$ for $i = 1, 2, \dots, p-1$ satisfying the following conditions:*

(a) $\alpha_n + \lambda_n \in [0,1]$ *and* $\beta_n^i + c_n^i \in [0,1]$, $n \geqslant 1$, $i = 1, 2, p-1$*;*

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n^1 = 0$ *and* $\lim_{n\to\infty} c_n^1 = 0$*;*

(c) $0 < \alpha_n < 1$, $n \geqslant 1$*;*

(d) $\lim_{n\to\infty} \frac{\lambda_n}{\alpha_n + \lambda_n} = 0$*;*

(e) $\sum_{n=1}^{\infty} (\alpha_n + \lambda_n) = \infty$.

Assume that the sequences $\{Tu_n\}$, $\{Ty_n^j\}$ or $\{(I-T)u_n\}$, $\{(I-T)y_n^j\}$ are bounded for $j = 1, 2$. For error term sequences $\{\omega_n\}$, $\{\xi_n\}$, $\{\eta_n^1\}$ bounded in K. If $u_0 = x_0 \in K$, then the following are equivalent:

(i) the Mann iterative sequence (2.18) converges strongly to the fixed point of T.

(ii) the Ishikawa iterative sequence (2.19) converges strongly to the fixed point of T.

(iii) the multi-step iterative sequence (2.20) converges strongly to the fixed point of T.

REMARK 2.6. The iteration parameters $\{\alpha_n\},\{\lambda_n\},\{\beta_n^i\},\{\eta_n^i\}$ $(i=1,\cdots,p-1)$ in Theorem 2.1 and Corollary 2.3–2.5 do not depend on any geometric structure of the Banach space E or on any property of the operator T.

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