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STABILITY OF A GENERALIZED JENSEN EQUATION ON RESTRICTED DOMAINS

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Abstract. In this paper, we establish the conditional stability of the generalized Jensen functional equation $f(ax+by) = ag(x)+bh(y)$ on various restricted domains such as inside balls, outside balls, and punctured spaces. In addition, we prove the orthogonal stability of this equation and study orthogonally generalized Jensen mappings on balls in inner product spaces.

1. Introduction

It is known that the problem of stability of functional equations originated from the following question of Ulam [30] posed in 1940: "Given an approximately linear mapping f, when does a linear mapping T estimating f exist?" In the next year, Hyers [9] gave an affirmative answer to the question of Ulam in the context of Banach spaces. The theorem of Hyers was extended by T. Aoki [1] for additive mappings in 1950 and by Th.M. Rassias [25] for linear mappings in 1978 by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p)$, where $\varepsilon > 0$ and $p \in [0,1)$ are constants. The paper [25] of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. In 1994, another generalization, the so-called generalized Hyers–Ulam–Rassias stability, was obtained by Găvruta [8]. During the last decades several stability problems of functional equations have been investigated. The reader is referred to [3, 5, 6, 7, 10, 11, 13, 16, 17, 18, 26] and references therein for more detailed information on stability of functional equations.

The generalized Jensen equation is $f(ax+by) = af(x)+bf(y)$ where f is a mapping between linear spaces and a,b are given positive rational numbers. It is easy to see that a mapping $f : \mathscr{X} \to \mathscr{Y}$ between linear spaces with $f(0) = 0$ satisfies the generalized Jensen equation for all $x,y \in \mathscr{X}$ if and only if it is additive; cf. [2] (see also [23, Theorem 6] in the case of Jensen's equation).

The first result on the stability of the following classical Jensen equation:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$$

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was given by Kominek [19]. The author who presumably investigated the stability problem on a restricted domain for the first time was Skof [29]. The stability of the Jensen equation and of its generalizations were studied by numerous mathematicians (cf., e.g., [4, 12, 15, 20, 24]. In this paper, by using the "direct method", we establish the Hyers–Ulam–Rassias stability of the generalized Jensen functional equation of Pexider type and the conditional stability on some certain restricted domains. Throughout the paper, $\mathscr{X}$ denotes a linear space and $\mathscr{Y}$ represents a Banach space. In addition, we assume a and b to be constant positive rational numbers.

2. Stability of generalized Jensen equation

The following theorem is a simple generalization of Theorem 2.1 of [2]. It can also be regarded as an extension of the main theorem of [14] and some results of [22].

THEOREM 2.1. *Let $f,\ g,\ h : \mathscr{X} \to \mathscr{Y}$ be mappings with $f(0) = g(0) = h(0) = 0$ for which there exists a function $\varphi : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ satisfying*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) < \infty$$

and define

$$\widetilde{\varphi}(x,y) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left[\varphi\left(2^n \frac{1}{a} x, 2^n \frac{1}{b} y\right) + \varphi\left(2^n \frac{1}{a} x, 0\right) + \varphi\left(0, 2^n \frac{1}{b} y\right) \right] \tag{2.1}$$

for all $x,y \in \mathscr{X}$. If f,g,h satisfy

$$\|f(ax+by) - ag(x) - bh(y)\| \leqslant \varphi(x,y) \tag{2.2}$$

for all $x,y \in \mathscr{X}$, then there exists a unique additive mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\begin{aligned} \|f(x) - T(x)\| &\leqslant \widetilde{\varphi}(x,x), \\ \|g(x) - T(x)\| &\leqslant \frac{1}{a}\varphi(x,0) + \frac{1}{a}\widetilde{\varphi}(ax,ax), \\ \|h(x) - T(x)\| &\leqslant \frac{1}{b}\varphi(0,x) + \frac{1}{b}\widetilde{\varphi}(bx,bx), \end{aligned}$$

for all $x \in \mathscr{X}$. Furthermore, if the mapping $\mu \mapsto f(\mu x)$ is continuous for each fixed $x \in \mathscr{X}$, then the additive mapping T is $\mathbb{R}$-linear.

Proof. Setting $y = 0$ in (2.2) we get

$$\|f(ax) - ag(x)\| \leqslant \varphi(x,0) \tag{2.3}$$

for all $x \in \mathscr{X}$. Setting $x = 0$ in (2.2) we obtain

$$\|f(by) - bh(y)\| \leqslant \varphi(0,y)$$

for all $y \in \mathscr{X}$. Then

$$\begin{aligned}\|f(ax+by)-f(ax)-f(by)\| &\leqslant \|f(ax+by)-ag(x)-bh(y)\| \\ &\quad + \|f(ax)-ag(x)\| + \|f(by)-bh(y)\| \\ &\leqslant \varphi(x,y)+\varphi(x,0)+\varphi(0,y)\end{aligned}$$

for all $x,y \in \mathscr{X}$. Replacing x and y by $\frac{1}{a}x$ and $\frac{1}{b}y$, respectively, we have

