Journal of
Mathematical
Inequalities
Volume 4, Number 2 (2010), 207–216

SOME MULTI–DIMENSIONAL OPIAL–TYPE INEQUALITIES ON TIME SCALES

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(*Communicated by A. Peterson*)

Abstract. In this paper, we not only give some time scale extensions of the results given in [J. Math. Anal. Appl. 126 (1987) 85–89] by Pachpatte, but also obtain some new results involving higher-order delta-derivatives by means of the generalized Taylor's formula on arbitrary time scales. We also apply the useful Muirhead's inequality to obtain interesting new results. Most of our results are new in particular time scales.

1. Introduction

In [10], Opial proved the following inequality:

$$\int_a^b |f(\xi)f'(\xi)|\mathrm{d}\xi \leqslant \frac{b-a}{4}\int_a^b |f'(\xi)|^2\mathrm{d}\xi,$$

for any $f \in C([a,b]_{\mathbb{R}},\mathbb{R})$ satisfying $f(a)=f(b)=0$.

After the publication of Opial's work [10] in 1960, a large number of papers have been published in the literature which deal with alternative proofs of various generalizations and their discrete analogues (see [1, 2]). Most of these results are very nicely illustrated in the book [2] by Agarwal and Pang which is devoted solely to continuous and discrete versions of Opial's inequalities.

In [4] Bohner and Kaymakçalan initiated the time scales unification of a continuos and a discrete analogues of a version of Opial's inequality and illustrated some applications of it to dynamic equations on time scales.

One of the most interesting generalizations of Opial's inequality is given by Patchpatte in [8], where the author extends the inequality to the multi-dimensional case involving gradient of a function instead of its first-order derivative. Our main aim in this paper is to generalize and extend these results to arbitrary time scales by using higher-order delta-derivatives in inequalities. To handle such a situation, the generalized Taylor's formula on time scales will be efficiently used. Thus we will be able to express a function by means of the generalized polynomials on time scales and its higher-order delta-derivatives.

In this paper, we will frequently use the following result, which can be found in [1, pp. 338].

Mathematics subject classification (2010): 34A40, 39A13, 26D15.
Keywords and phrases: Muirhead's inequality, Opial's inequality, Taylor's formula, time scales.

A USEFUL INEQUALITY. *Let $k \geqslant 2$ be an integer and $x_i \geqslant 0$ be reals for all $i \in [1,k]_{\mathbb{N}}$, then*

$$\left(\sum_{i=1}^{k} x_i\right)^{\alpha} \leqslant \left\{\begin{matrix} 1, & \text{if } 0 \leqslant \alpha \leqslant 1 \\ k^{\alpha-1}, & \text{if } \alpha \geqslant 1 \end{matrix}\right\} \sum_{i=1}^{k} x_i^{\alpha}. \tag{1}$$

Now, we state Muirhead's inequality, which enables us to obtain new interesting results (see [5, Sections 2.18 and 2.19]).

MUIRHEAD'S INEQUALITY. *Let S^k be the symmetry group of the set $[1,k]_{\mathbb{N}}$, where $k \in \mathbb{N}$, and $A := (\alpha_1, \dots, \alpha_k)$, $B := (\beta_1, \dots, \beta_k)$ be two vectors with nonnegative and nonincreasing entries (i.e., $\alpha_j \geqslant \alpha_{j+1} \geqslant 0$ and $\beta_j \geqslant \beta_{j+1} \geqslant 0$ for all $j \in [1,k)_{\mathbb{N}}$) and $\sum_{i=1}^{j} \alpha_i \geqslant \sum_{i=1}^{j} \beta_i$ for all $j \in [1,k)_{\mathbb{N}}$ and $\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \beta_i$. Then it is said that A majorizes B (we prefer the notation $A \triangleright B$), and the following inequality is true:*

$$\sum_{\pi \in S^k} \prod_{i=1}^{k} x_{\pi_i}^{\alpha_i} \geqslant \sum_{\pi \in S^k} \prod_{i=1}^{k} x_{\pi_i}^{\beta_i},$$

where π_i denotes the i-th component of the permutation π, and $x_i \geqslant 0$ holds for all $i \in [1,k]_{\mathbb{N}}$.

