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EXPONENTIAL CONVEXITY, POSITIVE SEMI–DEFINITE MATRICES AND FUNDAMENTAL INEQUALITIES

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Abstract. The first two chapters of the classical book [6] on inequalities are devoted to fundamental inequalities and positive definiteness. In this paper we obtain results which give connection between fundamental inequalities and positive definiteness using the notion of exponential convexity.

1. Introduction and preliminaries

A real symmetric matrix $\mathbf{A}$ is positive semi-definite, $A \geqslant 0$, if

$$\mathbf{b}A\mathbf{b}^t \geqslant 0 \tag{1.1}$$

for all row vectors $\mathbf{b}$. This definition may seem abstruse, but positive semi-definite matrices have a number of interesting properties. One of these is that all the eigenvalues of a positive semi-definite matrix are real and nonnegative.

As was noted in [6, p. 59–61] a very important positive semi-definite matrix is Grami matrix. The corresponding determinantal inequality is well known as Gram's inequality: Let $k \in \mathbb{N}$ and $x_1, \ldots, x_k$ be vectors in some Euclidean space and $\langle x_i, x_j \rangle$ denotes the inner product of two vectors x_i and x_j. Then Gram's inequality is

$$\det\left[\langle x_i, x_j \rangle\right]_{i,j=1}^{k} \geqslant 0 \tag{1.2}$$

(see [10]).

Gram's inequality can be seen in many books. For instance [21, p. 595–609], [17, p. 345–357] and [9, p. 176–187].

Gram's inequality for positive linear functionals is as follows. Let E be a nonempty set and L be a linear class of real valued functions $f : E \longrightarrow \mathbb{R}$ having the properties:

$$f, g \in L \Rightarrow af + bg \in L, \text{ forall } a, b \in \mathbb{R}, \tag{L1}$$

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$$\text{if } 1 \in L \Leftrightarrow f(t) = 1 \text{ for } t \in E, \text{ then } f \in L. \tag{L2}$$

A positive linear functional is a mapping $A : L \longrightarrow \mathbb{R}$ with properties

$$A(af + bg) = aA(f) + bA(g) \text{ for } f,g \in L,\ a,b \in \mathbb{R}, \tag{A1}$$

$$f \in L,\ f(t) \geqslant 0 \text{ on } E \Rightarrow A(f) \geqslant 0 \text{ (A is positive)}. \tag{A2}$$

If $A(1) = 1$ we say that A is normalized functional.

THEOREM 1. *Let A be a positive linear functional and $f_1, f_2, \ldots, f_n$ be real functions such that $f_i f_j \in L$ for all $i, j = 1, 2, \ldots, n$. Then the following inequality is valid*

$$[A(f_i f_j)]_{i,j=1}^{k} \geqslant 0, \tag{1.3}$$

for all $k \in \{1, \ldots, n\}$.

Proof. Suppose that $(u_1, \ldots, u_k) \in \mathbb{R}^k$, then we have

$$\sum_{i,j=1}^{k} u_i u_j f_i(x) f_j(x) = \left(\sum_{i=1}^{k} u_i f_i(x) \right)^2 \geqslant 0.$$

Then

$$A\left(\sum_{i,j=1}^{k} u_i u_j f_i f_j \right) = \sum_{i,j=1}^{k} u_i u_j A(f_i f_j) \geqslant 0. \tag{1.4}$$

From the last expression we have that $[A(f_i f_j)]_{i,j=1}^{k} \geqslant 0$ and (1.3) is valid. □

In this paper we show that we can use a lot of fundamental inequalities to obtain positive semi-definite matrices, that is we can give determinantal form of some fundamental inequalities. Very specific form of these determinantal forms enable us to interpret our results in a form of exponentially convex functions. This is a sub-class of convex functions introduced by Bernstein in [7] (see also [1], [20], [21]).

DEFINITION 1. A function $h : (a,b) \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \geqslant 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $x_i + x_j \in (a,b)$, $1 \leqslant i, j \leqslant n$.

PROPOSITION 1. *Let $h : (a,b) \to \mathbb{R}$. The following propositions are equivalent.*

(i) h is exponentially convex.

(ii) h is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2} \right) \geqslant 0,$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and every $x_i \in (a,b)$, $1 \leqslant i \leqslant n$.

COROLLARY 1. *If h is exponentially convex, then*

$$\det\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geqslant 0,$$

for all $n\in\mathbb{N}$, and all $x_i\in(a,b)$, $i=1,\ldots,n$.

COROLLARY 2. *If $h:(a,b)\to(0,\infty)$ is exponentially convex function, then h is a log-convex function:*

$$h\left(\frac{x+y}{2}\right)\leqslant\sqrt{h(x)h(y)},\ \textit{for all } x,y\in(a,b).$$

REMARK 1. In Definition 1 and Proposition 1 it could have been required properties of mesurability and finiteness almost everywhere instead of continuity because of the following theorem (see [12], p. 105, and [28]):

If a function $h:(a,b)\to\mathbb{R}\cup\{+\infty\}$ is measurable and finite almost everywhere and if

$$h\left(\frac{x+y}{2}\right)\leqslant\frac{h(x)+h(y)}{2}\quad(a<x,y<b),$$

then h is continuous function.

Finiteness almost everywhere is very mild condition for our applications, and all results in following sections can be restated under that assumption.

