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INEQUALITIES FOR TERMINATING $_{r+1}\phi_r$ AND APPLICATIONS

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Abstract. In this paper, we use the q-binomial formula to establish two inequalities for the terminating basic hypergeometric series $_{r+1}\phi_r$. Applications of the inequalities are also given.

1. Introduction and main results

q-Series, which are also called basic hypergeometric series, plays a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics, etc. Inequality technique is one of the useful tools in the study of special functions, see e.g. the papers [1, 4, 5, 6, 7, 8, 9]. In [1], the authors gave some inequalities for hypergeometric functions. In [8], the present author gave an inequality for nonterminating $_{r+1}\phi_r$. In this paper, we derive two inequalities for the terminating basic hypergeometric series $_{r+1}\phi_r$. Some applications of the inequalities are also given.

The main results of this paper are the following inequalities for terminating $_{r+1}\phi_r$:

THEOREM 1.1. *Suppose a_i, b_i be any real numbers such that $b_i < 1$ with $i = 1,2,\cdots,r$. Then for any positive integer n, we have*

$$\left|{}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right)\right| \leqslant (-|z|q^{-n}\prod_{i=1}^{r}M_i;q)_n, \tag{1.1}$$

where $M_i = \max\{1, \frac{|1-a_i|}{1-b_i}\}$ for $i = 1,2,\cdots,r$.

THEOREM 1.2. *Suppose a_i, b_i be any real numbers such that $a_i < 1$, $b_i < 1$ with $i = 1,2,\cdots,r$. Then for any positive integer n and $z < 0$, we have*

$${}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right) \geqslant (zq^{-n}\prod_{i=1}^{r}m_i;q)_n, \tag{1.2}$$

where $m_i = \min\{1, \frac{1-a_i}{1-b_i}\}$ for $i = 1,2,\cdots,r$.

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Before the proof of the theorems, we recall some definitions, notations and known results which will be used in this paper. Throughout the whole paper, it is supposed that $0<q<1$. The q-shifted factorials are defined as

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k). \tag{1.3}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1,a_2,...,a_m;q)_n=(a_1;q)_n(a_2;q)_n...(a_m;q)_n, \tag{1.4}$$

where n is an integer or ∞.

The most important theorem in the theory of basic hypergeometric series is the q-binomial theorem [2, 3]:

$$\sum_{k=0}^{\infty}\frac{(a;q)_k z^k}{(q;q)_k}=\frac{(az;q)_\infty}{(z;q)_\infty},\ \ |z|<1. \tag{1.5}$$

When $a=q^{-n}$, where n denotes a nonnegative integer, this reduces to

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k z^k}{(q;q)_k}=(zq^{-n};q)_n. \tag{1.6}$$

Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by [2, 3]

$${}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,...,a_{r+1}\\ b_1,b_2,...,b_r\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_{r+1};q)_n z^n}{(q,b_1,b_2,...,b_r;q)_n}. \tag{1.7}$$

We denote the sign function by

$$\operatorname{sign}x=\begin{cases}1, & \text{when } x>0,\\ 0, & \text{when } x=0,\\ -1, & \text{when } x<0.\end{cases} \tag{1.8}$$

It is obvious that $|x|=x\cdot\operatorname{sign}x$.

2. The proof of theorem 1.1

In this section, we use the terminating case of the q-binomial formula to prove theorem 1.1. In order to prove theorem 1.1, we need the following lemma:

LEMMA 2.1. *Let a,b be given real numbers such that $b<1$, and $0\leqslant t\leqslant 1$. Then we have*

$$\left|\frac{1-at}{1-bt}\right|\leqslant\max\left\{1,\ \frac{|1-a|}{1-b}\right\} \tag{2.1}$$

Proof. Let

$$f(t)=\frac{1-at}{1-bt},\ \ 0\leqslant t\leqslant 1,$$

then

$$f'(t)=\frac{b-a}{(1-bt)^2},\ \ 0\leqslant t\leqslant 1.$$

So $f(t)$ is a monotone function with respect to $0\leqslant t\leqslant 1$. Because of $f(0)=1$ and $f(1)=\frac{1-a}{1-b}$, (2.1) holds. □

Now, we give the proof of theorem 1.1.