One can easily see that Muirhead's inequality reduces to the well-known inequality between arithmetic and geometric means for $(2,0) \triangleright (1,1)$.

2. Preliminaries on time scales

Here, we quote some definitions and results from [3], which will be applied in our proofs.

DEFINITION 2.1. A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

DEFINITION 2.2. On an arbitrary time scale $\mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t,\infty)_{\mathbb{T}}$ for $t \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty,t)_{\mathbb{T}}$ for $t \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$. Here it is assumed that $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$.

DEFINITION 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided that it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exists (finite) at left-dense points of $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, and $C_{\mathrm{rd}}^1(\mathbb{T},\mathbb{R})$ denotes the set of functions whose delta-derivative belongs to $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. Similarly, the set $C_{\mathrm{rd}}^n(\mathbb{T},\mathbb{R})$ can be defined.

Now, we give the definition of the generalized Taylor Monomials as follows:

$$h_k(t,s) := \begin{cases} 1, & k = 0 \\ \displaystyle\int_s^t h_{k-1}(\xi,s)\Delta\xi, & k \in \mathbb{N} \end{cases}$$

for all $s,t \in \mathbb{T}$.

The following result can be obtained from [6, Theorem 3.3].

TAYLOR'S FORMULA. *Let $r \in \mathbb{N}$ and $f \in C^r_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, then for $s \in \mathbb{T}^{\kappa^{r-1}}$, we have*

$$f(t) = \sum_{i=0}^{r-1} h_i(t,s) f^{\Delta^i}(s) + \int_s^t h_{r-1}(t,\sigma(\xi)) f^{\Delta^r}(\xi)\Delta\xi$$

for all $t \in \mathbb{T}$.

Also, we need the following integral inequality (see [3, Theorem 6.13]).

HÖLDER'S INEQUALITY. *Let $a,b \in \mathbb{T}$. For $f,g \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, we have*

$$\int_a^b \big|f(\xi)g(\xi)\big|\Delta\xi \leqslant \left(\int_a^b \big|f(\xi)\big|^p \Delta\xi\right)^{\frac{1}{p}} \left(\int_a^b \big|g(\xi)\big|^q \Delta\xi\right)^{\frac{1}{q}},$$

where $p>1$ and $1/p+1/q=1$.

3. Main results

From now on, we make use of the following notations and definitions. Let $n \in \mathbb{N}$ satisfy $n \geqslant 2$, $\mathbb{T}_i$ be a time scale for all $i \in [1,n]_{\mathbb{N}}$, and $a_i, b_i \in \mathbb{T}_i$ satisfy $-\infty < a_i \leqslant b_i < \infty$ for all $i \in [1,n]_{\mathbb{N}}$. Set $\Omega_n := [a_1,b_1]_{\mathbb{T}_1} \times \cdots \times [a_n,b_n]_{\mathbb{T}_n}$, and for $f \in C_{\mathrm{rd}}(\Omega_n,\mathbb{R})$ (f belongs to $C_{\mathrm{rd}}([a_i,b_i]_{\mathbb{T}_i},\mathbb{R})$ for each $i \in [1,n]_{\mathbb{N}}$) denote

$$\int_{\Omega_n} f(\xi)\Delta\xi = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(\xi_1,\ldots,\xi_n)\Delta\xi_1 \cdots \Delta\xi_n, \quad \mathrm{grad}^r_n f(t) := \left(\frac{\partial^r}{\Delta t_1^r} f(t), \ldots, \frac{\partial^r}{\Delta t_n^r} f(t)\right)$$

and

$$\big|\big|\mathrm{grad}^r_n f(t)\big|\big| := \left(\sum_{i=1}^n \left|\frac{\partial^r}{\Delta t_i^r} f(t)\right|^2\right)^{\frac{1}{2}},$$

where $t = (t_1,t_2,\ldots,t_n) \in \Omega_n$ and $r \in \mathbb{N}$.