REMARK 2. In the next sections we cover results about log-convexity from [2, 3, 4, 5, 14, 15, 26, 27] and then extending them to exponential convexity.

2. Jensen's inequality

Jessen (see [22], p–47) gave the following generalization of Jensen's inequality for functionals.

THEOREM 2. *Let L satisfy $L1$, $L2$ on a nonempty set E, and assume that ϕ is a continuous convex function on an interval $I\subset\mathbb{R}$. If A is a linear positive functional with $A(1)=1$, then for all $f\in L$ such that $\phi(f)\in L$ we have $A(f)\in I$ and*

$$\phi(A(f))\leqslant A(\phi(f)). \tag{2.1}$$

LEMMA 1. [2] *Let us define the following family of functions*

$$\varphi_t(x)=\begin{cases}\frac{x^t}{t(t-1)}, & t\neq 0,1;\\ -\log x,\ t=0; & \\ x\log x, & t=1.\end{cases} \tag{2.2}$$

Then $\frac{d^2}{dx^2}\varphi_t(x)=x^{t-2}$, that is $x\mapsto\varphi_t(x)$ is convex on $(0,\infty)$ for every $t\in\mathbb{R}$.

THEOREM 3. *Let L satisfy properties $L1$, $L2$ on a nonempty set E. Let a positive function $f \in L$ be such that $f^r \in L$ for $r \in I \setminus \{0,1\}$, I is an interval from $\mathbb{R}$, let $\log f \in L$ if $r = 0 \in I$ and $f \log f \in L$ if $r = 1$. Let us define*

$$\Lambda_t = A(\varphi_t(f)) - \varphi_t(A(f)). \tag{2.3}$$

Then

(i) for every $n \in \mathbb{N}$ and for every $p_k \in I$, $k = 1,2,...,n$, the matrix $\left[\Lambda_{\frac{p_i+p_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite matrix. Particularly

$$\det\left[\Lambda_{\frac{p_i+p_j}{2}}\right]_{i,j=1}^n \geqslant 0; \tag{2.4}$$

(ii) if the function $t \mapsto \Lambda_t$ is continuous on I, then it is exponentially convex on I.

Proof. Consider the function

$$f(x) = \sum_{i,j}^n u_i u_j \varphi_{p_{ij}}(x)$$

for $x > 0$, $u_i \in \mathbb{R}$ and $p_{ij} \in I$ where $p_{ij} = \frac{p_i+p_j}{2}$. Then

$$f''(x) = \sum_{i,j}^n u_i u_j x^{p_{ij}-2} = \left(\sum_i^n u_i x^{\frac{p_i}{2}-1}\right)^2 \geqslant 0 \quad \text{for} \quad x > 0.$$

So f is a convex function. Therefore by applying (2.1) we get

$$\sum_{i,j=1}^n u_i u_j \Lambda_{p_{ij}} \geqslant 0,$$

concluding positive semi-definiteness. Assuming continuity and using Proposition 1 we have also exponential convexity of the function $t \mapsto \Lambda_t$. □

Let us note that the well known Jensen-Steffensen inequality is valid (see, for example [22], pp. 57–58).

THEOREM 4. *If $f : I \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1,...,x_m)$ is a real monotonic m-tuple such that $x_i \in I$ $(i = 1,...,m)$, and $p = (p_1,...,p_m)$ is a real n-tuple such that*

$$0 \leqslant P_k \leqslant P_m = 1 \ (k = 1,..,m), \text{ where } P_k = \sum_{i=1}^k p_i \tag{2.5}$$

is satisfied. Then

$$f\left(\sum_{i=1}^m p_i x_i\right) \leqslant \sum_{i=1}^m p_i f(x_i). \tag{2.6}$$

As in the proof of Theorem 3 we can get:

THEOREM 5. *Let* $(x_1,...,x_m)$ *be a monotonic m-tuple of positive numbers,* $p_i \in \mathbb{R}$ *such that conditions of Theorem* 4 *are valid and define the function*

$$\lambda_t = \sum_{i=1}^{n} p_i \varphi_t(x_i) - \varphi_t\left(\sum_{i=1}^{n} p_i x_i\right).$$

Then

(i) for every $n \in \mathbb{N}$ *and for every* $t_k \in \mathbb{R}$, $k = 1,2,...,n$, *the matrix* $\left[\lambda_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\lambda_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0; \tag{2.7}$$

(ii) the function $t \mapsto \lambda_t$ *is exponentially convex.*

Proof. It is easy to check continuity of the function $t \mapsto \lambda_t$. Now we complete the proof similarly as in Theorem 3. □

Moreover, we can also use related integral analogues of Jensen-Steffensen inequality and generalizations (see Jensen-Steffensen's, Jensen-Boas and Jensen-Brunk inequalities as well as Theorem 2.26 from [22], pp. 59–65).

LEMMA 2. [2] *Let us define the following family of functions*

$$\phi_t(x) = \begin{cases} \frac{1}{t^2}e^{tx}, & t \neq 0; \\ \frac{1}{2}x^2, & t = 0. \end{cases}$$

Then $\frac{d^2}{dx^2}\phi_t(x) = e^{tx}$, *that is* $x \mapsto \phi_t(x)$ *is a convex function on* $\mathbb{R}$ *for every* $t \in \mathbb{R}$.

