A NOTE ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

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Abstract. By means of the Ruscheweyh derivative we define a new class $\mathcal{BR}_{n}(m, \mu, \alpha)$ involving functions $f \in \mathcal{A}_n$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

1. Introduction and definitions

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in $U$, $n \in \mathbb{N} = \{1, 2, \ldots\}$.

Let $\mathcal{S}_n$ denote the subclass of functions that are univalent in $U$.

By $\mathcal{S}_n^*(\alpha)$ we denote a subclass of $\mathcal{A}_n$ consisting of starlike univalent functions of order $\alpha$, $0 \leq \alpha < 1$ which satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in U. \quad (1.2)$$

Further, a function $f$ belonging to $\mathcal{S}_n$ is said to be convex of order $\alpha$ in $U$, if and only if

$$\text{Re} \left( \frac{zf''(z)}{f'(z)} + 1 \right) > \alpha, \quad z \in U \quad (1.3)$$

for some $\alpha$, $0 \leq \alpha < 1$. We denote by $\mathcal{K}_n(\alpha)$ the class of functions in $\mathcal{S}_n$ which are convex of order $\alpha$ in $U$ and denote by $\mathcal{R}_n(\alpha)$ the class of functions in $\mathcal{A}_n$ which satisfy

$$\text{Re} f'(z) > \alpha, \quad z \in U. \quad (1.4)$$

It is well known that $\mathcal{K}_n(\alpha) \subset \mathcal{S}_n^*(\alpha) \subset \mathcal{S}_n$.

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If \( f \) and \( g \) are analytic functions in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \), if there is a function \( w \) analytic in \( U \), with \( w(0) = 0 \), \( |w(z)| < 1 \), for all \( z \in U \) such that \( f(z) = g(w(z)) \) for all \( z \in U \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subseteq g(U) \).

In [3] Ruscheweyh has defined the operator \( D^m : \mathcal{A}_n \to \mathcal{A}_n, \ n \in \mathbb{N}, \ m \in \mathbb{N} \cup \{0\}, \)

\[
D^0 f(z) = f(z) \\
D^1 f(z) = zf'(z) \\
(m + 1)D^{m+1} f(z) = z[D^m f(z)]' + mD^m f(z), \quad z \in U.
\]

We note that if \( f \in \mathcal{A}_n \), then

\[
D^m f(z) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} a_j z^j, \quad z \in U.
\]

To prove our main theorem we shall need the following lemma.

**Lemma 1.1.** [2] Let \( p \) be analytic in \( U \) with \( p(0) = 1 \) and suppose that

\[
\Re \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{1.5}
\]

Then \( \Re p(z) > \alpha \) for \( z \in U \) and \( 1/2 \leq \alpha < 1 \).

**2. Main results**

**Definition 2.1.** We say that a function \( f \in \mathcal{A}_n \) is in the class \( \mathcal{BR}_n(m, \mu, \alpha) \), \( n \in \mathbb{N}, \ m \in \mathbb{N} \cup \{0\}, \ \mu \geq 0, \ \alpha \in [0, 1) \) if

\[
\left| \frac{D^{m+1} f(z)}{z} \left( \frac{z}{D^m f(z)} \right)^\mu - 1 \right| < 1 - \alpha, \quad z \in U. \tag{2.1}
\]

**Remark 2.2.** The family \( \mathcal{BR}_n(m, \mu, \alpha) \) is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, \( \mathcal{BR}_n(0,1,\alpha) \equiv \mathcal{I}_n^*(\alpha) \), \( \mathcal{BR}_n(1,1,\alpha) \equiv \mathcal{K}_n(\alpha) \) and \( \mathcal{BR}_n(0,0,\alpha) \equiv \mathcal{R}_n(\alpha) \). Another interesting subclass is the special case \( \mathcal{BR}_1(0,2,\alpha) \equiv \mathcal{B}(\alpha) \) which has been introduced by Frasin and Darus [1] and also the class \( \mathcal{BR}_1(0,\mu,\alpha) \equiv \mathcal{B}(\mu,\alpha) \) which has been introduced by Frasin and Jahangiri [2].

In this note we provide a sufficient condition for functions to be in the class \( \mathcal{BR}_n(m, \mu, \alpha) \). Consequently, as a special case, we show that convex functions of order \( 1/2 \) are also members of the above defined family.
Theorem 2.3. For the function $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $1/2 \leq \alpha < 1$ if

$$(m + 2) \frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu(m + 1) \frac{D^{m+1}f(z)}{D^mf(z)} + \mu(m + 1) - (m + 2) < 1 - \beta z, \quad z \in U$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha}$$

then $f \in \mathcal{BR}_n(m, \mu, \alpha)$.

Proof. If we consider

$$p(z) = \frac{D^{m+1}f(z)}{z} \left( \frac{z}{D^mf(z)} \right)^\mu$$

then $p(z)$ is analytic in $U$ with $p(0) = 1$. A simple differentiation yields

$$\frac{zp'(z)}{p(z)} = (m + 2) \frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu(m + 1) \frac{D^{m+1}f(z)}{D^mf(z)} + \mu(m + 1) - (m + 2).$$

Using (2.2) we get

$$\Re \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}.$$

Thus, from Lemma 1.1 we deduce that

$$\Re \left\{ \frac{D^{m+1}f(z)}{z} \left( \frac{z}{D^mf(z)} \right)^\mu \right\} > \alpha.$$

Therefore, $f \in \mathcal{BR}_n(m, \mu, \alpha)$, by Definition 2.1.

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 2.4. If $f \in \mathcal{A}_n$ and

$$\Re \left\{ \frac{6zf'(z) + 6z^2f''(z) + z^3f'''(z)}{2zf'(z) + z^2f''(z)} - \frac{zf'''(z)}{f'(z)} \right\} > \frac{3}{2}, \quad z \in U$$

then

$$\Re \left\{ 1 + \frac{zf'''(z)}{f'(z)} \right\} > \frac{3}{2}, \quad z \in U.$$

That is, $f$ is convex of order $\frac{3}{2}$. 
COROLLARY 2.5. If \( f \in \mathcal{A}_n \) and

\[
\operatorname{Re} \left\{ \frac{4zf'(z) + 5z^2f''(z) + z^3f'''(z)}{2zf'(z) + z^2f''(z)} \right\} > \frac{1}{2}, \quad z \in U
\]

then

\[
\operatorname{Re} \left\{ f'(z) + \frac{1}{2}z^2f''(z) \right\} > \frac{1}{2}, \quad z \in U. \tag{2.7}
\]

COROLLARY 2.6. If \( f \in \mathcal{A}_n \) and

\[
\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in U
\]

then

\[
\operatorname{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{2.9}
\]

In another words, if the function \( f \) is convex of order \( \frac{1}{2} \) then \( f \in \mathcal{BR}_n(0,0,\frac{1}{2}) \equiv \mathcal{R}_n\left(\frac{1}{2}\right) \).

COROLLARY 2.7. If \( f \in \mathcal{A}_n \) and

\[
\operatorname{Re} \left\{ \frac{zf''(z) - zf'(z)}{f'(z)} \right\} > -\frac{3}{2}, \quad z \in U
\]

then \( f \) is starlike of order \( \frac{1}{2} \) hence \( f \in \mathcal{BR}_n(0,1,\frac{1}{2}) \).

REFERENCES


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