

## ON ALZER'S INEQUALITY AND ITS GENERALIZED FORMS

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*Abstract.* By using the theory of multiplicatively convex functions, we present some general forms of Alzer's inequality. As consequence, some relevant results in the literature are recovered and an open problem by J. S. Ume [12] is also solved.

### 1. Introduction

Several authors including Alzer [1], Sandor [2], Ume [3], and Kuang [4] proved the following inequality:

$$\frac{n}{n+1} \leq \left( \frac{\frac{1}{n} \sum_{i=1}^n i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} \quad (n = 1, 2, \dots), \quad (1.1)$$

where  $r > 0$ . The proof of this inequality involves the principle of mathematical induction and other analytical methods. Recently, Bennett [8] also proved it by using the *Ratio Principle* and showed that the inequality (1.1) holds for  $r < 0$ . It is easy to find that the inequality (1.1) can also be rephrased as *monotonic result*.

**THEOREM 1.1.** *If  $r > 0$ , then the sequence*

$$\frac{1^r + 2^r + \dots + n^r}{n^{r+1}} \quad (n = 1, 2, \dots)$$

*decreases with  $n$ .*

On the other hand, it should be noted that there are many generalizations of Alzer's inequality (1.1). See, for example, [5-13] and the references cited therein.

Qi [5] offered the first generalization of Alzer's inequality by replacing the sequence  $\{n\}_{n=1}^{\infty}$  with the sequences  $\{n+k\}_{n=1}^{\infty}$ , and obtained: Let  $n$  and  $m$  be natural numbers,  $k$  a nonnegative integer. Then

$$\frac{n+k}{n+m+k} \leq \left( \frac{\frac{1}{n} \sum_{i=k+1}^{n+k} i^r}{\frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r} \right)^{1/r}, \quad (1.2)$$

where  $r$  is any given positive real number.

Qi and Debnath[6] also considered "abstract" version of Alzer's inequality and established the following extension.

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**THEOREM 1.2.** ([6], Corollary 2.2) *Let  $n$  and  $m$  be natural numbers. Suppose that  $\{a_n\}_{n=1}^\infty$  is a positive and increasing sequence satisfying*

$$a_{n+1}^2 \geq a_n a_{n+2} \quad (\text{log - concave}), \tag{1.3}$$

$$\frac{a_{n+1} - a_n}{a_{n+1}^2 - a_n a_{n+2}} \geq \max \left\{ \frac{n+1}{a_{n+1}}, \frac{n+2}{a_{n+2}} \right\} \quad (n = 1, 2, \dots). \tag{1.4}$$

*Then, for any given positive number  $r$ , we have the inequality*

$$\frac{a_n}{a_{n+m}} \leq \left( \frac{\frac{1}{n} \sum_{i=1}^n a_i^r}{\frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r} \right)^{1/r}. \tag{1.5}$$

*The lower bound of (1.5) is best possible.*

Z. Xu and D. Xu [7] also obtained the inequality (1.5) under certain conditions different from those of Theorem 1.2. The authors proved the following

**THEOREM 1.3.** *Let  $n$  and  $m$  be natural numbers,  $r$  a positive number. Suppose that  $\{a_n\}_{n=1}^\infty$  is an increasing positive sequence satisfying (1.3) and*

$$\left( \frac{a_{n+1}}{a_n} \right)^n \leq \left( \frac{a_{n+2}}{a_{n+1}} \right)^{n+1} \quad (n = 1, 2, \dots), \tag{1.6}$$

*then we have the inequality (1.5).*

**REMARK 1.4.** The condition (1.4) is stronger than the hypotheses (1.6). This shows that Theorem 1.3, to some extent, improves Theorem 1.2.

As a matter of fact, if (1.3) and (1.4) hold, then we have

$$\frac{a_{n+1} - a_n}{a_{n+1}^2 - a_n a_{n+2}} \geq \max \left\{ \frac{n+1}{a_{n+1}}, \frac{n+2}{a_{n+2}} \right\} \geq \frac{n+1}{a_{n+1}}.$$

From this, one can easily deduce that

$$n \left( \frac{a_{n+1}}{a_n} - 1 \right) \leq (n+1) \left( \frac{a_{n+2}}{a_{n+1}} - 1 \right). \tag{1.7}$$

Using (1.7) and the arithmetic-geometric mean inequality, we obtain

$$\frac{a_{n+2}}{a_{n+1}} \geq \frac{n \left( \frac{a_{n+1}}{a_n} - 1 \right)}{n+1} + 1 = \frac{n}{n+1} \frac{a_{n+1}}{a_n} + \frac{1}{n+1} \geq \left( \frac{a_{n+1}}{a_n} \right)^{\frac{n}{n+1}}.$$

This implies that the inequality (1.6) holds.