THEOREM 6. *Theorems* 3 *and* 5 *are still valid if we set* $\varphi_t = \phi_t$.

Proof. Similar to the proof of Theorem 3. □

3. Some results of Aczél's type

The following version of Jensen's inequality is valid ([22], p. 124–125).

THEOREM 7. *Let* L *satisfy properties* $L1$, $L2$ *on a nonempty set* E, *and* A *satisfy conditions* $A1$, $A2$. *Let* I *be an interval,* $I \subset \mathbb{R}$, f *and* w *two arbitrary real functions defined on* E *such that* w, $wf \in L$. *Suppose* $0 < A(w) < u \in \mathbb{R}$, $\frac{(ua-A(wf))}{u-A(w)} \in I, a \in I$ *and* ψ *is a continuous convex function on* I *and* $w\psi(f) \in L$. *Then*

$$\psi\left(\frac{ua - A(wf)}{u - A(w)}\right) \geqslant \frac{u\psi(a) - A(w\psi(f))}{u - A(w)}. \tag{3.1}$$

Similarly to the proof of Theorem 3 we can prove the following theorem (see [2] for preliminary result on log-convexity)

THEOREM 8. *Let us suppose that the conditions of Theorem 7 are satisfied for an interval* $I \subseteq (0,+\infty)$ *and for function* $\psi = \varphi_t$ $(\varphi_t$ *is defined by* (2.2)$)$ *for every* $t \in \mathbb{R}$. *Let us define*

$$\Omega_t = \varphi_t\left(\frac{ua - A(wf)}{u - A(w)}\right) - \frac{u\varphi_t(a) - A(w\varphi_t(f))}{u - A(w)}. \tag{3.2}$$

Then

(i) *for every* $n \in \mathbb{N}$ *and for every* $p_k \in J$, $k = 1,2,...,n$, *the matrix* $\left[\Omega_{\frac{p_i+p_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\Omega_{\frac{p_i+p_j}{2}}\right]_{i,j=1}^{n} \geqslant 0; \tag{3.3}$$

(ii) *if the function* $t \mapsto \Omega_t$ *is continuous, then it is exponentially convex on* $\mathbb{R}$.

Let us note that the following converse of Jensen-Steffensen's inequality was given by J. E. Pečarić ([23], see also [22], pp. 83–84).

THEOREM 9. *Let* $I \subseteq \mathbb{R}$ *be some interval, let* $\mathbf{x} = (x_1, x_2, ..., x_m)$ *and* $(p_1, p_2, ..., p_m)$ *be the real m-tuples such that* $x_i \in I$ $(1 \leqslant i \leqslant m)$, $P_m = 1$, $\sum_{i=1}^{m} p_i x_i \in I$, $\mathbf{x}$ *is monotonic, and there exists an* $l \in \{1,2,...,n\}$ *such that*

$$P_k \leqslant 0 \ (k < l), \ \ 1 \leqslant P_{k-1} \ \ (k > l). \tag{3.4}$$

If $f : I \to \mathbb{R}$ *is a convex function, then*

$$f\left(\sum_{i=1}^{m} p_i x_i\right) \geqslant \sum_{i=1}^{m} p_i f(x_i). \tag{3.5}$$

We can use Theorem 9 in similar way for the proof of the following result.

THEOREM 10. *Let the conditions of Theorem 9 be satisfied for an interval I for function* $\varphi = \varphi_t$ $($*as is defined by* (2.2)$)$ *for t in some interval* $J \subseteq \mathbb{R}$. *Let us define*

$$\tilde{\lambda}_t = \varphi_t\Big(\sum_{i=1}^{m} p_i x_i\Big) - \sum_{i=1}^{m} p_i \varphi_t(x_i) \tag{3.6}$$

Then

(i) *for every* $n \in \mathbb{N}$ *and for every* $t_k \in \mathbb{R}$, $k = 1,2,...,n$, *the matrix* $\left[\tilde{\lambda}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\tilde{\lambda}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0; \tag{3.7}$$

(ii) *the function* $t \mapsto \tilde{\lambda}_t$ *is exponentially convex.*

REMARK 3. We can also use related integral analogues of Jensen-Steffensen inequality(see for example [22], pp. 84–87).

4. Some results of Mercer's type

The following version of Jessen's inequality is valid (see [8]).

THEOREM 11. *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a convex function on an interval $I = [m,M]$ $(-\infty < m < M < \infty)$. If A is a positive linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g), \varphi(m+M-g) \in L$ (so that $m \leqslant g(t) \leqslant M$ for all $t \in E$), we have the following variant of Jessen'n inequality*

$$\varphi(m+M-A(g)) \leqslant \varphi(m)+\varphi(M)-A(\varphi(g)). \tag{4.1}$$

As previously we can prove the following two theorems.

THEOREM 12. *Let the conditions of Theorem 11 be satisfied for an interval $I = [m,M] \subseteq (0,\infty)$ and for function $\varphi = \varphi_t$ (as is defined by (2.2)) for t in some interval $J \subseteq \mathbb{R}$. Let us define*

$$\tilde{\Omega}_t = \varphi_t(m)+\varphi_t(M)-A(\varphi_t(g))-\varphi_t(m+M-A(g)).$$

Then

(i) for every $n \in \mathbb{N}$ and for every $t_k \in J$, $k = 1,2,\ldots,n$, the matrix $\left[\tilde{\Omega}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite matrix. Particularly

$$\det\left[\tilde{\Omega}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^n \geqslant 0; \tag{4.2}$$

(ii) if the function $t \mapsto \tilde{\Omega}_t$ is continuous, then it is exponentially convex on J.