THEOREM 3.1. *Let p and q be real constants such that $p \geqslant 1$ and $q \geqslant 1$, and $f \in C^1_{\mathrm{rd}}(\Omega_n,\mathbb{R})$ (f belongs to $C^1_{\mathrm{rd}}([a_i,b_i]_{\mathbb{T}_i},\mathbb{R})$ for each $i \in [1,n]_{\mathbb{N}}$) vanish on the boundary $\partial\Omega_n$ of Ω_n. Then, the following inequality holds*

$$\int_{\Omega_n} \big|f(\xi)\big|^p \big|\big|\mathrm{grad}^1_n f(\xi)\big|\big|^q \Delta\xi \leqslant \frac{1}{2^p n}\left[\sum_{i=1}^n (b_i-a_i)^{\frac{p(p+q)}{q}}\right]^{\frac{q}{p+q}} \int_{\Omega_n} \big|\big|\mathrm{grad}^1_n f(\xi)\big|\big|^{p+q}\Delta\xi. \tag{2}$$

Proof. Clearly, because of the boundary condition on f, for all $t=(t_1,\dots,t_n)\in\Omega_n$ and all $i\in[1,n]_{\mathbb{N}}$, we have

$$\int_{a_i}^{t_i}\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\Delta\xi_i=f(t;t_i)-f(t;a_i)=f(t) \tag{3}$$

and similarly

$$\int_{t_i}^{b_i}\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\Delta\xi_i=-f(t), \tag{4}$$

where $f(t;s_i):=f(t_1,\dots,t_{i-1},s_i,t_{i+1},\dots,t_n)$ for all $i\in[1,n]_{\mathbb{N}}$. Therefore, from (3) and (4), we have

$$f(t)=\frac{1}{n}\sum_{i=1}^{n}\int_{a_i}^{t_i}\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\Delta\xi_i \tag{5}$$

and

$$f(t)=-\frac{1}{n}\sum_{i=1}^{n}\int_{t_i}^{b_i}\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\Delta\xi_i \tag{6}$$

for all $t\in\Omega_n$. Taking (5) and (6) into account, we see that

$$\begin{aligned}|f(t)|&=\frac{1}{2}\big(|f(t)|+|f(t)|\big)=\frac{1}{2n}\left(\left|\sum_{i=1}^{n}\int_{a_i}^{t_i}\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\Delta\xi_i\right|+\left|\sum_{i=1}^{n}\int_{t_i}^{b_i}\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\Delta\xi_i\right|\right)\\&\leqslant\frac{1}{2n}\left(\sum_{i=1}^{n}\int_{a_i}^{t_i}\left|\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\right|\Delta\xi_i+\sum_{i=1}^{n}\int_{t_i}^{b_i}\left|\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\right|\Delta\xi_i\right)\\&=\frac{1}{2n}\sum_{i=1}^{n}\int_{a_i}^{b_i}\left|\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\right|\Delta\xi_i\end{aligned} \tag{7}$$

holds for all $t\in\Omega_n$. Applying Hölder's inequality with the indices $(p+q)/(p+q-1)$ and $p+q$ to the right-hand side of (7), we get

$$|f(t)|\leqslant\frac{1}{2n}\sum_{i=1}^{n}\left[(b_i-a_i)^{\frac{p+q-1}{p+q}}\left(\int_{a_i}^{b_i}\left|\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\right|^{p+q}\Delta\xi_i\right)^{\frac{1}{p+q}}\right] \tag{8}$$

for all $t\in\Omega_n$. Raising both sides of (8) to p-th power, and applying (1), we have

$$|f(t)|^p\leqslant n^{p-1}\left(\frac{1}{2n}\right)^p\sum_{i=1}^{n}\left[(b_i-a_i)^{\frac{p(p+q-1)}{p+q}}\left(\int_{a_i}^{b_i}\left|\frac{\partial}{\Delta\xi_i}f(t;\xi_i)\right|^{p+q}\Delta\xi_i\right)^{\frac{p}{p+q}}\right] \tag{9}$$

for all $t \in \Omega_n$. Multiplying both sides of (9) by $|\nabla_n^1 f|^q$, we have

