A NOTE ON CLARKSON’S INEQUALITY IN THE REAL CASE

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Abstract. We present an elementary proof of the generalized Clarkson’s inequality in the real case.

In this note, we consider a generalized Clarkson’s inequality in the real case:

\[
(|a + b|^q + |a - b|^q)^{\frac{1}{q}} \leq C(|a|^p + |b|^p)^{\frac{1}{p}},
\]

(1)

for all \(a, b \in \mathbb{R}\). We put

\[
C_{p,q}(\mathbb{R}) = \sup_{a,b \in \mathbb{R}, |a|^p + |b|^p \neq 0} \frac{(|a + b|^q + |a - b|^q)^{\frac{1}{q}}}{(|a|^p + |b|^p)^{\frac{1}{p}}} = \sup_{t \in [0,1]} \frac{((1+t)^q + (1-t)^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}.
\]

In [3], L. Maligranda and N. Saburova computed the best constant \(C = C_{p,q}(\mathbb{R})\) in the inequality (1) for all \(0 < p, q < \infty\). By Theorem 2.1 in [3], we have

THEOREM 1. Let \(0 < p, q < \infty\). Then the best constant \(C_{p,q}(\mathbb{R})\) in equality (1) is:

1. If \(0 < p, q \leq 2\), then \(C_{p,q}(\mathbb{R}) = 2^{1/q}\).
2. If \(2 \leq p < \infty\) and \(0 < q \leq 1\), then \(C_{p,q}(\mathbb{R}) = 2^{1/q}\).
3. If \(2 \leq q < \infty\) and \(1/p + 1/q \geq 1\), then \(C_{p,q}(\mathbb{R}) = 2^{1/q}\).
4. If \(2 \leq q < \infty\) and \(1/p + 1/q \leq 1\), then \(C_{p,q}(\mathbb{R}) = 2^{1-1/p}\).
5. If \(1 < q < 2 < p < \infty\), then \(\max\{2^{1-1/p}, 2^{1/q}\} < C_{p,q}(\mathbb{R}) < 2^{1/q-1/p+1/2}\).

In Theorem 2.5 in [2], K. Kuriyama, M. Miyagi, M. Okada and T. Miyoshi gave the elementary proof of the case that \(1 < p \leq 2\) and \(q > 1\).

Our aim in this note is to present an elementary proof of Theorem 1(5) (cf. [4, 5]). Suppose that \(0 < q < 2 < p < \infty\). We define a function \(f\) from \([0,1]\) into \(\mathbb{R}\) by

\[
f(t) = \frac{((1+t)^q + (1-t)^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}
\]
for \( t \) with \( 0 \leq t \leq 1 \). Then the derivative of \( f \) is

\[
f'(t) = \frac{((1+t)^q+(1-t)^q)^{\frac{1}{p}-1}}{(1+t^p)^{1+\frac{1}{p}}} \{((1+t)^{q-1}-(1-t)^{q-1})(1+t^p)-t^{p-1}((1+t)^{q}+(1-t)^{q})\}.
\]

If \( 0 < q \leq 1 \), then \( f'(t) < 0 \) and so \( f \) is decreasing on \([0, 1]\). Therefore

\[
C_{p,q}(\mathbb{R}) = f(0) = 2^{\frac{1}{p}}.
\]

In the case that \( 1 < q < 2 < p < \infty \), we will prove that there exists a unique \( t_0 \in (0, 1) \) at which the function \( f \) has its maximum. That is, we have

**Lemma 2.** If \( 1 < q < 2 < p < \infty \), then there exists a unique \( t_0 \in (0, 1) \) such that

\[
C_{p,q}(\mathbb{R}) = f(t_0).
\]

Moreover, \( \max\{2^{1-1/p}, 2^{1/q}\} < C_{p,q}(\mathbb{R}) < 2^{1/q-1/p+1/2} \).