THEOREM 13. *Let the conditions of Theorem 11 be satisfied for an interval $I = [m,M]$ for function $\varphi = \phi_t$ (as is defined Lemma 2) for s in some interval in $J \subseteq \mathbb{R}$. Let us define*

$$\hat{\Omega}_t = \phi_t(m)+\phi_t(M)-A(\phi_t(g))-\phi_t(m+M-A(g)).$$

Then

(i) for every $n \in \mathbb{N}$ and for every $t_k \in J$, $k = 1,2,\ldots,n$, the matrix $\left[\hat{\Omega}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite matrix. Particularly

$$\det\left[\hat{\Omega}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^n \geqslant 0; \tag{4.3}$$

(ii) if the function $t \mapsto \hat{\Omega}_t$ is continuous, then it is exponentially convex on J.

5. Levinson's inequality

It is well known that Ky-Fan's inequality can be obtained from the Levinson inequality (see [18]; see also [22], p. 71).

THEOREM 14. *Let $a>0$ be any real number and let f be a real valued $3-$convex function on $[0,2a]$. Then for $0<x_i<a,\ p_i>0,\ i=1,\dots,m$ we have*

$$\frac{1}{P_m}\sum_{i=1}^{m}p_if(x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^{m}p_ix_i\right)\leqslant\frac{1}{P_m}\sum_{i=1}^{m}p_if(2a-x_i)-f\left(\frac{1}{P_m}\sum_{i=1}^{m}p_i(2a-x_i)\right). \tag{5.1}$$

LEMMA 3. [3] *Let us define the following family of functions*

$$\varphi_t(x)=\begin{cases}\frac{x^t}{t(t-1)(t-2)}, & t\neq 0,1,2;\\ \frac{1}{2}\log x, & t=0;\\ -x\log x, & t=1;\\ \frac{1}{2}x^2\log x, & t=2.\end{cases} \tag{5.2}$$

Then $\frac{d^3}{dx^3}\varphi_t(x)=x^{t-3}$, that is $x\mapsto\varphi_t(x)$ is $3-$convex on $(0,\infty)$ for every $t\in\mathbb{R}$.

THEOREM 15. *Define the function*

$$\xi_t=\frac{1}{P_m}\sum_{i=1}^{m}p_i\Big(\varphi_t(2a-x_i)-\varphi_t(x_i)\Big)-\varphi_t(2a-\overline{x})+\varphi_t(\overline{x}), \tag{5.3}$$

for $m-$tuples $(x_1,\dots,x_m)$ and $(p_1,\dots,p_m)$ as in Theorem 14 and for the function $f=\varphi_t$ (as is defined by (5.2)). Then

(i) for every $n\in\mathbb{N}$ and for every $t_k\in\mathbb{R}$, $k=1,2,\dots,n$, the matrix $\left[\xi_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly

$$\det\left[\xi_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n}\geqslant 0; \tag{5.4}$$

(ii) the function $t\mapsto\xi_t$ is exponentially convex on $\mathbb{R}$.

Proof. Consider the function

$$f(x)=\sum_{i,j}^{n}u_iu_j\xi_{t_{ij}}(x)$$

for $x>0$, $u_i\in\mathbb{R}$ and $t_{ij}\in I$ where $t_{ij}=\frac{t_i+t_j}{2}$. Then

$$f'''(x)=\sum_{i,j}^{n}u_iu_jx^{t_{ij}-3}=\left(\sum_{i}^{n}u_ix^{\frac{t_i}{2}-\frac{3}{2}}\right)^2\geqslant 0\quad\text{for}\quad x>0.$$

So $f(x)$ is a 3-convex function. Therefore by applying inequality (5.1) we get

$$\sum_{i,j=1}^{n} u_i u_j \xi_{t_{ij}} \geqslant 0.$$

It can be easily checked that the function $t \mapsto \xi_t$ is continuous function concluding that it is exponentially convex function. □

In [25] the third author proved the following result.

THEOREM 16. *Let $a > 0$ be any real number and let f be a real valued $3-$convex function on $[0,2a]$ and $x_i \in [0,2a]$ $(1 \leqslant i \leqslant m)$. Then*

$$\frac{1}{P_m}\sum_{i=1}^{m} p_i f(x_i) - f\Big(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\Big) \leqslant \frac{1}{P_m}\sum_{i=1}^{m} p_i f(a+x_i) - f\Big(\frac{1}{P_m}\sum_{i=1}^{m} p_i (a+x_i)\Big). \tag{5.5}$$

THEOREM 17. *Let $a > 0$ be any real number. Define the function*

$$\rho_s = \frac{1}{P_m}\sum_{i=1}^{m} p_i \Big(\varphi_s(a+x_i) - \varphi_s(x_i)\Big) - \varphi_s(a+\overline{x}) + \varphi_s(\overline{x}), \tag{5.6}$$

for $m-$tuples $(x_1,\ldots,x_m)$ and $(p_1,\ldots,p_m)$ as in Theorem 16 and for the function $f = \varphi_t$ (as is defined by (5.2)). Then

(i) for every $n \in \mathbb{N}$ and for every $t_k \in \mathbb{R}$, $k = 1,2,...,n$, the matrix $\left[\rho_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly

$$\det\left[\rho_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0; \tag{5.7}$$

(ii) the function $t \mapsto \rho_t$ is exponentially convex on $\mathbb{R}$.