$$|f(t)|^p |\nabla_n^1 f(t)|^q \leqslant \frac{1}{2^p n} \sum_{i=1}^{n} \left[(b_i - a_i)^{\frac{p(p+q-1)}{p+q}} |\nabla_n^1 f(t)|^q \left(\int_{a_i}^{b_i} \left| \frac{\partial}{\Delta \xi_i} f(t;\xi_i) \right|^{p+q} \Delta \xi_i \right)^{\frac{p}{p+q}} \right] \tag{10}$$

for all $t \in \Omega_n$. Here we would like to note that

$$\int_{a_i}^{b_i} \left| \frac{\partial}{\Delta \xi_i} f(t;\xi_i) \right|^{p+q} \Delta \xi_i$$

does not depend on i-th component t_i. Integrating both sides of (10) on Ω_n with respect to t, we obtain

$$\begin{aligned}
&\int_{\Omega_n} |f(\xi)|^p \left|\left|\mathrm{grad}_n^1 f(\xi)\right|\right|^q \Delta \xi \\
&\quad \leqslant \frac{1}{2^p n} \sum_{i=1}^{n} \left[(b_i - a_i)^{\frac{p(p+q-1)}{p+q}} \int_{\Omega_n} \left|\left|\mathrm{grad}_n^1 f(\xi)\right|\right|^q \left(\int_{a_i}^{b_i} \left| \frac{\partial}{\Delta \zeta_i} f(\xi;\zeta_i) \right|^{p+q} \Delta \zeta_i \right)^{\frac{p}{p+q}} \Delta \xi \right] \\
&\quad \leqslant \frac{1}{2^p n} \sum_{i=1}^{n} \left[(b_i - a_i)^{\frac{p(p+q-1)}{p+q}} \left(\int_{\Omega_n} \left|\left|\mathrm{grad}_n^1 f(\xi)\right|\right|^{p+q} \Delta \xi \right)^{\frac{q}{p+q}} \right. \\
&\qquad \left. \times \left(\int_{\Omega_n} \int_{a_i}^{b_i} \left| \frac{\partial}{\Delta \zeta_i} f(\xi;\zeta_i) \right|^{p+q} \Delta \zeta_i \Delta \xi \right)^{\frac{p}{p+q}} \right] \\
&\quad = \frac{1}{2^p n} \sum_{i=1}^{n} \left[(b_i - a_i)^{\frac{p(p+q-1)}{p+q}} (b_i - a_i)^{\frac{p}{p+q}} \left(\int_{\Omega_n} \left|\left|\mathrm{grad}_n^1 f(\xi)\right|\right|^{p+q} \Delta \xi \right)^{\frac{q}{p+q}} \right. \\
&\qquad \left. \times \left(\int_{\Omega_n} \left| \frac{\partial}{\Delta \xi_i} f(\xi) \right|^{p+q} \Delta \xi \right)^{\frac{p}{p+q}} \right] \\
&\quad = \frac{1}{2^p n} \sum_{i=1}^{n} \left[(b_i - a_i)^p \left(\int_{\Omega_n} \left|\left|\mathrm{grad}_n^1 f(\xi)\right|\right|^{p+q} \Delta \xi \right)^{\frac{q}{p+q}} \left(\int_{\Omega_n} \left| \frac{\partial}{\Delta \xi_i} f(\xi) \right|^{p+q} \Delta \xi \right)^{\frac{p}{p+q}} \right].
\end{aligned} \tag{11}$$

Now, consider

$$\begin{aligned}
\int_{\Omega_n} \sum_{i=1}^{n} \left| \frac{\partial}{\Delta \xi_i} f(\xi) \right|^{p+q} \Delta \xi &= \int_{\Omega_n} \left(\left(\sum_{i=1}^{n} \left| \frac{\partial}{\Delta \xi_i} f(\xi) \right|^{p+q} \right)^{\frac{2}{p+q}} \right)^{\frac{p+q}{2}} \Delta \xi \\
&\leqslant \int_{\Omega_n} \left|\left|\mathrm{grad}_n^1 f(\xi)\right|\right|^{p+q} \Delta \xi,
\end{aligned} \tag{12}$$

where we have applied (1) to the right-hand side on passing to the last step. Applying Hölder's inequality and (1) to the right-hand side of (11), and taking (12) into account, we have