Recently, Bennett [8] announced without proof the following generalization of Alzer’s inequality as Theorem 14 of [8], from which one can see that the condition (1.3) may be superfluous.

THEOREM 1.5. ([8], Theorem 14) *If the sequences*

$$a_n \quad \text{and} \quad \left( \frac{a_{n+1}}{a_n} \right)^n,$$

*both increase (respectively decrease) with  $n$ , then the sequence*

$$\frac{\left( \frac{1}{n} \sum_{k=1}^n a_k^r \right)^{\frac{1}{r}}}{a_n} \quad (r \neq 0) \quad (1.8)$$

*decreases (respectively increases) with  $n$ .*

It should be noted that Niculescu [10] systemically stated the theory of multiplicatively convex functions, which is similar to that of classical convex functions. However, they differ from each other. In many cases, the inequalities based on multiplicatively convexity are better than the direct application of the usual inequalities of convexity. For the more details, the reader is referred to [10] and [11, p. 65–87]. Here we still cite the following conclusions for the sake of convenience.

DEFINITION 1.6. Suppose that  $I$  is a subinterval of  $(0, \infty)$ . A function  $f : I \rightarrow (0, \infty)$  is called multiplicatively convex if for all  $x, y \in I$  and  $\lambda \in [0, 1]$ ,

$$f(x^{1-\lambda}y^\lambda) \leq f(x)^{1-\lambda}f(y)^\lambda. \quad (1.9)$$

If (1.9) is strict for all  $x \neq y$  and  $\lambda \in (0, 1)$ , then  $f$  is said to be strictly multiplicatively convex.

If the inequality in (1.9) is reversed, then  $f$  is said to be multiplicatively concave. If inequality (1.9) is reversed and strict for all  $x \neq y$  and  $\lambda \in (0, 1)$ , then  $f$  is said to be strictly multiplicatively concave.

THEOREM 1.7. *Let  $f : I \rightarrow (0, \infty)$  be a twice differentiable function defined on a subinterval of  $(0, \infty)$ . Then  $f$  is multiplicatively convex (concave) if and only if*

$$x[f(x)f''(x) - f'^2(x)] + f(x)f'(x) \geq 0 (\leq 0), \quad \text{for all } x \in I.$$

*The corresponding variants for the strictly multiplicatively convex (concave) functions also work.*

Our main purpose of this paper is to present some new generalizations of Alzer's inequality by using the theory of multiplicatively convex functions. The presented results contain some relevant results in the literature as their special cases.

## 2. Main results

In this section, we investigate the monotonicity of some sequences involving multiplicatively convex function and convex sequence. Some new generalized versions of Alzer's inequality are presented. As consequences, some general forms of Alzer's inequality in the literature are recovered.

**THEOREM 2.1.** *Let  $f$  be a positive function defined in  $(0, 1]$ . Suppose that  $\{a_n\}_{n=1}^\infty$  is an increasing positive sequence such that the sequence  $\left\{\left(\frac{a_{n+1}}{a_n}\right)^n\right\}_{n=1}^\infty$  increases.*

(1) *If  $f$  is an increasing and multiplicatively convex (concave) function, then the sequence*

$$\left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{1}{n}} \quad (n = 1, 2, \dots)$$

*is decreasing with  $n$ . That is*

$$\left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right)\right)^{\frac{1}{n+1}}. \tag{2.1}$$

(2) *If  $f$  is decreasing and multiplicatively convex (concave), then the sequence*

$$\left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{1}{n}} \quad (n = 1, 2, \dots)$$

*increases with  $n$ .*

*Proof.* Here we only give the proof of (1), since that of (2) is similar and we omit it.