Proof. The proof is quite similar to the proof of Theorem 15. □

6. Power sums

Let $(x_1,...,x_m)$ be a positive m-tuple. The well-known inequality for power sums of order s and r, for $s > r > 0$ ([22], p–164), states that

$$\left(\sum_{i=1}^{m} x_i^s\right)^{1/s} < \left(\sum_{i=1}^{m} x_i^r\right)^{1/r}. \tag{6.1}$$

Moreover, if $(p_1,...,p_m)$ is a positive m-tuple such that $p_i \geqslant 1$ $(i=1,...,m)$, then for $s > r > 0$ ([22], p–165), we have

$$\left(\sum_{i=1}^{m} p_i x_i^s\right)^{1/s} < \left(\sum_{i=1}^{m} p_i x_i^r\right)^{1/r}. \tag{6.2}$$

Let us note that (6.2) can also be obtained from the following theorem ([22], p–152):

THEOREM 18. *Let $(x_1,...,x_m)$ and $(p_1,...,p_m)$ be two non-negative m-tuples such that $x_i \in (0,a]$ $(i=1,...,m)$ and*

$$\textstyle\sum_{i=1}^{m} p_i x_i \geqslant x_j, \;\; for \;\; j=1,...,m \;\; and \;\; \sum_{i=1}^{m} p_i x_i \in (0,a]. \tag{6.3}$$

If $f(x)/x$ is an increasing function on $(0,a]$, then

$$f\left(\sum_{i=1}^{m} p_i x_i\right) \geqslant \sum_{i=1}^{m} p_i f(x_i). \tag{6.4}$$

LEMMA 4. [26] *Let us define the following family of functions*

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t-1}, & t \neq 1; \\ x \log x, & t = 1. \end{cases}$$

Then $x \mapsto \frac{\varphi_t(x)}{x}$ is a strictly increasing function on $(0,\infty)$ for every $t \in \mathbb{R}$.

THEOREM 19. *Define the function*

$$\Delta_t = \Delta_t(\mathbf{x};\mathbf{p}) = \varphi_t\left(\sum_{i=1}^{m} p_i x_i\right) - \sum_{i=1}^{m} p_i \varphi_t(x_i) \tag{6.5}$$

for $m-$tuples $(x_1,\ldots,x_m)$ and $(p_1,\ldots,p_m)$ $(n \geqslant 2)$ as in Theorem 18 and for function $f = \varphi_t$ (as is defined in Lemma 4). Then

(i) for every $n \in \mathbb{N}$ and for every $t_k \in \mathbb{R}$, $k = 1,2,...,n$, the matrix $\left[\Delta_{\frac{t_i+t_j}{2}}\right]$ is a positive semi-definite matrix. Particularly

$$\det\left[\Delta_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0;$$

(ii) the function $t \mapsto \Delta_t$ is exponentially convex on $\mathbb{R}$.

Proof. (i) Consider the function

$$f(x) = \sum_{i,j}^{n} u_i u_j \varphi_{t_{ij}}(x)$$

for $x > 0,\ u_i \in \mathbb{R}$ and $t_{ij} \in \mathbb{R}$ where $t_{ij} = \frac{t_i+t_j}{2}$. Then

$$\left(\frac{f(x)}{x}\right)' = \sum_{i,j}^{n} u_i u_j x^{t_{ij}-2} = \left(\sum_{i}^{n} u_i x^{\frac{t_i}{2}-1}\right)^2 \geqslant 0.$$

So $\frac{f(x)}{x}$ is an increasing function. After we apply inequality (6.4) to function f we get

$$\sum_{i,j}^{n} u_i u_j \Delta_{t_{ij}} \geqslant 0.$$

(ii) Since $\lim_{t\to 1} \Delta_t = \Delta_1,\ t \mapsto \Delta_t$ is continuous function, concluding its exponential convexity on $\mathbb{R}$. □

The following similar result is also valid ([22], p–153):

THEOREM 20. *Let $f(x)/x$ be an increasing function on $(0,+\infty)$. If $0 < x_1 \leqslant \ldots \leqslant x_m$ and*

(i) if there exist an $l(\leqslant n)$ such that

$$\bar{P}_1 \geqslant \bar{P}_2 \geqslant \ldots \geqslant \bar{P}_l \geqslant 1,\ \ \bar{P}_{l+1} = \ldots = \bar{P}_m = 0, \tag{6.6}$$

where $P_k = \sum_{i=1}^{k} p_i$, $\bar{P}_k = P_m - P_{k-1}$ $(k = 2,\ldots,m)$ and $\bar{P}_1 = P_m$, then (6.4) holds.

(ii) If there exists an $l(\leqslant n)$ such that

$$0 \leqslant \bar{P}_1 \leqslant \bar{P}_2 \leqslant \ldots \leqslant \bar{P}_l \leqslant 1,\ \ \bar{P}_{l+1} = \ldots = \bar{P}_m = 0, \tag{6.7}$$

then the reverse of inequality in (6.4) holds.