$$
\begin{aligned}
&\int_{\Omega_n} |f(\xi)|^p \big|\big|\mathrm{grad}_n^1 f(\xi)\big|\big|^q \Delta\xi \\
&\quad \leqslant \frac{1}{2^p n} \sum_{i=1}^{n} \left[(b_i - a_i)^{\frac{p(p+q)}{q}} \left(\int_{\Omega_n} \big|\big|\mathrm{grad}_n^1 f(\xi)\big|\big|^{p+q} \Delta\xi \right)^{\frac{q}{p+q}} \left(\int_{\Omega_n} \left| \frac{\partial}{\Delta\xi_i} f(\xi) \right|^{p+q} \Delta\xi \right)^{\frac{p}{p+q}} \right] \\
&\quad = \frac{1}{2^p n} \left(\sum_{i=1}^{n} (b_i - a_i)^{\frac{p(p+q)}{p+q}} \int_{\Omega_n} \big|\big|\mathrm{grad}_n^1 f(\xi)\big|\big|^{p+q} \Delta\xi \right)^{\frac{q}{p+q}} \left(\sum_{i=1}^{n} \int_{\Omega_n} \left| \frac{\partial}{\Delta\xi_i} f(\xi) \right|^{p+q} \Delta\xi \right)^{\frac{p}{p+q}} \\
&\quad \leqslant \frac{1}{2^p n} \left(\sum_{i=1}^{n} (b_i - a_i)^{\frac{p(p+q)}{p+q}} \right)^{\frac{q}{p+q}} \left(\int_{\Omega_n} \big|\big|\mathrm{grad}_n^1 f(\xi)\big|\big|^{p+q} \Delta\xi \right)^{\frac{q}{p+q}} \\
&\qquad \times \left(\int_{\Omega_n} \big|\big|\mathrm{grad}_n^1 f(\xi)\big|\big|^{p+q} \Delta\xi \right)^{\frac{p}{p+q}} \\
&\quad = \frac{1}{2^p n} \left(\sum_{i=1}^{n} (b_i - a_i)^{\frac{p(p+q)}{p+q}} \right)^{\frac{q}{p+q}} \int_{\Omega_n} \big|\big|\mathrm{grad}_n^1 f(\xi)\big|\big|^{p+q} \Delta\xi.
\end{aligned}
$$

The proof is therefore completed. □

The following result considers several functions.

THEOREM 3.2. *Let m be an integer satisfying $m \geqslant 2$, p_i and q_i be real constants such that $p_i \geqslant 1$ and $q_i \geqslant 1$ for all $i \in [1,m]_{\mathbb{N}}$, and $f_i \in C^1_{\mathrm{rd}}(\Omega_n,\mathbb{R})$ vanish on the boundary $\partial\Omega_n$ of Ω_n for all $i \in [1,k]_{\mathbb{N}}$. Then, the following inequality holds*

$$
\begin{aligned}
\int_{\Omega_n} \prod_{i=1}^{k} \big|f_i(\xi)\big|^{p_i} \big|\big|\mathrm{grad}_n^1 f_i(\xi)\big|\big|^{q_i} \Delta\xi \leqslant \frac{1}{kn} \sum_{i=1}^{k} &\left[\frac{1}{2^{kp_i}} \left(\sum_{j=1}^{n} (b_j - a_j)^{\frac{kp_i(p_i+q_i)}{q_i}} \right)^{\frac{q_i}{p_i+q_i}} \right. \\
&\left. \times \int_{\Omega_n} \big|\big|\mathrm{grad}_n^1 f_i(\xi)\big|\big|^{k(p_i+q_i)} \Delta\xi \right].
\end{aligned} \tag{13}
$$

Proof. Applying the well-known inequality concerning the arithmetic mean and geometric mean, we get

$$
\begin{aligned}
\int_{\Omega_n} \prod_{i=1}^{k} \big|f_i(\xi)\big|^{p_i} \big|\big|\mathrm{grad}_n^1 f_i(\xi)\big|\big|^{q_i} \Delta\xi &= \int_{\Omega_n} \left(\prod_{i=1}^{k} \big|f_i(\xi)\big|^{kp_i} \big|\big|\mathrm{grad}_n^1 f_i(\xi)\big|\big|^{kq_i} \right)^{\frac{1}{k}} \Delta\xi \\
&\leqslant \frac{1}{k} \sum_{i=1}^{k} \int_{\Omega_n} \big|f_i(\xi)\big|^{kp_i} \big|\big|\mathrm{grad}_n^1 f_i(\xi)\big|\big|^{kq_i} \Delta\xi.
\end{aligned} \tag{14}
$$