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varphi\left(\frac{1}{a}x, \frac{1}{b}y\right) + \varphi\left(\frac{1}{a}x, 0\right) + \varphi\left(0, \frac{1}{b}y\right).$$

By the well-known theorem of Găvruta [8] there exists a unique additive mapping $T : \mathscr{X} \to \mathscr{Y}$ given by $T(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$ such that

$$\|f(x)-T(x)\| \leqslant \widetilde{\varphi}(x,x) \tag{2.4}$$

for all $x \in \mathscr{X}$, where $\widetilde{\varphi}$ is given by (2.1). Since T is additive, we have $T(\alpha x) = \alpha T(x)$ for all rational numbers α and $x \in \mathscr{X}$. It follows from (2.3) and (2.4) that

$$\begin{aligned}\|g(x)-T(x)\| &\leqslant \left\|g(x)-\frac{1}{a}f(ax)\right\| + \left\|\frac{1}{a}f(ax)-T(x)\right\| \\ &\leqslant \frac{1}{a}\varphi(x,0) + \frac{1}{a}\widetilde{\varphi}(ax,ax)\end{aligned}$$

for all $x \in \mathscr{X}$. In a similar way we obtain the following inequality

$$\|h(x)-T(x)\| \leqslant \frac{1}{b}\varphi(0,x) + \frac{1}{b}\widetilde{\varphi}(bx,bx)$$

for all $x \in \mathscr{X}$.

If the mapping $\mu \mapsto f(\mu x)$ is continuous for each fixed $x \in \mathscr{X}$, then the linearity of the mapping T can be deduced by the same reasoning as in the proof of the main theorem of [25].

COROLLARY 2.2. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a mapping with $f(0) = 0$, and there exist constants $\varepsilon, \delta \geqslant 0$ and $p \in [0,1)$ such that*

$$\|f(ax+by)-af(x)-bf(y)\| \leqslant \varepsilon + \delta(\|x\|^p + \|y\|^p),$$

for all $x,y \in \mathscr{X}$. Then there exists a unique additive mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x)-T(x)\| \leqslant 3\varepsilon + \left[\left(\frac{1}{a}\right)^p + \left(\frac{1}{b}\right)^p\right] \frac{2\delta\|x\|^p}{2-2^p}$$

for all $x \in \mathscr{X}$.

Proof. Define $\varphi(x,y) = \varepsilon + \delta(\|x\|^p + \|y\|^p)$ and apply Theorem 2.1.

3. Asymptotic behavior of generalized Jensen equation

We start this section with investigating the stability of generalized Jensen equation outside a ball. The results in this section are generalizations of Theorem 3.3 and Corollary 3.4 of [15].

THEOREM 3.1. *Let $d > 0$, $\varepsilon > 0$, and $f : \mathscr{X} \to \mathscr{Y}$ be a mapping with $f(0) = 0$ such that*

$$\|f(ax+by) - af(x) - bf(y)\| \leqslant \varepsilon,$$

for all $x, y \in \mathscr{X}$ with $\|x\| + \|y\| \geqslant d$. Then there exists a unique additive mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x) - T(x)\| \leqslant 15\varepsilon$$

for all $x \in \mathscr{X}$.

Proof. Let $x, y \in \mathscr{X}$ with $\|x\| + \|y\| < d$. If $x = y = 0$, we choose a $z \in \mathscr{X}$ with $\|z\| = d$, otherwise

$$z := \begin{cases} \left(1 + \frac{d}{\|x\|}\right) x \text{ if } \|x\| \geqslant \|y\| \\ \left(1 + \frac{d}{\|y\|}\right) y \text{ if } \|x\| \leqslant \|y\|. \end{cases}$$

Then one can easily verify the following inequalities:

$$\left\|\left(2 + \frac{b}{a}\right) z + \frac{b}{a} y\right\| + \left\|\frac{a}{b} x - \left(1 + \frac{2a}{b}\right) z\right\| \geqslant d,$$

$$\|x\| + \|z\| \geqslant d,$$

$$\left\|2\left(1 + \frac{b}{a}\right) z\right\| + \|y\| \geqslant d,$$

$$\left\|2\left(1 + \frac{b}{a}\right) z\right\| + \left\|\frac{a}{b} x - \left(1 + \frac{2a}{b}\right) z\right\| \geqslant d,$$

$$\left\|\left(2 + \frac{b}{a}\right) z + \frac{b}{a} y\right\| + \|z\| \geqslant d.$$

It follows that

$$\begin{aligned} &\|f(ax+by) - af(x) - bf(y)\| \\ &\quad \leqslant \left\|f(ax+by) - af\left(\left(2 + \frac{b}{a}\right) z + \frac{b}{a} y\right) - bf\left(\frac{a}{b} x - \left(1 + \frac{2a}{b}\right) z\right)\right\| \\ &\qquad + \|f(ax+bz) - af(x) - bf(z)\| \\ &\qquad + \left\|f(2(a+b)z+by) - af\left(2\left(1 + \frac{b}{a}\right) z\right) - bf(y)\right\| \end{aligned}$$

$$
\begin{aligned}
&+\left\|-f(ax+bz)+af\left(2\left(1+\frac{b}{a}\right)z\right)+bf\left(\frac{a}{b}x-\left(1+\frac{2a}{b}\right)z\right)\right\| \\
&+\left\|-f(2(a+b)z+by)+af\left(\left(2+\frac{b}{a}\right)z+\frac{b}{a}y\right)+bf(z)\right\| \\
\leqslant\ & 5\varepsilon.
\end{aligned}
$$