Now make two applications of Theorem 20 :

THEOREM 21. *Define the function*

$$\phi_t = \varphi_t\left(\sum_{i=1}^{m} p_i x_i\right) - \sum_{i=1}^{m} p_i \varphi_t(x_i)$$

for $n-$tuples $(x_1,\ldots,x_m)$ and $(p_1,\ldots,p_m)$ $(m \geqslant 2)$ as in (i)-part of Theorem 20 and for function $f = \varphi_t$ (as is defined in Lemma 4). Then

(i) For every $n \in \mathbb{N}$ and for every $t_k \in \mathbb{R},\ k = 1,2,\ldots,n,$ the matrix $\left[\phi_{\frac{t_i+t_j}{2}}\right]$ is a positive semi-definite matrix. Particularly

$$\det\left[\phi_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0; \tag{6.8}$$

(ii) the function $t \mapsto \phi_t$ *is exponentially convex on* $\mathbb{R}$.

THEOREM 22. *Define the function*

$$\bar{\phi}_t = \sum_{i=1}^{m} p_i \varphi_t(x_i) - \varphi_t\left(\sum_{i=1}^{m} p_i x_i\right)$$

for $m-$*tuples* $(x_1,\ldots,x_m)$ *and* $(p_1,\ldots,p_m)$ $(n \geqslant 2)$ *as in (ii)-part of Theorem* 20 *and for function* $f = \varphi_t$ *(as is defined in Lemma* 4*). Then*

(i) For every $n \in \mathbb{N}$ *and for every* $t_k \in \mathbb{R}$, $k = 1,2,...,n$, *the matrix* $\left[\bar{\phi}_{\frac{t_i+t_j}{2}}\right]$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\bar{\phi}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0. \tag{6.9}$$

(ii) The function $t \mapsto \bar{\phi}_t$ *is exponentially convex on* $\mathbb{R}$.

Let us note that power sums inequalities can be also obtained by using related inequalities for convex functions.

LEMMA 5. [27] *Let us define the following family of functions*

$$\tau_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 1; \\ x \log x, & t = 1. \end{cases}$$

where $t \in \mathbb{R}^+$ *and* $x \geqslant 0$. *Then* $x \mapsto \tau_t(x)$ *is convex function on* $[0,\infty)$ *for every* $t \in \mathbb{R}^+$.

THEOREM 23. *Let* $(x_1,\ldots,x_m)$ *and* $(p_1,\ldots,p_m)$ *be two positive* m*-tuples such that condition* (6.3) *is satisfied. Let*

$$\overline{\Delta}_t = \overline{\Delta}_t(\mathbf{x};\mathbf{p}) = \frac{\Delta_t}{t}$$

be function defined on $(0,\infty)$ *where the function* $t \mapsto \Delta_t$ *is defined in* (6.5)*. Then*

(i) For every $n \in \mathbb{N}$ *and for every* $t_k \in (0,\infty)$, $k = 1,2,...,n$, *the matrix* $\left[\overline{\Delta}_{\frac{t_i+t_j}{2}}\right]$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\overline{\Delta}_{\frac{t_i+t_j}{2}}\right]_{i,j}^{n} \geqslant 0.$$

(ii) The function $t \mapsto \overline{\Delta}_t$ *is exponentially convex on* $(0,\infty)$.

Proof. (i) For $x>0$, $u_i\in\mathbb{R}$ and $t_i\in(0,\infty)$, $i=1,\dots,n$ let us consider the function

$$f(x)=\sum_{i,j}^{n}u_iu_j\tau_{t_{ij}}(x)$$

where $t_{ij}=\frac{t_i+t_j}{2}$ and τ is defined as in Lemma 5. Then

$$f''(x)=\sum_{i,j}^{n}u_iu_jx^{t_{ij}-2}=\left(\sum_{i}^{n}u_ix^{\frac{t_i}{2}-1}\right)^2\geqslant 0.$$

So f is convex function. After we apply Petrović inequality (see [22], p. 154)

$$f\left(\sum_{i=1}^{m}p_ix_i\right)\leqslant\sum_{i=1}^{m}p_if(x_i)+\left(1-\sum_{i=1}^{m}p_i\right)f(0)$$

we get

$$\sum_{i,j}^{n}u_iu_j\overline{\Delta}_{t_{ij}}\geqslant 0.$$

(ii) Since $\lim\limits_{t\to 1}\overline{\Delta}_t=\Delta_1$, $t\mapsto\overline{\Delta}_t$ is continuous function on $(0,\infty)$, concluding its exponential convexity on $(0,\infty)$. □

COROLLARY 3. *Let $(x_1,\dots,x_m)$ and $(p_1,\dots,p_m)$ be two positive m-tuples such that condition* (6.3) *is satisfied, and let $s\in(0,\infty)$. Let*

$$\Theta_t^s=\begin{cases}\frac{1}{t(t-s)}\left\{\left(\sum_{i=1}^n p_ix_i^s\right)^{\frac{t}{s}}-\sum_{i=1}^n p_ix_i^t\right\}, & t\neq s;\\ \frac{1}{s^2}\left\{\left(\sum_{i=1}^n p_ix_i^s\right)\log\left(\sum_{i=1}^n p_ix_i^s\right)-s\sum_{i=1}^n p_ix_i^s\log x_i\right\}, & t=s.\end{cases}$$

be function defined on $(0,\infty)$. Then

(i) for every $n\in\mathbb{N}$ and for every $t_k\in(0,\infty)$, $k=1,2,...,n$, the matrix $\left[\Theta^s_{\frac{t_i+t_j}{2}}\right]$ is a positive semi-definite matrix. Particularly

$$\det\left[\Theta^s_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n}\geqslant 0. \tag{6.10}$$

(ii) the function $t\mapsto\Theta_t^s$ is exponentially convex on $(0,\infty)$.