Applying Theorem 3.1 on the right-hand side of (14), we see that (13) is true. $\square$

THEOREM 3.3. *Let p and q be real constants such that $p \geqslant 1$ and $q \geqslant 1$, r be a fixed integer satisfying $r \geqslant 1$, and $f \in C_{\mathrm{rd}}^{r}(\Omega_n, \mathbb{R})$ and each of its partial delta-derivatives up to the order $r-1$ vanish on the boundary $\partial\Omega_n$ of Ω_n. Then, the following inequality holds*

$$\begin{aligned}
&\int_{\Omega_n} |f(\xi)|^p \left|\left|\operatorname{grad}_n^r f(\xi)\right|\right|^q \Delta\xi \\
&\qquad \leqslant \frac{1}{2^p n} \left[\sum_{i=1}^{n} (b_i - a_i)^{\frac{p}{p+q}} \left(\int_{a_i}^{b_i} \left| h_{r-1}(t_i, \sigma(\xi_i)) \right|^{\frac{p+q}{p+q-1}} \Delta\xi_i \right)^{\frac{p(p+q-1)}{p+q}} \right]^{\frac{q}{p+q}} \qquad (15) \\
&\qquad \times \int_{\Omega_n} \left|\left|\operatorname{grad}_n^r f(\xi)\right|\right|^{p+q} \Delta\xi.
\end{aligned}$$

Proof. Considering Taylor's formula given in Section 2 together with the boundary conditions, we have

$$\int_{a_i}^{t_i} h_{r-1}(t_i, \sigma(\xi_i)) \frac{\partial^r}{\Delta\xi_i^r} f(t;\xi_i) \Delta\xi_i = f(t) \tag{16}$$

and similarly

$$\int_{t_i}^{b_i} h_{r-1}(t_i, \sigma(\xi_i)) \frac{\partial^r}{\Delta\xi_i^r} f(t;\xi_i) \Delta\xi_i = -f(t), \tag{17}$$

for all $t = (t_1, \ldots, t_n) \in \Omega_n$ and all $i \in [1,n]_{\mathbb{N}}$. From (16) and (17), we get

$$\begin{aligned}
|f(t)| &= \frac{1}{2n} \left(\left| \sum_{i=1}^{n} \int_{a_i}^{t_i} h_{r-1}(t_i, \sigma(\xi_i)) \frac{\partial^r}{\Delta\xi_i^r} f(t;\xi_i) \Delta\xi_i \right| \right. \\
&\quad \left. + \left| \sum_{i=1}^{n} \int_{t_i}^{b_i} h_{r-1}(t_i, \sigma(\xi_i)) \frac{\partial^r}{\Delta\xi_i^r} f(t;\xi_i) \Delta\xi_i \right| \right) \\
&\leqslant \frac{1}{2n} \left(\sum_{i=1}^{n} \int_{a_i}^{t_i} \left| h_{r-1}(t_i, \sigma(\xi_i)) \right| \left| \frac{\partial^r}{\Delta\xi_i^r} f(t;\xi_i) \right| \Delta\xi_i \right. \\
&\quad \left. + \sum_{i=1}^{n} \int_{t_i}^{b_i} \left| h_{r-1}(t_i, \sigma(\xi_i)) \right| \left| \frac{\partial^r}{\Delta\xi_i^r} f(t;\xi_i) \right| \Delta\xi_i \right) \\
&\leqslant \frac{1}{2n} \sum_{i=1}^{n} \int_{a_i}^{t_i} \left| h_{r-1}(t_i, \sigma(\xi_i)) \right| \left| \frac{\partial^r}{\Delta\xi_i^r} f(t;\xi_i) \right| \Delta\xi_i
\end{aligned}$$

holds for all $t \in \Omega_n$. Following similar steps to that of Theorem 3.1, one can easily obtain (15). $\square$

Finally, in the following result, we combine Theorems 3.2 and 3.3. Since the proof is very simple, we omit it.