Hence

$$\|f(ax+by)-af(x)-bf(y)\| \leqslant 5\varepsilon$$

holds for all $x,y \in \mathscr{X}$. Using Corollary 2.2 (with $\delta = 0$), we conclude the existence of a unique additive mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x)-T(x)\| \leqslant 15\varepsilon$$

for all $x \in \mathscr{X}$.

Now we are ready to study the asymptotic behavior of the generalized Jensen equation.

COROLLARY 3.2. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping with $f(0) = 0$. Then f is additive if and only if*

$$\|f(ax+by)-af(x)-bf(y)\| \to 0 \quad as \quad \|x\|+\|y\| \to \infty. \tag{3.1}$$

Proof. Let (3.1) be satisfied. Then there is a sequence $\{\delta_n\}$ monotonically decreasing to zero such that

$$\|f(ax+by)-af(x)-bf(y)\| \leqslant \delta_n \tag{3.2}$$

for all $x,y \in \mathscr{X}$ with $\|x\|+\|y\| \geqslant n$. Applying (3.2) and Theorem 3.1 we obtain a sequence $\{T_n\}$ of unique additive mappings from $\mathscr{X}$ into $\mathscr{Y}$ such that

$$\|f(x)-T_n(x)\| \leqslant 15\delta_n$$

for all $x \in \mathscr{X}$. The uniqueness of T_n implies that $T_n = T_{n+j}$ for all $j \in \mathbb{N}$. Hence by letting n tend to ∞ we infer that f is additive. The reverse statement is obvious.

4. Stability on the punctured space

In this section we prove the stability of the Jensen equation of Pexider type on the punctured space $\mathscr{X}_0 := \mathscr{X} \setminus \{0\}$. In particular, in the case of $a = b = 1$, we obtain some results on stability of the Pexiderized Cauchy equation restricted to the punctured space.

PROPOSITION 4.1. *Let $f,g,h:\mathscr{X}\to\mathscr{Y}$ be mappings with $f(0)=g(0)=h(0)=0$ for which there exists a function $\varphi:\mathscr{X}_0\times\mathscr{X}_0\to[0,\infty)$ satisfying*

$$\begin{aligned}\widetilde{\varphi}(x,y) &:= \frac{2}{3}\sum_{n=0}^{\infty}3^{-n}\Bigg[\varphi\left(\frac{3^{n+1}}{2}x,\frac{-3^n}{2}y\right)+\frac{1}{2}\varphi\left(\frac{3^{n+1}}{2}x,\frac{3^{n+1}}{2}y\right)\\ &\quad+\frac{1}{2}\varphi\left(\frac{3^{n+1}}{2}x,\frac{-3^{n+1}}{2}y\right)+\frac{1}{2}\varphi\left(\frac{3^n}{2}x,\frac{3^n}{2}y\right)+\frac{1}{2}\varphi\left(\frac{3^n}{2}x,\frac{-3^n}{2}y\right)\Bigg]\\ &< \infty\end{aligned}$$

and

$$\|f(ax+by)-ag(x)-bh(y)\|\leqslant\varphi\left(x,\frac{b}{a}y\right) \tag{4.1}$$

for all $x,y\in\mathscr{X}_0$. If h is an odd mapping, then there exists a unique additive mapping $A:\mathscr{X}\to\mathscr{Y}$ such that

$$\|f(x)-A(x)\|\leqslant\widetilde{\varphi}\left(\frac{1}{a}x,\frac{1}{a}x\right),$$

$$\|g(x)-A(x)\|\leqslant\frac{1}{2a}\,\varphi(x,x)+\frac{1}{2a}\,\varphi(x,-x)+\frac{1}{2a}\,\widetilde{\varphi}(2x,2x)$$

and

$$\|h(x)-A(x)\|\leqslant\frac{1}{2b}\,\varphi\left(\frac{b}{a}x,\frac{b}{a}x\right)+\frac{1}{2b}\,\varphi\left(\frac{b}{a}x,\frac{-b}{a}x\right)+\frac{1}{2b}\,\widetilde{\varphi}\left(\frac{2b}{a}x,\frac{2b}{a}x\right)$$

for all $x\in\mathscr{X}$.

Proof. Replacing y by $\frac{a}{b}y$ in (4.1) we get

$$\left\|f(a(x+y))-ag(x)-bh\left(\frac{a}{b}y\right)\right\|\leqslant\varphi(x,y)$$

for all $x,y\in\mathscr{X}$. Define the mappings F, G, H by $F(x):=f(ax)$, $G(x):=ag(x)$ and $H(x):=bh\left(\frac{a}{b}x\right)$. Then

$$\|F(x+y)-G(x)-H(y)\|\leqslant\varphi(x,y) \tag{4.2}$$

for all $x,y\in\mathscr{X}_0$.