Proof. If $z=0$, it is easy to see that (1.1) holds. Now we suppose $z\neq 0$. From lemma 2.1, we know

$$\left|\frac{(a_i;q)_k}{(b_i;q)_k}\right|=\left|\frac{1-a_i}{1-b_i}\cdot\frac{1-a_iq}{1-b_iq}\cdot\cdots\cdot\frac{1-a_iq^{k-1}}{1-b_iq^{k-1}}\right|\leqslant M_i^k, \tag{2.2}$$

where $b_{r+1}=0$, $M_i=\max\{1,\frac{|1-a_i|}{1-b_i}\}$ for $i=1,2,\cdots,r+1$.

Hence,

$$\left|\frac{(a_1,a_2,\cdots,a_r;q)_k}{(b_1,b_2,\cdots,b_r;q)_k}(-\text{sign } z)^k\right|=\left|\frac{(a_1,a_2,\cdots,a_r;q)_k}{(b_1,b_2,\cdots,b_r;q)_k}\right|\leqslant(\prod_{i=1}^r M_i)^k. \tag{2.3}$$

It is obvious that

$$\frac{(q^{-n};q)_k(-|z|)^k}{(q;q)_k}>0,\quad k=1,2,\cdots,n.$$

Multiplying both sides of (2.3) by

$$\frac{(q^{-n};q)_k(-|z|)^k}{(q;q)_k}$$

gives

$$\left|\frac{(q^{-n},a_1,a_2,\cdots,a_r;q)_kz^k}{(q,b_1,b_2,\cdots,b_r;q)_k}\right|\leqslant\frac{(q^{-n};q)_k}{(q;q)_k}(-|z|\prod_{i=1}^r M_i)^k. \tag{2.4}$$

So, we have

$$\begin{aligned}\left|{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\cdots,a_r,q^{-n}\\ b_1,b_2,\cdots,b_r\end{matrix};q,z\right)\right|&=\left|\sum_{k=0}^n\frac{(q^{-n},a_1,a_2,\cdots,a_r;q)_kz^k}{(q,b_1,b_2,\cdots,b_r;q)_k}\right|\\&\leqslant\sum_{k=0}^n\left|\frac{(q^{-n},a_1,a_2,\cdots,a_r;q)_kz^k}{(q,b_1,b_2,\cdots,b_r;q)_k}\right|\\&\leqslant\sum_{k=0}^n\frac{(q^{-n};q)_k}{(q;q)_k}(-|z|\prod_{i=1}^r M_i)^k.\end{aligned} \tag{2.5}$$

Using the q-binomial theorem (1.6) one obtains

$$\sum_{k=0}^n\frac{(q^{-n};q)_k}{(q;q)_k}(-|z|\prod_{i=1}^r M_i)^k=(-|z|q^{-n}\prod_{i=1}^r M_i;q)_n. \tag{2.6}$$

Substituting (2.6) into (2.5), we get (1.1). □

Noticing $(-|z|\prod_{i=1}^{r} M_i;q)_\infty > 1$, we have

COROLLARY 1. Suppose a_i, b_i be any real numbers such that $b_i < 1$ with $i = 1,2,\cdots,r$. Then for any positive integer n, we have

$$\left|{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\cdots,a_r,q^{-n}\\ b_1,b_2,\cdots,b_r\end{matrix};q,z\right)\right| \leqslant (-|z|q^{-n}\prod_{i=1}^{r} M_i;q)_\infty, \tag{2.7}$$

where $M_i = \max\{1, \frac{|1-a_i|}{1-b_i}\}$ for $i = 1,2,\cdots,r$.