THEOREM 3.4. *Let m be a fixed integer satisfying $m \geqslant 2$, p_i and q_i be real constants such that $p_i \geqslant 1$ and $q_i \geqslant 1$ for all $i \in [1,m]_{\mathbb{N}}$, r be a fixed integer satisfying $r \geqslant 1$, and $f_i \in C^r_{\mathrm{rd}}(\Omega_n, \mathbb{R})$ and each of their partial derivatives up to the order $r-1$ vanish on the boundary $\partial\Omega_n$ of Ω_n for all $i \in [1,k]_{\mathbb{N}}$. Then, the following inequality holds*

$$\int\limits_{\Omega_n} \prod_{i=1}^{k} |f_i(\xi)|^{p_i} \big|\big|\mathrm{grad}_n^r f_i(\xi)\big|\big|^{q_i} \Delta\xi$$
$$\leqslant \frac{1}{kn} \sum_{i=1}^{n} \left[\frac{1}{2^{kp_i}} \left(\sum_{j=1}^{k} (b_j - a_j)^{\frac{p_j}{p_j+q_j}} \left(\int\limits_{a_j}^{b_j} |h_{r-1}(t_j, \sigma(\xi_j))|^{\frac{k(p_j+q_j)}{k(p_j+q_j)-1}} \Delta\xi_j \right)^{\frac{k(p_j+q_j)-1}{k(p_j+q_j)}} \right)^{\frac{q_i}{p_i+q_i}} \right.$$
$$\left. \times \int\limits_{\Omega_n} \big|\big|\mathrm{grad}_n^r f_i(\xi)\big|\big|^{k(p_i+q_i)} \Delta\xi \right]. \tag{18}$$

4. Final comments

Theorems 3.2 and 3.4 in Section 3 can be extended by using Muirhead's inequality. We state these results as corollaries below.

COROLLARY 4.1. *In addition to assumptions of Theorem 3.2, suppose that there exists $B = (\beta_1, \ldots, \beta_k)$ with nonnegative and nonincreasing entries such that $\sum_{i=1}^{j} \beta_i \geqslant j$ for all $j \in [1,k)_{\mathbb{N}}$ and $\sum_{i=1}^{k} \beta_i = k$. Then the right-hand side of (13) can be replaced by*

$$\frac{1}{k!kn} \sum_{\pi \in S^k} \sum_{i=1}^{k} \left[\frac{1}{2^{k\beta_i p_{\pi_i}}} \left(\sum_{j=1}^{n} (b_j - a_j)^{\frac{k\beta_i p_{\pi_i}(p_i+q_{\pi_i})}{q_{\pi_i}}} \right)^{\frac{q_{\pi_i}}{p_{\pi_i}+q_{\pi_i}}} \int\limits_{\Omega_n} \big|\big|\mathrm{grad}_n^1 f_{\pi_i}(\xi)\big|\big|^{k\beta_i(p_{\pi_i}+q_{\pi_i})} \Delta\xi \right].$$

Proof. In the proof of Theorem 3.2 before applying the arithmetic mean and geometric mean inequality, we apply Muirhead's inequality to get

$$\int\limits_{\Omega_n} \prod_{i=1}^{k} |f_i(\xi)|^{p_i} \big|\big|\mathrm{grad}_n^1 f_i(\xi)\big|\big|^{q_i} \Delta\xi$$
$$= \frac{1}{k!} \int\limits_{\Omega_n} \sum_{\pi \in S^k} \prod_{i=1}^{k} |f_{\pi_i}(\xi)|^{p_{\pi_i}} \big|\big|\mathrm{grad}_n^1 f_{\pi_i}(\xi)\big|\big|^{q_{\pi_i}} \Delta\xi$$