Replacing both x and y by $\frac{x}{2}$ in (4.2) we get

$$\left\|F(x)-G\left(\frac{x}{2}\right)-H\left(\frac{x}{2}\right)\right\|\leqslant\varphi\left(\frac{x}{2},\frac{x}{2}\right) \tag{4.3}$$

for all $x\in\mathscr{X}_0$. Replacing x by $\frac{x}{2}$ and y by $\frac{-x}{2}$, respectively, in (4.2) we obtain

$$\left\|G\left(\frac{x}{2}\right)-H\left(\frac{x}{2}\right)\right\|\leqslant\varphi\left(\frac{x}{2},\frac{-x}{2}\right) \tag{4.4}$$

for all $x \in \mathscr{X}_0$. It follows from (4.3) and (4.4) that

$$\left\| F(x) - 2H\left(\frac{x}{2}\right) \right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, \frac{-x}{2}\right) \tag{4.5}$$

and

$$\left\| F(x) - 2G\left(\frac{x}{2}\right) \right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, \frac{-x}{2}\right) \tag{4.6}$$

for all $x \in \mathscr{X}_0$. Replacing x by $\frac{3x}{2}$ and y by $\frac{-x}{2}$, respectively, in (4.2) we have

$$\left\| F(x) - G\left(\frac{3x}{2}\right) + H\left(\frac{x}{2}\right) \right\| \leqslant \varphi\left(\frac{3x}{2}, \frac{-x}{2}\right) \tag{4.7}$$

for all $x \in \mathscr{X}_0$. Using (4.5), (4.6) and (4.7) we infer that

$$\begin{aligned}
\left\| F(x) - \frac{1}{3}F(3x) \right\| &= \frac{2}{3} \left\| F(x) - \frac{1}{2}F(3x) + \frac{1}{2}F(x) \right\| \\
&\leqslant \frac{2}{3} \left\| F(x) - G\left(\frac{3x}{2}\right) + H\left(\frac{x}{2}\right) \right\| \\
&\qquad + \frac{2}{3} \left\| G\left(\frac{3x}{2}\right) - \frac{1}{2}F(3x) \right\| + \frac{2}{3} \left\| \frac{1}{2}F(x) - H\left(\frac{x}{2}\right) \right\| \\
&\leqslant \frac{2}{3}\varphi\left(\frac{3x}{2}, \frac{-x}{2}\right) + \frac{1}{2}\varphi\left(\frac{3x}{2}, \frac{3x}{2}\right) + \frac{1}{2}\varphi\left(\frac{3x}{2}, \frac{-3x}{2}\right) \\
&\qquad + \frac{1}{2}\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{2}\varphi\left(\frac{x}{2}, \frac{-x}{2}\right) \\
&=: \psi(x)
\end{aligned} \tag{4.8}$$

for all $x \in \mathscr{X}_0$. Using (4.8) and the induction, one can prove that

$$\left\| F(x) - \frac{1}{3^n}F(3^n x) \right\| \leqslant \sum_{k=0}^{n-1} 3^{-k}\psi(3^k x) \tag{4.9}$$

for all n and all $x \in \mathscr{X}_0$, and also

$$\left\| \frac{1}{3^n}F(3^n x) - \frac{1}{3^m}F(3^m x) \right\| \leqslant \sum_{k=m}^{n-1} 3^{-k}\psi(3^k x)$$

for all $m < n$ and all $x \in \mathscr{X}_0$. Since $\mathscr{Y}$ is complete we deduce that the sequence $\left\{\frac{1}{3^n}F(3^n x)\right\}$ is convergent. Therefore we can define the mapping $T : \mathscr{X} \to \mathscr{Y}$ by

$$T(x) := \lim_{n\to\infty} \frac{1}{3^n}F(3^n x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n a x). \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\|f(ax) - T(x)\| = \|F(x) - T(x)\| \leqslant \widetilde{\varphi}(x,x) \tag{4.11}$$

and so

$$\left\|f(x)-T\left(\frac{1}{a}x\right)\right\| \leqslant \widetilde{\varphi}\left(\frac{1}{a}x,\frac{1}{a}x\right) \tag{4.12}$$

for all $x \in \mathscr{X}$. Note that $T(0)=0$.