3. The proof of theorem 1.2

In this section, we use the terminating case of the q-binomial formula again to prove theorem 1.2. First, we give the following lemma:

LEMMA 3.1. *Let a,b be given real numbers such that $a < 1$ and $b < 1$. Then for $0 \leqslant t \leqslant 1$, we have*

$$\frac{1-at}{1-bt} \geqslant \min\left\{1,\ \frac{1-a}{1-b}\right\}. \tag{3.1}$$

Proof. Let

$$f(t) = \frac{1-at}{1-bt},\ \ 0 \leqslant t \leqslant 1,$$

then

$$f'(t) = \frac{b-a}{(1-bt)^2},\ \ 0 \leqslant t \leqslant 1.$$

So $f(t)$ is a monotone function with respect to $0 \leqslant t \leqslant 1$. Because of $f(0) = 1$ and $f(1) = \frac{1-a}{1-b} > 0$, the inequality (3.1) holds. □

Now, we give the proof of theorem 1.2.

Proof. From lemma 3.1, we know

$$\frac{(a_i;q)_k}{(b_i;q)_k} = \frac{1-a_i}{1-b_i}\cdot\frac{1-a_iq}{1-b_iq}\cdot\cdots\cdot\frac{1-a_iq^{k-1}}{1-b_iq^{k-1}} \geqslant m_i^k, \tag{3.2}$$

where $m_i = \min\{1, \frac{1-a_i}{1-b_i}\}$ for $i = 1,2,\cdots,r$.

Hence,

$$\frac{(a_1,a_2,\cdots,a_r;q)_k}{(b_1,b_2,\cdots,b_r;q)_k} \geqslant (\prod_{i=1}^{r} m_i)^k. \tag{3.3}$$

It is obvious that, when $z < 0$

$$\frac{(q^{-n};q)_k z^k}{(q;q)_k} > 0,\ \ \ k = 1,2,\cdots,n.$$

Multiplying both sides of (3.3) by

$$\frac{(q^{-n};q)_k z^k}{(q;q)_k}$$

gives

$$\frac{(q^{-n},a_1,a_2,\cdots,a_r;q)_k z^k}{(q,b_1,b_2,\cdots,b_r;q)_k} \geqslant \frac{(q^{-n};q)_k}{(q;q)_k}(z\prod_{i=1}^{r} m_i)^k. \tag{3.4}$$

So, we have

$$\begin{aligned} {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right) &= \sum_{k=0}^{n} \frac{(q^{-n},a_1,a_2,\cdots,a_r;q)_k z^k}{(q,b_1,b_2,\cdots,b_r;q)_k} \\ &\geqslant \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}(z\prod_{i=1}^{r} m_i)^k. \end{aligned} \tag{3.5}$$

Using the q-binomial theorem (1.6) one obtains

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}(z\prod_{i=1}^{r} m_i)^k = (zq^{-n}\prod_{i=1}^{r} m_i;q)_n. \tag{3.6}$$

Substituting (3.6) into (3.5), we get (1.2). □

4. Some applications of the inequality

Convergence of a given q-series is an important problem in the study of q-series. In this section, we first use the inequality (1.1) to give the following sufficient condition for convergence of q-series.

THEOREM 4.1. *Suppose a_i, b_i be any real numbers such that $b_i < 1$ and let $M_i = \max\{1, \frac{|1-a_i|}{1-b_i}\}$ for $i = 1,2,\cdots,r$. Further, let $\{c_n\}$ be any sequence of numbers. If*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}q^{-(n+1)}}{c_n}\right| = \lambda < \frac{1}{|z|\prod_{i=1}^{r} M_i}, \tag{4.1}$$

then the q-series

$$\sum_{n=0}^{\infty} c_n \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right) \tag{4.2}$$

converges absolutely.

Proof. Multiplying both sides of (1.1) by $|c_n|$ one gets

$$\left|c_n \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right)\right| \leqslant |c_n|(-|z|q^{-n}\prod_{i=1}^{r} M_i;q)_n. \tag{4.3}$$

¿From (4.1), it is easy to know

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = 0. \tag{4.4}$$

Hence,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{|c_{n+1}|(-|z|q^{-(n+1)}\prod_{i=1}^{r}M_i;q)_{n+1}}{|c_n|(-|z|q^{-n}\prod_{i=1}^{r}M_i;q)_n} \\
&= \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| + |z|\prod_{i=1}^{r}M_i\cdot\lim_{n\to\infty}\left|\frac{c_{n+1}q^{-(n+1)}}{c_n}\right| < 1.
\end{aligned} \tag{4.5}$$