Proof. The proof follows from Theorem 23 after substitutions $x_i\to x_i^s$, $t\to t/s$. □

7. Results on integral inequalities

Let us consider the following result from [22, page 159].

THEOREM 24. *Let $t_0 \in [a,b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation and*

$$G(t) := \int_a^t dg(x), \ \ \overline{G}(t) := \int_t^b dg(x).$$

(a) If

$$0 \leqslant G(t) \leqslant 1 \ \ for \ \ a \leqslant t \leqslant t_0, \ \ 0 \leqslant \overline{G}(t) \leqslant 1 \ \ for \ \ t_0 \leqslant t \leqslant b, \tag{7.1}$$

then for every convex function $f : I \to \mathbb{R}$ such that $h(x) \in I$ for all $x \in [a,b]$, we have

$$\int_a^b f(h(t))\, dg(t) \geqslant f\left(\int_a^b h(t) dg(t)\right) + \left(\int_a^b dg(t) - 1\right) f(0). \tag{7.2}$$

(b) If $\int_a^b h(t) dg(t) \in I$ and either

$$\begin{aligned} &there\ exists\ an\ s \leqslant t_0\ such\ that\ G(t) \leqslant 0\ \ for\ \ t < s, \\ &\qquad G(t) \geqslant 1\ \ for\ \ s \leqslant t \leqslant t_0\ \ and\ \overline{G}(t) \leqslant 0\ \ for\ t > t_0 \end{aligned} \tag{7.3}$$

or

$$\begin{aligned} &there\ exists\ an\ s \geqslant t_0\ such\ that\ G(t) \leqslant 0\ \ for\ t < t_0, \\ &\qquad \overline{G}(t) \geqslant 1\ for\ t_0 < t < s,\ \ and\ \overline{G}(t) \leqslant 0\ for\ t \geqslant s, \end{aligned} \tag{7.4}$$

then for every convex function $f : I \to \mathbb{R}$ such that $h(x) \in I$ for all $x \in [a,b]$, the reverse of the inequality in (7.2) *holds.*

In the sequel we extend results from [27] from log-convexity to exponential convexity.

The proofs of the following two theorems are quite similar to proof Theorem 23 and therefore they are omitted.

THEOREM 25. *Let $t_0 \in [a,b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation that satisfies* (7.3) *or* (7.4)*. Let $t \mapsto \Lambda_t$ be function defined on $(0,\infty)$ as*

$$\Lambda_t = \Lambda_t(a,b,h,g) = \int_a^b \varphi_t(h(x))\, dg(x) - \varphi_t\left(\int_a^b h(x)\, dg(x)\right), \tag{7.5}$$

where φ is defined as in Lemma 5*. Then* *(i) For every* $n \in \mathbb{N}$ *and for every* $t_k \in (0,\infty)$, $k = 1,2,...,n$, *the matrix* $\left[\Lambda_{\frac{t_i+t_j}{2}}\right]$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\Lambda_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0. \tag{7.6}$$

(ii) The function $t \mapsto \Lambda_t$ *is exponentially convex on* $(0,\infty)$.

THEOREM 26. *Let* $t_0 \in [a,b]$ *be fixed,* h *be continuous and monotonic with* $h(t_0) = 0$, g *be a function of bounded variation that satisfies* (7.1). *Let* $t \mapsto \widetilde{\Lambda}_t$ *be function defined on* $(0,\infty)$ *as*

$$\widetilde{\Lambda}_t = \Lambda_t(a,b,h,g) = \varphi_t\left(\int_a^b h(x)\,dg(x)\right) - \int_a^b \varphi_t(h(x))\,dg(x), \tag{7.7}$$

where φ *is defined as in Lemma* 5. *Then*

(i) For every $n \in \mathbb{N}$ *and for every* $t_k \in (0,\infty)$, $k = 1,2,...,n$, *the matrix* $\left[\widetilde{\Lambda}_{\frac{t_i+t_j}{2}}\right]$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\widetilde{\Lambda}_{\frac{t_i+t_j}{2}}\right]_{i,j=1}^{n} \geqslant 0. \tag{7.8}$$

(ii) The function $t \mapsto \widetilde{\Lambda}_t$ *is exponentially convex on* $(0,\infty)$.

8. Steffensen's inequality

The well-known Steffensen inequality reads as follows:

THEOREM 27. *Suppose that* f *is decreasing and* g *is integrable on* $[a,b]$ *with* $0 \leqslant g \leqslant 1$ *and* $\lambda = \int_a^b g(t)dt$. *Then*

$$\int_{b-\lambda}^{b} f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt. \tag{8.1}$$

The inequalities are reversed for an increasing function f.

In the sequel we extend results from [15] from log-convexity to exponential convexity. We will need the following lemma.