Using (4.2), (4.5) and (4.6) we have

$$\begin{aligned}\left\|2F\left(\frac{x+y}{2}\right)-F(x)-F(y)\right\| &\leqslant \varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(\frac{x}{2},\frac{-x}{2}\right)\\ &\quad +2\varphi\left(\frac{x}{2},\frac{y}{2}\right)+\varphi\left(\frac{y}{2},\frac{y}{2}\right)+\varphi\left(\frac{y}{2},\frac{-y}{2}\right)\end{aligned}$$

for all $x,y \in \mathscr{X}_0$, whence

$$\begin{aligned}&\left\|\frac{2}{3^n}F\left(\frac{3^n(x+y)}{2}\right)-\frac{1}{3^n}F(3^nx)-\frac{1}{3^n}F(3^ny)\right\|\\ &\quad\leqslant \frac{1}{3^n}\varphi\left(\frac{3^nx}{2},\frac{3^nx}{2}\right)+\frac{1}{3^n}\varphi\left(\frac{3^nx}{2},\frac{-3^nx}{2}\right)+\frac{2}{3^n}\varphi\left(\frac{3^nx}{2},\frac{3^ny}{2}\right)\\ &\quad\quad+\frac{1}{3^n}\varphi\left(\frac{3^ny}{2},\frac{3^ny}{2}\right)+\frac{1}{3^n}\varphi\left(\frac{3^ny}{2},\frac{-3^ny}{2}\right)\end{aligned}$$

for all $x \in \mathscr{X}_0$. Letting $n\to\infty$ and noting to the fact that the right hand side tends to zero, we conclude that T satisfies Jensen's equation and so it is additive. By a known strategy one can easily establish the uniqueness of T; cf. [21]. It follows from (4.12) that

$$\left\|f(x)-\frac{1}{a}T(x)\right\| \leqslant \widetilde{\varphi}\left(\frac{1}{a}x,\frac{1}{a}x\right)$$

for all $x \in \mathscr{X}$.

Using(4.6) and (4.11), we obtain

$$\begin{aligned}\|ag(x)-T(x)\| = \|G(x)-T(x)\| &\leqslant \left\|G(x)-\frac{1}{2}F(2x)\right\|+\left\|\frac{1}{2}F(2x)-\frac{1}{2}T(2x)\right\|\\ &\leqslant \frac{1}{2}\varphi(x,x)+\frac{1}{2}\varphi(x,-x)+\frac{1}{2}\widetilde{\varphi}(2x,2x),\end{aligned}$$

and so

$$\left\|g(x)-\frac{1}{a}T(x)\right\| \leqslant \frac{1}{2a}\varphi(x,x)+\frac{1}{2a}\varphi(x,-x)+\frac{1}{2a}\widetilde{\varphi}(2x,2x)$$

for all $x \in \mathscr{X}$. Similarly by applying (4.11) and (4.5) we have

$$\left\|bh\left(\frac{a}{b}x\right)-T(x)\right\| = \|H(x)-T(x)\| \leqslant \frac{1}{2}\varphi(x,x)+\frac{1}{2}\varphi(x,-x)+\frac{1}{2}\widetilde{\varphi}(2x,2x),$$

and so

$$\left\|h(x)-\frac{1}{a}T(x)\right\| \leqslant \frac{1}{2b}\varphi\left(\frac{b}{a}x,\frac{b}{a}x\right)+\frac{1}{2b}\varphi\left(\frac{b}{a}x,\frac{-b}{a}x\right)+\frac{1}{2b}\widetilde{\varphi}\left(\frac{2b}{a}x,\frac{2b}{a}x\right)$$

for all $x \in \mathscr{X}$.

With $A(x) := \frac{1}{a}T(x)$, we conclude that our assertions are true.

PROPOSITION 4.2. *Let $f,\ g,\ h : \mathscr{X} \to \mathscr{Y}$ be mappings with $f(0) = g(0) = h(0) = 0$ for which there exists a function $\varphi : \mathscr{X}_0 \times \mathscr{X}_0 \to [0,\infty)$ such that*

$$\|f(ax+by) - ag(x) - bh(y)\| \leqslant \varphi\left(x, \frac{b}{a}y\right) \tag{4.13}$$

for all $x,y \in \mathscr{X}_0$. If h is an even mapping, then

$$\|f(x)\| \leqslant \varphi\left(\frac{x}{2a}, \frac{x}{2a}\right) + \varphi\left(\frac{x}{2a}, \frac{-x}{2a}\right),$$

and

$$\left\|g(x) + \frac{b}{a}h\left(\frac{a}{b}x\right)\right\| \leqslant \frac{1}{a}\,\varphi(x,-x)$$

for all $x \in \mathscr{X}$.

Proof. Replacing y by $\frac{a}{b}y$ in (4.13) we get

$$\left\|f(a(x+y)) - ag(x) - bh\left(\frac{a}{b}y\right)\right\| \leqslant \varphi(x,y)$$

for all $x,y \in \mathscr{X}$. Define the mappings F, G, H by $F(x) := f(ax)$, $G(x) := ag(x)$ and $H(x) := bh\left(\frac{a}{b}x\right)$. Then

$$\|F(x+y) - G(x) - H(y)\| \leqslant \varphi(x,y) \tag{4.14}$$

for all $x,y \in \mathscr{X}_0$.