By the ratio test, (4.5) shows that the series

$$\sum_{n=0}^{\infty} c_n(-|z|q^{-n}\prod_{i=1}^{r}M_i;q)_n$$

is absolutely convergent. From (4.3), it is sufficient to establish that (4.2) is absolutely convergent. □

EXAMPLE 4.2. Suppose a_i, b_i be any real numbers such that $b_i < 1$ with $i = 1,2,\cdots,r$. Because of

$$\lim_{n\to\infty}\frac{q^{(n+1)^2}q^{-(n+1)}}{q^{n^2}} = \lim_{n\to\infty} q^n = 0, \tag{4.6}$$

the q-series

$$\sum_{n=0}^{\infty} q^{n^2}\,{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\cdots,a_r,q^{-n}\\ b_1,b_2,\cdots,b_r\end{matrix};q,z\right) \tag{4.7}$$

converges absolutely.

Then we give the following result from inequality (1.2).

THEOREM 4.3. *Suppose a_i, b_i, z be any real numbers such that $z < 0, a_i < 1$, $b_i < 1$ and let $m_i = \min\{1, \frac{1-a_i}{1-b_i}\}$ for $i = 1,2,\cdots,r$. Further, let $\{c_n\}$ be any sequence of numbers. If*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}q^{-(n+1)}}{c_n}\right| = \lambda > \frac{1}{|z|\prod_{i=1}^{r}m_i}, \quad or \quad \lim_{n\to\infty}\left|\frac{c_{n+1}q^{-(n+1)}}{c_n}\right| = \infty, \tag{4.8}$$

then the q-series

$$\sum_{n=0}^{\infty} c_n\cdot{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\cdots,a_r,q^{-n}\\ b_1,b_2,\cdots,b_r\end{matrix};q,z\right) \tag{4.9}$$

diverges.

Proof. Multiplying both sides of (1.2) by $|c_n|$ one gets

$$|c_n|_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right) \geqslant |c_n|(zq^{-n}\prod_{i=1}^{r} m_i;q)_n. \tag{4.10}$$

Since

$$\begin{aligned} &\lim_{n\to\infty} \frac{|c_{n+1}|(zq^{-(n+1)}\prod_{i=1}^{r} m_i;q)_{n+1}}{|c_n|(zq^{-n}\prod_{i=1}^{r} m_i;q)_n} \\ &\quad = \lim_{n\to\infty}\left\{\left|\frac{c_{n+1}}{c_n}\right| + |z|\prod_{i=1}^{r} m_i \cdot \left|\frac{c_{n+1}q^{-(n+1)}}{c_n}\right|\right\} \\ &\quad \geqslant |z|\prod_{i=1}^{r} m_i \cdot \lim_{n\to\infty}\left|\frac{c_{n+1}q^{-(n+1)}}{c_n}\right| > 1, \end{aligned} \tag{4.11}$$

there exists a integer N_0 such that, when $n > N_0$,

$$\begin{aligned} |c_n|\cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right) &\geqslant |c_n|(zq^{-n}\prod_{i=1}^{r} m_i;q)_n \\ &> |c_{N_0}|(zq^{-N_0}\prod_{i=1}^{r} m_i;q)_{N_0} > 0. \end{aligned} \tag{4.12}$$

Therefore the series in (4.9) diverges. □

EXAMPLE 4.4. Suppose a_i, b_i,z be any real numbers such that $z < 0, a_i < 1$, $b_i < 1$ with $i = 1,2,\cdots,r$. Because of

$$\lim_{n\to\infty} \frac{q^{n+1}q^{-(n+1)}}{q^n} = \lim_{n\to\infty} q^{-n} = \infty, \tag{4.13}$$

the q-series

$$\sum_{n=0}^{\infty} q^n {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_r,q^{-n} \\ b_1,b_2,\cdots,b_r \end{matrix};q,z\right) \tag{4.14}$$

diverges.

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