$$\leqslant \frac{1}{k!} \int_{\Omega_n} \sum_{\pi \in S^k} \prod_{i=1}^{k} \left| f_{\pi_i}(\xi) \right|^{\beta_i p_{\pi_i}} \left| \left| \operatorname{grad}_n^1 f_{\pi_i}(\xi) \right| \right|^{\beta_i q_{\pi_i}} \Delta\xi$$

$$= \frac{1}{k!} \sum_{\pi \in S^k} \int_{\Omega_n} \left(\prod_{i=1}^{k} \left| f_{\pi_i}(\xi) \right|^{k\beta_i p_{\pi_i}} \left| \left| \operatorname{grad}_n^1 f_{\pi_i}(\xi) \right| \right|^{k\beta_i q_{\pi_i}} \right)^{\frac{1}{k}} \Delta\xi$$

$$\leqslant \frac{1}{k!k} \sum_{\pi \in S^k} \sum_{i=1}^{k} \int_{\Omega_n} \left| f_{\pi_i}(\xi) \right|^{k\beta_i p_{\pi_i}} \left| \left| \operatorname{grad}_n^1 f_{\pi_i}(\xi) \right| \right|^{k\beta_i q_{\pi_i}} \Delta\xi.$$

The rest of the proof is similar to that of Theorem 3.2, and we therefore omit. □

COROLLARY 4.2. *In addition to assumptions of Theorem* 3.4*, suppose that there exists* $B = (\beta_1, \ldots, \beta_k)$ *with nonnegative and nonincreasing entries such that* $\sum_{i=1}^{j} \beta_i \geqslant j$ *for all* $j \in [1,k)_{\mathbb{N}}$ *and* $\sum_{i=1}^{k} \beta_i = k$*. Then the right-hand side of* (18) *can be replaced by*

$$\frac{1}{k!kn} \sum_{\pi \in S^k} \sum_{i=1}^{k} \left[\frac{1}{2^{k\beta_i p_{\pi_i}}} \left(\sum_{j=1}^{k} (b_j - a_j)^{\frac{p_j}{p_j+q_j}} \right.\right.$$
$$\left. \times \left(\int_{a_j}^{b_j} \left| h_{r-1}(t_j, \sigma(\xi_j)) \right|^{\frac{k\beta_i(p_j+q_j)}{k\beta_i(p_j+q_j)-1}} \Delta\xi_j \right)^{\frac{k\beta_i(p_j+q_j)-1}{k\beta_i(p_j+q_j)}} \right)^{\frac{q_{\pi_i}}{p_{\pi_i}+q_{\pi_i}}}$$
$$\left. \times \int_{\Omega_n} \left| \left| \operatorname{grad}_n^r f_{\pi_i}(\xi) \right| \right|^{k\beta_i(p_{\pi_i}+q_{\pi i})} \Delta\xi \right].$$

Acknowledgement. The authors are thankful to the referee for his/her careful reading of the manuscript and valuable comments.

REFERENCES

[1] D. S. MITRINOVIC, *Analytic Inequalities*, Springer-Verlag, New York, 1970.

[2] R. P. AGARWAL AND P. Y. H. PANG, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Academic Publishers, Dordrecht, 1995.

[3] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Boston, MA: Birkhäuser Boston Inc., 2001.

[4] M. BOHNER AND B. KAYMAKÇALAN, *Opial inequalities on time scales*, Ann. Polon. Math., **77**, 1 (2001), 11–20.

[5] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1988.

[6] B. KARPUZ AND U. M. ÖZKAN, *Some generalizations for Opial's inequality involving several functions and their derivatives of arbitrary order on arbitrary time scales*, Math. Inequal. Appl., (2009). (Accepted for publication)

[7] B. G. PACHPATTE, *On Opial-type integral inequalities*, J. Math. Anal. Appl., **120**, 2 (1986), 547–556.

[8] B. G. PACHPATTE, *On multidimensional Opial-type inequalities*, J. Math. Anal. Appl., **126**, 1 (1987), 85–89.

[9] B. G. PACHPATTE, *A note on Opial and Wirtinger type discrete inequalities*, J. Math. Anal. Appl., **127**, 2 (1987), 470–474.
[10] Z. OPIAL, *Sur une inégalité*, Ann. Polon. Math., **8** (1960), 29–32.

(Received February 11, 2009)

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