Replacing both x and y by $\frac{x}{2}$ in (4.14) we get

$$\left\|F(x) - G\left(\frac{x}{2}\right) - H\left(\frac{x}{2}\right)\right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{4.15}$$

for all $x \in \mathscr{X}_0$. Replacing x by $\frac{x}{2}$ and y by $\frac{-x}{2}$, respectively, in (4.14) we obtain

$$\left\|G\left(\frac{x}{2}\right) + H\left(\frac{x}{2}\right)\right\| \leqslant \varphi\left(\frac{x}{2}, \frac{-x}{2}\right) \tag{4.16}$$

for all $x \in \mathscr{X}_0$. It follows from (4.15) and (4.16) that

$$\begin{aligned}\|f(ax)\| \ = \ \|F(x)\| &\leqslant \left\|F(x) - G\left(\frac{x}{2}\right) - H\left(\frac{x}{2}\right)\right\| + \left\|G\left(\frac{x}{2}\right) + H\left(\frac{x}{2}\right)\right\| \\ &\leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, \frac{-x}{2}\right),\end{aligned}$$

and so

$$\|f(x)\| \leqslant \varphi\left(\frac{x}{2a}, \frac{x}{2a}\right) + \varphi\left(\frac{x}{2a}, \frac{-x}{2a}\right).$$

In addition

$$\left\|g(x) + \frac{b}{a} h\left(\frac{a}{b} x\right)\right\| = \frac{1}{a}\|G(x) + H(x)\| \leqslant \frac{1}{a}\varphi(x, -x)$$

for all $x \in \mathscr{X}$.

THEOREM 4.3. *Let* $\varepsilon > 0$ *and let* $f : \mathscr{X} \to \mathscr{Y}$ *be a mapping with* $f(0) = 0$ *satisfying*

$$\|f(ax + by) - af(x) - bf(y)\| \leqslant \varepsilon \tag{4.17}$$

for all $x, y \in \mathscr{X}_0$. *Then there exists a unique additive mapping* $A : \mathscr{X} \to \mathscr{Y}$ *such that*

$$\|f(x) - A(x)\| \leqslant \min\left\{3\varepsilon, \frac{5\varepsilon}{2a}, \frac{5\varepsilon}{2b}\right\} + 2\varepsilon$$

for all $x \in \mathscr{X}$.

Proof. Replacing x, y by $-x$, $-\frac{ay}{b}$ in (4.17), respectively, we get

$$\left\|f(a(-x - y)) - af(-x) - bf\left(-\frac{a}{b} y\right)\right\| \leqslant \varepsilon \tag{4.18}$$

for all $x, y \in \mathscr{X}_0$. Using the odd and even parts f^o, f^e of f and (4.17) and (4.18) we obtain

$$\|f^o(ax + by) - af^o(x) - bf^o(y)\| \leqslant \varepsilon$$

and

$$\|f^e(ax + by) - af^e(x) - bf^e(y)\| \leqslant \varepsilon$$

for all $x \in \mathscr{X}_0$. Then Propositions 4.1 and 4.2 give us a unique additive mapping A such that

$$\|f^o(x) - A(x)\| \leqslant \min\left\{3\varepsilon, \frac{5\varepsilon}{2a}, \frac{5\varepsilon}{2b}\right\}$$

and

$$\|f^e(x)\| \leqslant 2\varepsilon$$

for all $x \in \mathscr{X}$. Hence

$$\|f(x) - A(x)\| \leqslant \|f^o(x) - A(x)\| + \|f^e(x)\| \leqslant \min\left\{3\varepsilon, \frac{5\varepsilon}{2a}, \frac{5\varepsilon}{2b}\right\} + 2\varepsilon$$

for all $x \in \mathscr{X}$.

5. Stability on orthogonality spaces

Let us recall the orthogonality in the sense of Rätz; cf. [27]. Suppose $\mathscr{X}$ is a real vector space with $\dim \mathscr{X} \geqslant 2$ and $\perp$ is a binary relation on $\mathscr{X}$ with the following properties:

(O1) $x \perp 0$, $0 \perp x$ for all $x \in \mathscr{X}$;

(O2) if $x, y \in \mathscr{X} \setminus \{0\}$, $x \perp y$, then x, y are linearly independent;

(O3) if $x, y \in \mathscr{X}$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) if P is a 2-dimensional subspace of $\mathscr{X}$, $x \in P$ and $\lambda \in \mathbb{R}_+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(\mathscr{X}, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are as follows.

(i) The trivial orthogonality on a vector space $\mathscr{X}$ defined by (O1), and for non-zero elements $x, y \in \mathscr{X}$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(\mathscr{X}, \langle .,. \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff–James orthogonality on a normed space $(\mathscr{X}, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geqslant \|x\|$ for all $\lambda \in \mathbb{R}$.

Let $\mathscr{X}$ be a vector space (an orthogonality space) and $(\mathscr{Y}, +)$ be an abelian group. Then a mapping $f : \mathscr{X} \to \mathscr{Y}$ is called

(i) *orthogonally additive* if it satisfies the additive functional equation for all $x, y \in \mathscr{X}$ with $x \perp y$;

(ii) *orthogonally quadratic* if it satisfies the quadratic functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in \mathscr{X}$ with $x \perp y$;

(iii) *orthogonally generalized Jensen* if it satisfies the generalized Jensen functional equation for all $x, y \in \mathscr{X}$ with $x \perp y$.