LEMMA 6. *Let us define the following family of functions*

$$\eta_t(x) = \begin{cases} \frac{x^t}{t}, & t \neq 0; \\ \log x, & t = 0. \end{cases} \tag{8.2}$$

Then $\frac{d}{dx}\eta_t(x) = x^{t-1}$, *that is* $x \mapsto \eta_t(x)$ *is increasing function on* $(0,\infty)$ *for every* $t \in \mathbb{R}$.

Let x,y be fixed real numbers and $x<y$. Let f be decreasing, positive function, and g is integrable function on $[x,y]$ with $0\leqslant g\leqslant 1$ and $\lambda=\int_x^y g(t)dt$. Let us define functions ϕ and ψ on $\mathbb{R}$ with

$$\phi(r)=\begin{cases}\frac{1}{r}\left(\int\limits_x^{x+\lambda} f^r(t)dt-\int\limits_x^y f^r(t)g(t)dt\right), & r\neq 0;\\ \int\limits_x^{x+\lambda}\log f(t)dt-\int\limits_x^y g(t)\log f(t)dt, & r=0.\end{cases} \tag{8.3}$$

$$\psi(r)=\begin{cases}\frac{1}{r}\left(\int\limits_x^y f^r(t)g(t)dt-\int\limits_{y-\lambda}^y f^r(t)dt\right), & r\neq 0;\\ \int\limits_x^y g(t)\log f(t)dt-\int\limits_{y-\lambda}^y\log f(t)dt, & r=0.\end{cases} \tag{8.4}$$

THEOREM 28. *Let ϕ and ψ be two functions defined in* (8.3) *and* (8.4)*.*

(i) For every $n\in\mathbb{N}$ and for every $t_k\in\mathbb{R}$, $k=1,2,\ldots,n$, the matrices $\left[\phi\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n$ and $\left[\psi\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n$ are positive semi-definite matrices.
Particularly

$$\det\left[\phi\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n\geqslant 0, \tag{8.5}$$

and

$$\det\left[\psi\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n\geqslant 0. \tag{8.6}$$

(ii) The functions ϕ and ψ are exponentially convex functions on $(0,\infty)$.

Proof. (i) For $v>0$, $u_i\in\mathbb{R}$ and $t_i\in\mathbb{R}$, $i=1,\ldots n$ let us consider the function

$$h(v)=\sum_{i,j}^n u_iu_j\eta_{t_{ij}}(v)$$

where $t_{ij}=\frac{t_i+t_j}{2}$ and η is defined as in Lemma 6. Then

$$h'(v)=\sum_{i,j}^n u_iu_jv^{t_{ij}-1}=\left(\sum_i^n u_iv^{\frac{t_i-1}{2}}\right)^2\geqslant 0.$$

So h is the increasing function and hence $h\circ f$ is the decreasing function. After we apply Steffensen's inequality (8.1) for function $h\circ f$ we get

$$\sum_{i,j}^n u_iu_j\phi(t_{ij})\geqslant 0$$

and

$$\sum_{i,j}^{n} u_i u_j \psi(t_{ij}) \geqslant 0.$$

(ii) Since $\lim_{t \to 0} \phi(t) = \phi(0)$ and $\lim_{t \to 0} \psi(t) = \psi(0)$, ϕ and ψ are continuous functions on $\mathbb{R}$ concluding their exponential convexity on $\mathbb{R}$. $\square$

Similarly, using Steffensen's inequality, we can extend results on log-convexity from [14] to exponential convexity(assumptions on function g are same from beginning of this section):

THEOREM 29. *Let $0 < x < y < \infty$, and let $\widehat{\phi}(r)$, $\widehat{\psi}(r)$ are defined as*

$$\widehat{\phi}(r) = \begin{cases} \frac{1}{r-1}\left(\int\limits_x^y t^{r-1} g(t)dt - \frac{(x+\lambda)^r - x^r}{r}\right), & r \neq 0,1; \\ \ln\left(\frac{x+\lambda}{x}\right) - \int\limits_x^y \frac{g(t)}{t} dt, & r = 0; \\ \int\limits_x^y g(t) \log t dt - (x+\lambda)\log(x+\lambda) + x \log x, & r = 1, \end{cases} \tag{8.7}$$

and

$$\widehat{\psi}(r) = \begin{cases} \frac{1}{r-1}\left(\frac{y^r - (y-\lambda)^r}{r} - \int\limits_x^y t^{r-1} g(t)dt\right), & r \neq 0,1; \\ \int\limits_x^y \frac{g(t)}{t} dt - \log\left(\frac{y}{y-\lambda}\right), & r = 0; \\ y \log y - (y-\lambda)\log(y-\lambda) - \int\limits_x^y g(t) \log t dt - \lambda, & r = 1. \end{cases} \tag{8.8}$$

(i) For every $n \in \mathbb{N}$ and for every $t_k \in \mathbb{R}$, $k = 1,2,\ldots,n$, the matrices $\left[\widehat{\phi}\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n$ and $\left[\widehat{\psi}\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n$ are positive semi-definite matrices. Particularly

$$\det\left[\widehat{\phi}\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n \geqslant 0, \tag{8.9}$$

and

$$\det\left[\widehat{\psi}\left(\frac{t_i+t_j}{2}\right)\right]_{i,j}^n \geqslant 0, \tag{8.10}$$

(i) The functions $\widehat{\phi}$ and $\widehat{\psi}$ are exponentially convex on $\mathbb{R}$.

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