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A NOTE ON CLARKSON'S INEQUALITY IN THE REAL CASE

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Abstract. We present an elementary proof of the generalized Clarkson's inequality in the real case.

In this note, we consider a generalized Clarkson's inequality in the real case:

$$(|a+b|^q+|a-b|^q)^{\frac{1}{q}} \leqslant C(|a|^p+|b|^p)^{\frac{1}{p}}, \tag{1}$$

for all $a,b\in\mathbb{R}$. We put

$$C_{p,q}(\mathbb{R}) = \sup_{a,b\in\mathbb{R},|a|^p+|b|^p\neq 0} \frac{(|a+b|^q+|a-b|^q)^{\frac{1}{q}}}{(|a|^p+|b|^p)^{\frac{1}{p}}} = \sup_{t\in[0,1]} \frac{((1+t)^q+(1-t)^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}.$$

In [3], L. Maligranda and N. Sabourova computed the best constant $C=C_{p,q}(\mathbb{R})$ in the inequality (1) for all $0<p,\ q<\infty$. By Theorem 2.1 in [3], we have

THEOREM 1. *Let $0<p,\ q<\infty$. Then the best constant $C_{p,q}(\mathbb{R})$ in equality (1) is:*

(1) *If $0<p,\ q\leqslant 2$, then $C_{p,q}(\mathbb{R})=2^{1/q}$.*
(2) *If $2\leqslant p<\infty$ and $0<q\leqslant 1$, then $C_{p,q}(\mathbb{R})=2^{1/q}$.*
(3) *If $2\leqslant q<\infty$ and $1/p+1/q\geqslant 1$, then $C_{p,q}(\mathbb{R})=2^{1/q}$.*
(4) *If $2\leqslant q<\infty$ and $1/p+1/q\leqslant 1$, then $C_{p,q}(\mathbb{R})=2^{1-1/p}$.*
(5) *If $1<q<2<p<\infty$, then $\max\{2^{1-1/p},2^{1/q}\}<C_{p,q}(\mathbb{R})<2^{1/q-1/p+1/2}$.*

In Theorem 2.5 in [2], K. Kuriyama, M. Miyagi, M. Okada and T. Miyoshi gave the elementary proof of the case that $1<p\leqslant 2$ and $q>1$.

Our aim in this note is to present an elementary proof of Theorem 1(5) (cf. [4, 5]). Suppose that $0<q<2<p<\infty$. We define a function f from $[0,1]$ into $\mathbb{R}$ by

$$f(t) = \frac{((1+t)^q+(1-t)^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}$$

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for t with $0 \leqslant t \leqslant 1$. Then the derivative of f is

$$f'(t) = \frac{((1+t)^q+(1-t)^q)^{\frac{1}{q}-1}}{(1+t^p)^{1+\frac{1}{p}}}\{((1+t)^{q-1}-(1-t)^{q-1})(1+t^p)-t^{p-1}((1+t)^q+(1-t)^q)\}.$$

If $0 < q \leqslant 1$, then $f'(t) < 0$ and so f is decreasing on $[0,1]$. Therefore

$$C_{p,q}(\mathbb{R}) = f(0) = 2^{\frac{1}{q}}.$$

In the case that $1 < q < 2 < p < \infty$, we will prove that there exists a unique $t_0 \in (0,1)$ at which the function f has its maximum. That is, we have

LEMMA 2. *If $1 < q < 2 < p < \infty$, then there exists a unique $t_0 \in (0,1)$ such that*

$$C_{p,q}(\mathbb{R}) = f(t_0).$$

Moreover, $\max\{2^{1-1/p}, 2^{1/q}\} < C_{p,q}(\mathbb{R}) < 2^{1/q-1/p+1/2}$.

Proof. It is clear that the derivative of f is

$$f'(t) = \frac{((1+t)^q+(1-t)^q)^{\frac{1}{q}-1}}{(1+t^p)^{1+\frac{1}{p}}}\{(1+t)^{q-1}(1-t^{p-1})-(1-t)^{q-1}(1+t^{p-1})\}.$$