Since the sequences  $\{a_n\}_{n=1}^\infty$  and  $\left\{\left(\frac{a_{n+1}}{a_n}\right)^n\right\}_{n=1}^\infty$  increase, then, for  $i = 1, \dots, n$ , we obtain

$$\left(\frac{a_{i+1}}{a_{n+1}}\right)^{\frac{i}{n+1}} \left(\frac{a_i}{a_{n+1}}\right)^{1-\frac{i}{n+1}} = \frac{a_i}{a_{n+1}} \left(\frac{a_{i+1}}{a_i}\right)^{\frac{i}{n+1}} \leq \frac{a_i}{a_{n+1}} \left(\frac{a_{n+1}}{a_n}\right)^{\frac{n}{n+1}} \leq \frac{a_i}{a_n}. \tag{2.2}$$

We also have

$$\left(\frac{a_{i-1}}{a_n}\right)^{\frac{i-1}{n}} \left(\frac{a_i}{a_n}\right)^{1-\frac{i-1}{n}} = \frac{a_i}{a_n} \left(\frac{a_{i-1}}{a_i}\right)^{\frac{i-1}{n}} \geq \frac{a_i}{a_{n+1}}, i = 2, \dots, n. \tag{2.3}$$

Since  $f$  is increasing, from (2.2) and (2.3), we get

$$f\left(\frac{a_i}{a_n}\right) \geq f\left(\left(\frac{a_{i+1}}{a_{n+1}}\right)^{\frac{i}{n+1}} \left(\frac{a_i}{a_{n+1}}\right)^{1-\frac{i}{n+1}}\right), i = 1, 2, \dots, n, \tag{2.4}$$

and

$$f\left(\frac{a_i}{a_{n+1}}\right) \leq f\left(\left(\frac{a_{i-1}}{a_n}\right)^{\frac{i-1}{n}} \left(\frac{a_i}{a_n}\right)^{1-\frac{i-1}{n}}\right), i = 2, \dots, n. \tag{2.5}$$

If  $f$  is multiplicatively concave, then

$$f\left(\left(\frac{a_{i+1}}{a_{n+1}}\right)^{\frac{i}{n+1}} \left(\frac{a_i}{a_{n+1}}\right)^{1-\frac{i}{n+1}}\right) \geq \left(f\left(\frac{a_{i+1}}{a_{n+1}}\right)\right)^{\frac{i}{n+1}} \left(f\left(\frac{a_i}{a_{n+1}}\right)\right)^{1-\frac{i}{n+1}}. \tag{2.6}$$

A combination of (2.4) with (2.6) leads to

$$f\left(\frac{a_i}{a_n}\right) \geq \left(f\left(\frac{a_{i+1}}{a_{n+1}}\right)\right)^{\frac{i}{n+1}} \left(f\left(\frac{a_i}{a_{n+1}}\right)\right)^{1-\frac{i}{n+1}}. \tag{2.7}$$

Multiplying the two sides of (2.7) from  $i = 1$  to  $n$ , we have

$$\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right) \geq \left(\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right)\right)^{\frac{n}{n+1}}.$$

This shows that inequality (2.1) holds.

If  $f$  is multiplicatively convex, then

$$f\left(\left(\frac{a_{i-1}}{a_n}\right)^{\frac{i-1}{n}} \left(\frac{a_i}{a_n}\right)^{1-\frac{i-1}{n}}\right) \leq \left(f\left(\frac{a_{i-1}}{a_n}\right)\right)^{\frac{i-1}{n}} \left(f\left(\frac{a_i}{a_n}\right)\right)^{1-\frac{i-1}{n}}, \quad i = 2, \dots, n.$$

This together with (2.5) yields

$$f\left(\frac{a_i}{a_{n+1}}\right) \leq \left(f\left(\frac{a_{i-1}}{a_n}\right)\right)^{\frac{i-1}{n}} \left(f\left(\frac{a_i}{a_n}\right)\right)^{1-\frac{i-1}{n}}, \quad i = 2, \dots, n. \tag{2.8}$$

It is easy to see that

$$f\left(\frac{a_1}{a_{n+1}}\right) \leq f\left(\frac{a_1}{a_n}\right), \tag{2.9}$$

and

$$f\left(\frac{a_{n+1}}{a_{n+1}}\right) = f\left(\frac{a_n}{a_n}\right) = f(1). \tag{2.10}$$

From (2.8) – (2.10), we can deduce that

$$\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right) \leq \left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{n+1}{n}},$$

and hence, as desired, the inequality (2.1) also follows. The proof is complete.  $\square$

**THEOREM 2.2.** *Let  $f : (0, 1] \rightarrow [1, +\infty)$  be a real-valued function and  $\{a_n\}_{n=1}^\infty$  an increasing positive sequence such that the sequence  $\left\{\left(\frac{a_{n+1}}{a_n}\right)^{a_n}\right\}_{n=1}^\infty$  increases. Then the following statements are valid.*