In [21] the orthogonal stability of various functional equations was established. In particular, the following theorem was proved; cf. Theorem 1 of [21].

THEOREM 5.1. *Suppose $(\mathscr{X}, \perp)$ is an orthogonality space and $(\mathscr{Y}, \|.\|)$ is a real Banach space. Let $f, g, h : \mathscr{X} \to \mathscr{Y}$ be mappings fulfilling*

$$\|f(x+y) - g(x) - h(y)\| \leqslant \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in \mathscr{X}$ with $x \perp y$. Then there exist exactly a quadratic mapping $Q : \mathscr{X} \to \mathscr{Y}$ and an additive mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x)-f(0)-Q(x)-T(x)\| \leqslant \frac{68}{3}\varepsilon,$$
$$\|g(x)-g(0)-Q(x)-T(x)\| \leqslant \frac{80}{3}\varepsilon,$$
$$\|h(x)-h(0)-Q(x)-T(x)\| \leqslant \frac{80}{3}\varepsilon$$

for all $x \in \mathscr{X}$.

Now we are ready to provide another result on orthogonal stability of functional equations.

THEOREM 5.2. *Suppose that* $f,g,h : \mathscr{X} \to \mathscr{Y}$ *is a mapping with* $f(0) = g(0) = h(0) = 0$ *satisfying*

$$\|f(ax+by)-ag(x)-bh(y)\| \leqslant \varepsilon$$

for all $x,y \in \mathscr{X}$ *with* $x \perp y$. *Then there exist a unique additive mapping* $T : \mathscr{X} \to \mathscr{Y}$ *and a unique quadratic mapping* $Q : \mathscr{X} \to \mathscr{Y}$ *such that*

$$\|f(x)-T(x)-Q(x)\| \leqslant 68\varepsilon,$$
$$\|g(x)-T(x)-Q(x)\| \leqslant 80\varepsilon,$$
$$\|h(x)-T(x)-Q(x)\| \leqslant 80\varepsilon$$

for all $x \in \mathscr{X}$.

Proof. Using the same argument as in the proof of Theorem 2.1 we conclude that $\|f(x+y)-g(x)-h(x)\| \leqslant 3\varepsilon$. Then the result is followed from Theorem 5.1.

6. Orthogonally generalized Jensen mappings on balls in inner product spaces

Sikorska showed that if f is an orthogonally additive mapping on an open ball $\mathbf{B}$ of a real inner product space $\mathscr{X}$ into a real sequentially complete linear topological space $\mathscr{Y}$ then there exist additive mappings $T : \mathscr{X} \to \mathscr{Y}$ and $b : \mathbb{R}_+ \to \mathscr{Y}$ such that $f(x) = T(x) + b(\|x\|^2)$ for all $x \in \mathbf{B}$; cf. Corollary 1 of [28]. By an orthogonally generalized Jensen mapping we mean a mapping $f : \mathscr{X} \to \mathscr{Y}$ such that $f(ax+by) = af(x)+bf(y)$ holds for all $x,y \in \mathscr{X}$ with $x \perp y$. We can extend Sikorska's result to the orthogonally generalized Jensen mappings as follows.

THEOREM 6.1. *If* $\mathbf{B}$ *is an open ball of a real inner product space* $\mathscr{X}$, $\mathscr{Y}$ *is a real sequentially complete linear topological space, and* $f : \mathbf{B} \to \mathscr{Y}$ *is orthogonally generalized Jensen with* $f(0) = 0$, *then there exist additive mappings* $T : \mathscr{X} \to \mathscr{Y}$ *and* $b : \mathbb{R}_+ \to \mathscr{Y}$ *such that* $f(x) = T(x) + b\left(\|x\|^2\right)$ *for all* $x \in \mathbf{B}$.

Proof. With $y = 0$, we have

$$f(ax+b0) = af(x)+bf(0) = af(x),$$

and therefore $f\left(\frac{1}{a}x\right) = \frac{1}{a}f(x)$. Similarly $f\left(\frac{1}{b}x\right) = \frac{1}{b}f(x)$. Hence

$$f(x+y) = af\left(\frac{1}{a}x\right) + bf\left(\frac{1}{b}y\right) = f(x) + f(y)$$

for all $x, y \in \mathbf{B}$ with $x \perp y$. This implies that f is orthogonally additive on $\mathbf{B}$. Now the assertion follows from the Sikorska's result.

If we are restricted to the punctured ball $\mathbf{B}\backslash\{0\}$ then by following the same strategy as in Lemma 1 of [28] we obtain the following result in the case that $a = b > \frac{1}{\sqrt{2}}$. The investigation of situation in the general case is left for further research.