For simplicity, we put $\alpha = p-1$ and $\beta = q-1$, respectively. We define a function f_1 from $[0,1]$ into $\mathbb{R}$ by

$$f_1(t) = (1+t)^\beta(1-t^\alpha) - (1-t)^\beta(1+t^\alpha)$$

for t with $0 \leqslant t \leqslant 1$. We also define

$$f_2(t) = \log((1+t)^\beta(1-t^\alpha)) - \log((1-t)^\beta(1+t^\alpha))$$

for t with $0 \leqslant t < 1$. Note that for any t, $f_2(t) \geqslant 0$ if and only if $f'(t) \geqslant 0$. Since

$$f_2(t) = \beta\log(1+t) + \log(1-t^\alpha) - \beta\log(1-t) - \log(1+t^\alpha),$$

we have $f_2(0) = 0$ and $\lim_{t\to 1-0} f_2(t) = -\infty$. Since the derivative of f_2 is

$$f_2'(t) = \frac{2(\beta - \beta x^{2\alpha} - \alpha t^{2\alpha-1} + \alpha t^{2\alpha+1})}{(1+t)(1-t)(1-t^\alpha)(1+t^\alpha)},$$

we put $f_3(t) = \beta - \beta t^{2\alpha} - \alpha t^{2\alpha-1} + \alpha t^{2\alpha+1}$. Then the derivative of f_3 is

$$f_3'(t) = \alpha t^{\alpha-2}\{(2\alpha+1)t^2 - 2\beta t - (2\alpha-1)\}.$$

Table 1

t	0		t_2		t_1		1
$f_3'(t)$		$-$		$-$	0	$+$	
$f_3(t)$	β	$+$	0	$-$		$-$	0
$f_2(t)$	0	$\nearrow$	max	$\searrow$		$\searrow$	$-\infty$

We put

$$f_4(t) = (2\alpha+1)t^2 - 2\beta t - (2\alpha - 1).$$

Since $f_4(0) = -2\alpha + 1 < 0$ and $f_4(1) = 2(1-\beta) > 0$, there exists a unique element $t_1 \in (0,1)$ such that $f_4(t_1) = f_3'(t_1) = 0$. Since $f_3(0) = \beta$ and $f_3(1) = 0$, the function f_3 has a minimum at t_1 and we have $f_3'(t) < 0$ on $(0,t_1)$, $f_3'(t) > 0$ on $(t_1,1)$. Since $f_3(t_1) < 0$, there exists a unique element $t_2 \in (0,t_1)$ such that $f_3(t_2) = 0$. Since $f_2'(t_2) = f_3(t_2) = 0$, by Table 1, f_2 has a unique maximum at t_2. Since $f_2(0) = 0$ and $\lim_{t\to 1-0} f_2(t) = -\infty$, there exists $t_0 \in (t_2,1)$ such that $f_2(t_0) = 0$. Since f_1 and f_2 have the same signature on $[0,1)$, we have $f_1(t_0) = 0$, $f_1(t) > 0$ on $(0,t_0)$ and $f_1(t) < 0$ on $(t_0,1)$.

Table 2

t	0		t_0		1
$f_2(t)$	0	$+$	0	$-$	0
$f_1(t)$	0	$+$	0	$-$	0
$f'(t)$	0	$+$	0	$-$	0
$f(t)$	$2^{1/q}$	$\nearrow$	$f(t_0)$	$\searrow$	$2^{1-1/p}$

Then f is increasing on $[0,t_0]$ and decreasing on $[t_0,1]$. This implies that f has the unique maximum at t_0. Therefore we have $C_{p,q}(\mathbb{R}) = f(t_0)$ and

$$\max\{2^{1/q}, 2^{1-1/p}\} = \max\{f(0), f(1)\} < f(t_0) = C_{p,q}(\mathbb{R}).$$

Since $1 < q < 2 < p < \infty$, we have for $t \in (0,1)$, by the Hölder inequality,

$$\begin{aligned}((1+t)^q + (1-t)^q)^{1/q} &\leqslant 2^{1/q-1/2}((1+t^2)^2 + (1-t)^2)^{1/2} \\ &= 2^{1/q}(1+t^2)^{1/2} < 2^{1/q-1/p+1/2}(1+t^p)^{1/p}.\end{aligned}$$

Thus, we have $f(t_0) < 2^{1/q-1/p+1/2}$. This completes the proof. □

REFERENCES

[1] J. A. CLARKSON, *Uniformly convex spaces*, Trans. Amer. Math. Soc., **40** (1936), 396–414.

[2] K. KURIYAMA, M. MIYAGI, M. OKADA AND T. MIYOSHI, *Elementary proof of Clarkson's inequalities and their generalization*, Memories of the Faculty of Engineering, Yamaguchi Univ., **48** (1997), 119–125.

[3] L. MALIGRANDA AND N. SABOUROVA, *On Clarkson's inequaity in the real case*, Math. Nachr., **280** (2007), 1363–1375.

[4] K. MITANI, K.-S. SAITO AND T. SUZUKI, *On the calculation of the James constant of Lorentz sequence spaces*, J. Math. Anal. Appl., **343** (2008), 310–314.

[5] K. MITANI AND K.-S. SAITO, *Dual of two dimensional Lorentz sequence spaces*, Nonlinear Analysis, **71** (2009), 5238–5247.

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