(1) *If  $f$  is an increasing and multiplicatively convex (concave) function and  $\{a_n\}_{n=0}^\infty$  is convex sequence, i.e.,  $a_{n-1} + a_{n+1} \geq 2a_n (n = 1, 2, \dots)$ , where  $a_0 = 0$ , then the sequence*

$$\left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{1}{a_n}} \quad (n = 1, 2, \dots)$$

decreases with  $n$ . That is

$$\left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{1}{a_n}} \geq \left(\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right)\right)^{\frac{1}{a_{n+1}}}. \tag{2.11}$$

(2) If  $f$  is decreasing and multiplicatively convex (concave) and  $\{a_n\}_{n=0}^\infty$  is concave sequence, i.e.,  $a_{n-1} + a_{n+1} \leq 2a_n (n = 1, 2, \dots)$ , then the sequence

$$\left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{1}{a_n}} \quad (n = 1, 2, \dots)$$

is increasing with  $n$ .

*Proof.* As the proofs are similar, here we only give the proof of (1) and that of (2) is omitted.

Since the sequences  $\{a_n\}_{n=1}^\infty$  and  $\left\{\left(\frac{a_{n+1}}{a_n}\right)^{a_n}\right\}_{n=1}^\infty$  increase, then, for  $i = 1, \dots, n$ , we have

$$\left(\frac{a_{i+1}}{a_{n+1}}\right)^{\frac{a_i}{a_{n+1}}} \left(\frac{a_i}{a_{n+1}}\right)^{1-\frac{a_i}{a_{n+1}}} = \frac{a_i}{a_{n+1}} \left(\frac{a_{i+1}}{a_i}\right)^{\frac{a_i}{a_{n+1}}} \leq \frac{a_i}{a_{n+1}} \left(\frac{a_{n+1}}{a_n}\right)^{\frac{a_n}{a_{n+1}}} \leq \frac{a_i}{a_n}. \tag{2.12}$$

Similarly,

$$\left(\frac{a_{i-1}}{a_n}\right)^{\frac{a_{i-1}}{a_n}} \left(\frac{a_i}{a_n}\right)^{1-\frac{a_{i-1}}{a_n}} = \frac{a_i}{a_n} \left(\frac{a_{i-1}}{a_i}\right)^{\frac{a_{i-1}}{a_n}} \geq \frac{a_i}{a_{n+1}}, i = 2, \dots, n. \tag{2.13}$$

Since  $f$  is increasing, from (2.12) and (2.13), we obtain

$$f\left(\frac{a_i}{a_n}\right) \geq f\left(\left(\frac{a_{i+1}}{a_{n+1}}\right)^{\frac{a_i}{a_{n+1}}} \left(\frac{a_i}{a_{n+1}}\right)^{1-\frac{a_i}{a_{n+1}}}\right), i = 1, 2, \dots, n, \tag{2.14}$$

and

$$f\left(\frac{a_i}{a_{n+1}}\right) \leq f\left(\left(\frac{a_{i-1}}{a_n}\right)^{\frac{a_{i-1}}{a_n}} \left(\frac{a_i}{a_n}\right)^{1-\frac{a_{i-1}}{a_n}}\right), i = 2, \dots, n. \tag{2.15}$$

If  $f$  is multiplicatively concave, then

$$f\left(\left(\frac{a_{i+1}}{a_{n+1}}\right)^{\frac{a_i}{a_{n+1}}} \left(\frac{a_i}{a_{n+1}}\right)^{1-\frac{a_i}{a_{n+1}}}\right) \geq \left(f\left(\frac{a_{i+1}}{a_{n+1}}\right)\right)^{\frac{a_i}{a_{n+1}}} \left(f\left(\frac{a_i}{a_{n+1}}\right)\right)^{1-\frac{a_i}{a_{n+1}}}. \tag{2.16}$$

A combination of (2.14) with (2.16) leads to

$$f\left(\frac{a_i}{a_n}\right) \geq \left(f\left(\frac{a_{i+1}}{a_{n+1}}\right)\right)^{\frac{a_i}{a_{n+1}}} \left(f\left(\frac{a_i}{a_{n+1}}\right)\right)^{1-\frac{a_i}{a_{n+1}}}. \tag{2.17}$$

Multiplying the two sides of (2.17) from  $i = 1$  to  $n$ , respectively, we have

$$\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right) \geq \left(\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right)\right)^{1 - \frac{a_i}{a_{n+1}} + \frac{a_{i-1}}{a_{n+1}}} \tag{2.18}$$

Since  $\{a_n\}_{n=0}^\infty$  is convex sequence, it is easy to verify that

$$a_{n+1} + a_{i-1} - a_i \geq a_n, i = 1, 2, \dots, n. \tag{2.19}$$

Note that  $f(x) \geq 1$ , from (2.18) and (2.19), it follows that

$$\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right) \geq \left(\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right)\right)^{\frac{a_n}{a_{n+1}}}.$$

This shows that inequality (2.11) holds.