THEOREM 6.2. *If $\mathbf{B}$ is an open ball of a real inner product space $\mathscr{X}$ of dimension greater than 1, $\mathscr{Y}$ is a real sequentially complete linear topological space, and $f : \mathbf{B}\backslash\{0\} \to \mathscr{Y}$ is orthogonally generalized Jensen mapping with parameters $a = b > \frac{1}{\sqrt{2}}$, then there exist additive mappings $T : \mathscr{X} \to \mathscr{Y}$ and $b : \mathbb{R}_+ \to \mathscr{Y}$ such that $f(x) = T(x) + b(\|x\|^2)$ for all $x \in \mathbf{B}\backslash\{0\}$.*

Proof. First note that if f is a generalized Jensen mapping with parameters $a = b \geqslant 1$, then $f(\lambda(x+y)) = \lambda f(x) + \lambda f(y)$ for some $\lambda \geqslant 1$ and all $x, y \in \mathbf{B}\backslash\{0\}$ such that $x \perp y$.

Step (I) — *the case that f is odd:* Let $x \in \mathbf{B}\backslash\{0\}$. There exists $y_0 \in \mathbf{B}\backslash\{0\}$ such that $x \perp y_0$, $x + y_0 \perp x - y_0$. Since $\frac{x+y_0}{2\lambda}$, $\frac{x-y_0}{2\lambda}$, $\frac{x}{2\lambda^2}$, $\frac{y_0}{2\lambda^2} \in \mathbf{B}$, we have

$$\begin{aligned} f(x) &= f(x) - \lambda f\left(\frac{x+y_0}{2\lambda}\right) - \lambda f\left(\frac{x-y_0}{2\lambda}\right) \\ &\quad + \lambda f\left(\frac{x+y_0}{2\lambda}\right) - \lambda^2 f\left(\frac{x}{2\lambda^2}\right) - \lambda^2 f\left(\frac{y_0}{2\lambda^2}\right) \\ &\quad + \lambda f\left(\frac{x-y_0}{2\lambda}\right) - \lambda^2 f\left(\frac{x}{2\lambda^2}\right) - \lambda^2 f\left(\frac{-y_0}{2\lambda^2}\right) \\ &\quad + 2\lambda^2 f\left(\frac{x}{2\lambda^2}\right) = 2\lambda^2 f\left(\frac{x}{2\lambda^2}\right). \end{aligned}$$

It follows that

$$f(x) = 2\lambda^2 f\left((2\lambda^2)^{-1}x\right) \tag{6.1}$$

for all $x \in \mathbf{B}\backslash\{0\}$. Since

$$\left(2\lambda^2\right)^{n+m} f((2\lambda^2)^{-n-m}x) = \left(2\lambda^2\right)^n f((2\lambda^2)^{-n}x)$$

for all $x \in \mathbf{B}\backslash\{0\}$, and all $m, n \in \mathbb{N}$ with $(2\lambda^2)^{-n}x \in \mathbf{B}\backslash\{0\}$, we can well define the mapping $T : \mathscr{X} \to \mathscr{Y}$ by $T(x) := (2\lambda^2)^n f((2\lambda^2)^{-n}x)$ where n is an integer such that $(2\lambda^2)^{-n}x \in \mathbf{B}\backslash\{0\}$. Clearly, T is an extension of f to $\mathscr{X}$. The mapping T is an odd orthogonally additive mapping. To see this, let $x, y \in \mathscr{X}$ with $x \perp y$. Then there exists a positive integer n such that $(2\lambda^2)^{-n}x$, $(2\lambda^2)^{-n}y$, $(2\lambda^2)^{-n}(x+y)$, $(2\lambda^2)^{-n}(x-y) \in \mathbf{B}\setminus\{0\}$, and by using (6.1) we obtain

$$\begin{aligned} T(x)+T(y) &= (2\lambda^2)^n f((2\lambda^2)^{-n}x) + (2\lambda^2)^n f((2\lambda^2)^{-n}y) \\ &= (2\lambda^2)^{n+1} f((2\lambda^2)^{-n-1}x) + (2\lambda^2)^{n+1} f((2\lambda^2)^{-n-1}y) \\ &= (2\lambda^2)^{n+1} f((2\lambda^2)^{-n-1}(x+y)) \\ &= (2\lambda^2)^n f((2\lambda^2)^{-n}(x+y)) \\ &= T(x+y). \end{aligned}$$

Hence $T(x)+T(y) = T(x+y)$ for all $x,y \in \mathscr{X}$. By Corollary 7 of [27], T is additive and $f = T|_{\mathbf{B}\setminus\{0\}}$.

Step (II) — *the case that f is even:* Using the same notation and the same reasoning as step (I), one can show that $f(x) = f(y_0)$ and the mapping $Q : \mathscr{X} \to \mathscr{Y}$ defined by $Q(x) := (4\lambda^2)^n f((2\lambda^2)^{-n}x)$ is even orthogonally additive. By Corollaries 7 and 10 of [27], Q is quadratic and there exists an additive mapping $b : \mathbb{R}_+ \to \mathscr{Y}$ such that $Q(x) = b(\|x\|^2)$ for all $x \in \mathbf{B}$. In addition, $f = Q|_{\mathbf{B}\setminus\{0\}}$

Step (III) — *the general case:* If f is an arbitrary mapping, then f can be expressed as $f = f^o + f^e$ where f^o and f^e are the odd and even parts of f, respectively. Now the result can be deduced from Steps (I) and (II).

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