**Proof.** It is clear that the derivative of \( f \) is

\[
f'(t) = \frac{((1+t)^q+(1-t)^q)^{\frac{1}{p}-1}}{(1+t^p)^{1+\frac{1}{p}}} \{((1+t)^{q-1}-(1-t)^{q-1})(1+t^p)-t^{p-1}((1+t)^{q}+(1-t)^{q})\}.
\]

For simplicity, we put \( \alpha = p-1 \) and \( \beta = q-1 \), respectively. We define a function \( f_1 \) from \([0, 1]\) into \( \mathbb{R} \) by

\[
f_1(t) = (1+t)^\beta (1-t^\alpha) - (1-t)^\beta (1+t^\alpha)
\]

for \( t \) with \( 0 \leq t \leq 1 \). We also define

\[
f_2(t) = \log((1+t)^\beta (1-t^\alpha)) - \log((1-t)^\beta (1+t^\alpha))
\]

for \( t \) with \( 0 \leq t < 1 \). Note that for any \( t \), \( f_2(t) \geq 0 \) if and only if \( f'(t) \geq 0 \). Since

\[
f_2(t) = \beta \log(1+t) + \log(1-t^\alpha) - \beta \log(1-t) - \log(1+t^\alpha),
\]

we have \( f_2(0) = 0 \) and \( \lim_{t \to -1} f_2(t) = -\infty \). Since the derivative of \( f_2 \) is

\[
f_2'(t) = \frac{2(\beta - \beta x^{2\alpha} - \alpha t^{2\alpha-1} + \alpha t^{2\alpha+1})}{(1+t)(1-t)(1-t^\alpha)(1+t^\alpha)},
\]

we put \( f_3(t) = \beta - \beta t^{2\alpha} - \alpha t^{2\alpha-1} + \alpha t^{2\alpha+1} \). Then the derivative of \( f_3 \) is

\[
f_3'(t) = \alpha t^{\alpha-2} \{(2\alpha + 1)t^2 - 2\beta t - (2\alpha - 1)\}.
\]
We put
\[ f_4(t) = (2\alpha + 1)t^2 - 2\beta t - (2\alpha - 1). \]
Since \( f_4(0) = -2\alpha + 1 < 0 \) and \( f_4(1) = 2(1 - \beta) > 0 \), there exists a unique element \( t_1 \in (0, 1) \) such that \( f_4(t_1) = f_3'(t_1) = 0 \). Since \( f_3(0) = \beta \) and \( f_3(1) = 0 \), the function \( f_3 \) has a minimum at \( t_1 \) and we have \( f_3'(t) < 0 \) on \((0, t_1)\), \( f_3'(t) > 0 \) on \((t_1, 1)\). Since \( f_3(t_1) < 0 \), there exists a unique element \( t_2 \in (0, t_1) \) such that \( f_3(t_2) = 0 \). Since \( f_2'(t_2) = f_3(t_2) = 0 \), by Table 1, \( f_2 \) has a unique maximum at \( t_2 \). Since \( f_2(0) = 0 \) and \( \lim_{t \to 1^-} f_2(t) = -\infty \), there exists \( t_0 \in (t_2, 1) \) such that \( f_2(t_0) = 0 \). Since \( f_1 \) and \( f_2 \) have the same signature on \([0, 1]\), we have \( f_1(t_0) = 0 \), \( f_1(t) > 0 \) on \((0, t_0)\) and \( f_1(t) < 0 \) on \((t_0, 1)\).

**Table 1**

<table>
<thead>
<tr>
<th></th>
<th>( t )</th>
<th>0</th>
<th>( t_2 )</th>
<th>( t_1 )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_3'(t) )</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>( f_3(t) )</td>
<td>( \beta )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( f_2(t) )</td>
<td>0</td>
<td>/</td>
<td>max</td>
<td>( \searrow )</td>
<td>( \searrow )</td>
</tr>
</tbody>
</table>

Then \( f \) is increasing on \([0, t_0]\) and decreasing on \([t_0, 1]\). This implies that \( f \) has the unique maximum at \( t_0 \). Therefore we have \( C_{p,q}(\mathbb{R}) = f(t_0) \) and

\[ \max\{2^{1/q}, 2^{1-1/p}\} = \max\{f(0), f(1)\} < f(t_0) = C_{p,q}(\mathbb{R}). \]

Since \( 1 < q < 2 < p < \infty \), we have for \( t \in (0, 1) \), by the Hölder inequality,

\[
((1+t)^q + (1-t)^q)^{1/q} \leq 2^{1/q-1/2}((1+t)^2 + (1-t)^2)^{1/2} \\
= 2^{1/2}(1+t^2)^{1/2} < 2^{1/q-1/p+1/2}(1+t^p)^{1/p}.
\]

Thus, we have \( f(t_0) < 2^{1/q-1/p+1/2} \). This completes the proof. \( \Box \)
REFERENCES


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