If  $f$  is multiplicatively convex, then

$$f\left(\left(\frac{a_{i-1}}{a_n}\right)^{\frac{a_{i-1}}{a_n}} \left(\frac{a_i}{a_n}\right)^{1 - \frac{a_{i-1}}{a_n}}\right) \leq \left(f\left(\frac{a_{i-1}}{a_n}\right)\right)^{\frac{a_{i-1}}{a_n}} \left(f\left(\frac{a_i}{a_n}\right)\right)^{1 - \frac{a_{i-1}}{a_n}}, i = 2, \dots, n.$$

This together with (2.15) yields

$$f\left(\frac{a_i}{a_{n+1}}\right) \leq \left(f\left(\frac{a_{i-1}}{a_n}\right)\right)^{\frac{a_{i-1}}{a_n}} \left(f\left(\frac{a_i}{a_n}\right)\right)^{1 - \frac{a_{i-1}}{a_n}}, i = 2, \dots, n. \tag{2.20}$$

Obviously, we have

$$f\left(\frac{a_1}{a_{n+1}}\right) \leq f\left(\frac{a_1}{a_n}\right), \tag{2.21}$$

and

$$f\left(\frac{a_{n+1}}{a_{n+1}}\right) = f\left(\frac{a_n}{a_n}\right) = f(1). \tag{2.22}$$

It follows from (2.20) – (2.22) that

$$\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right) \leq \left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{1 - \frac{a_{i-1}}{a_n} + \frac{a_i}{a_n}}. \tag{2.23}$$

Since  $f(x) \geq 1$ , from (2.19) and (2.23), we can deduce that

$$\prod_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right) \leq \left(\prod_{i=1}^n f\left(\frac{a_i}{a_n}\right)\right)^{\frac{a_{n+1}}{a_n}},$$

and hence, as desired, the inequality (2.11) also holds and so the proof is complete.  $\square$

From Lemma 2.3.4 of [11, p. 79], we can easily obtain: If a function  $f$  is increasing (decreasing) and multiplicatively convex (concave), then the function  $1/f$  is decreasing (increasing) and multiplicatively concave (convex). Therefore, it is easy to find that Theorem 2.2 is equivalent to the following result.

THEOREM 2.2' .. Let  $f : (0, 1] \rightarrow (0, 1]$  be a real-valued function and  $\{a_n\}_{n=1}^\infty$  an increasing positive sequence such that the sequence  $\left\{ \left( \frac{a_{n+1}}{a_n} \right)^{a_n} \right\}_{n=1}^\infty$  increases.

(1) If  $f$  is an increasing and multiplicatively convex (concave) function and  $\{a_n\}_{n=0}^\infty$  is concave sequence, where  $a_0 = 0$ , then the sequence

$$\left( \prod_{i=1}^n f \left( \frac{a_i}{a_n} \right) \right)^{\frac{1}{a_n}} \quad (n = 1, 2, \dots)$$

decreases with  $n$ ;

(2) If  $f$  is decreasing and multiplicatively convex (concave) and  $\{a_n\}_{n=0}^\infty$  is convex sequence, then the sequence

$$\left( \prod_{i=1}^n f \left( \frac{a_i}{a_n} \right) \right)^{\frac{1}{a_n}} \quad (n = 1, 2, \dots)$$

is increasing with  $n$ .

COROLLARY 2.3. ([8], Theorem 8) If the sequences

$$a_n \quad \text{and} \quad \left( \frac{a_{n+1}}{a_n} \right)^n,$$

both increase (respectively decrease) with  $n$ , then the sequence

$$\frac{\left( \frac{1}{n} \sum_{i=1}^n a_i^r \right)^{\frac{1}{r}}}{a_n} \quad (r \neq 0) \tag{2.24}$$

decreases (respectively increases) with  $n$ .

*Proof.* (Increasing case). Differentiating the function  $f(x) = \exp(x^r)$  ( $x > 0$ ), we obtain

$$f'(x) = rf(x)x^{r-1}, \quad f''(x) = f(x)(r(r-1)x^{r-2} + r^2x^{2r-2}).$$

Thus,

$$x[f(x)f''(x) - (f'(x))^2] + f(x)f'(x) = (rf(x))^2x^{r-1} > 0.$$

By Theorem 1.2, it is easy to see that  $f$  is increasing and multiplicatively convex for  $r > 0$ , and decreasing and multiplicatively convex for  $r < 0$ .

Taking  $f(x) = \exp(x^r)$  ( $x > 0$ ) in Theorem 2.1 and noticing that

$$\left( \prod_{i=1}^n f \left( \frac{a_i}{a_n} \right) \right)^{\frac{1}{n}} = \exp \left( \frac{\sum_{i=1}^n a_i^r}{na_n^r} \right),$$

we can deduce that the sequence

$$\frac{\sum_{i=1}^n a_i^r}{na_n^r} \quad (n = 1, 2, \dots)$$



decreases for  $r > 0$ , and increases for  $r < 0$ , which shows that the sequence

$$\frac{(\frac{1}{n} \sum_{i=1}^n a_i^r)^{\frac{1}{r}}}{a_n} \quad (n = 1, 2, \dots)$$

decreases with  $n$ .

(Decreasing case). We apply the above version (increasing case) with the sequence  $\{a_n\}_{n=1}^{\infty}$  replaced by the sequence  $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ .  $\square$

REMARK 2.4. If taking  $a_n = n + k$  ( $n = 1, 2, \dots$ ) in (2.24), where  $k$  is a nonnegative integer, one can easily see that inequality (1.2) holds for any given real number  $r (\neq 0)$ .

If taking  $f(x) = \exp(x^r)$  ( $0 < x \leq 1$ ) in Theorem 2.2, we can establish

COROLLARY 2.5. *Let  $\{a_n\}_{n=1}^{\infty}$  be an increasing positive sequence such that the sequence  $\left\{ \left( \frac{a_{n+1}}{a_n} \right)^{a_n} \right\}_{n=1}^{\infty}$  increases. Then*

(1) *If  $\{a_n\}_{n=0}^{\infty}$  is a convex sequence, where  $a_0 = 0$ , then, for any  $r > 0$ , the sequence*

$$\frac{a_1^r + a_2^r + \dots + a_n^r}{a_n^{r+1}} \quad (n = 1, 2, \dots) \tag{2.25}$$

*decreases with  $n$ ;*

(2) *If  $\{a_n\}_{n=0}^{\infty}$  is a concave sequence, then, for any  $r < 0$ , the sequence*

$$\frac{a_1^r + a_2^r + \dots + a_n^r}{a_n^{r+1}} \quad (n = 1, 2, \dots) \tag{2.26}$$

*is increasing with  $n$ .*

REMARK 2.6. It is easy to see that the sequence  $\{n\}_{n=0}^{\infty}$  satisfies all conditions of Corollary 2.5. Taking  $a_n = n$ , we see that Corollary 2.5 contains Theorem 10' of [8].

Ume[12] considered yet another variant on Alzer's inequality, but his results are too complicated to be described in details here. He showed that the sequence

$$\frac{\sum_{i=k+1}^{k+n} i^{rp}}{n^p(k+n)^{rp}} \quad (n = 1, 2, \dots)$$

decreases with  $n$ , where  $k$  is a nonnegative integer,  $r > 0$ , and  $p = 1$  or  $p \geq 2$ . And he asked whether this assertion continues to hold for all  $1 < p < 2$ . We shall give positive answer.

COROLLARY 2.7. *Let  $n$  be a natural number and  $k$  a nonnegative integer. Then*

(1) *When  $p \geq 1$  and  $r > 0$ , the sequence*

$$\frac{\sum_{i=k+1}^{k+n} i^{rp}}{n^p(k+n)^{rp}} \quad (n = 1, 2, \dots)$$

decreases with  $n$ ;

(2) When  $0 < p \leq 1$  and  $r < 0$ , the sequence

$$\frac{\sum_{i=k+1}^{k+n} i^r p}{(k+n)^{p(r+1)}} \quad (n = 1, 2, \dots)$$

increases with  $n$ .

*Proof.* It is easy to see that the sequence  $(0, (k+1)^p, (k+2)^p, \dots)$  is convex for  $p \geq 1$ , and concave for  $0 < p \leq 1$ . Since the sequence  $\{\frac{(n+l)^p}{n^p}\}_{n=1}^{\infty}$  ( $p > 0$ ) is decreasing, taking  $a_n = n+k$  ( $n = 1, 2, \dots$ ) in Corollary 2.5, we have completed the proof.  $\square$

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