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A NOTE ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

ALINA ALB LUPAŞ AND ADRIANA CĂTAŞ

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Abstract. By means of the Ruscheweyh derivative we define a new class $\mathscr{BR}_n(m,\mu,\alpha)$ involving functions $f \in \mathscr{A}_n$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

1. Introduction and definitions

Let $\mathscr{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and $\mathscr{H}(U)$ the space of holomorphic functions in U, $n \in \mathbb{N} = \{1,2,...\}$.

Let $\mathscr{S}_n$ denote the subclass of functions that are univalent in U.

By $\mathscr{S}_n^*(\alpha)$ we denote a subclass of $\mathscr{A}_n$ consisting of starlike univalent functions of order α, $0 \leqslant \alpha < 1$ which satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{1.2}$$

Further, a function f belonging to $\mathscr{S}_n$ is said to be convex of order α in U, if and only if

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in U \tag{1.3}$$

for some α, $(0 \leqslant \alpha < 1)$. We denote by $\mathscr{K}_n(\alpha)$ the class of functions in $\mathscr{S}_n$ which are convex of order α in U and denote by $\mathscr{R}_n(\alpha)$ the class of functions in $\mathscr{A}_n$ which satisfy

$$\operatorname{Re} f'(z) > \alpha, \quad z \in U. \tag{1.4}$$

It is well known that $\mathscr{K}_n(\alpha) \subset \mathscr{S}_n^*(\alpha) \subset \mathscr{S}_n$.

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If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In [3] Ruscheweyh has defined the operator $D^m : \mathscr{A}_n \to \mathscr{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= z f'(z) \\ (m+1) D^{m+1} f(z) &= z [D^m f(z)]' + m D^m f(z), \quad z \in U. \end{aligned}$$

We note that if $f \in \mathscr{A}_n$, then

$$D^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} a_j z^j, \ z \in U.$$

To prove our main theorem we shall need the following lemma.

LEMMA 1.1. [2] *Let p be analytic in U with $p(0) = 1$ and suppose that*

$$\operatorname{Re}\left(1 + \frac{z p'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{1.5}$$

Then $\operatorname{Re} p(z) > \alpha$ *for* $z \in U$ *and* $1/2 \leqslant \alpha < 1$.

2. Main results

DEFINITION 2.1. We say that a function $f \in \mathscr{A}_n$ is in the class $\mathscr{BR}_n(m, \mu, \alpha)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geqslant 0$, $\alpha \in [0, 1)$ if

$$\left| \frac{D^{m+1} f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^{\mu} - 1 \right| < 1 - \alpha \qquad z \in U. \tag{2.1}$$

REMARK 2.2. The family $\mathscr{BR}_n(m, \mu, \alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathscr{BR}_n(0, 1, \alpha) \equiv \mathscr{S}_n^*(\alpha)$, $\mathscr{BR}_n(1, 1, \alpha) \equiv \mathscr{K}_n(\alpha)$ and $\mathscr{BR}_n(0, 0, \alpha) \equiv \mathscr{R}_n(\alpha)$. Another interesting subclass is the special case $\mathscr{BR}_1(0, 2, \alpha) \equiv \mathscr{B}(\alpha)$ which has been introduced by Frasin and Darus [1] and also the class $\mathscr{BR}_1(0, \mu, \alpha) \equiv \mathscr{B}(\mu, \alpha)$ which has been introduced by Frasin and Jahangiri [2].

In this note we provide a sufficient condition for functions to be in the class $\mathscr{BR}_n(m, \mu, \alpha)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

THEOREM 2.3. *For the function* $f \in \mathscr{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geqslant 0$, $1/2 \leqslant \alpha < 1$ *if*

$$(m+2)\frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu(m+1)\frac{D^{m+1}f(z)}{D^m f(z)} + \mu(m+1) - (m+2) \prec 1 - \beta z, \ z \in U \tag{2.2}$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha}$$

then $f \in \mathscr{BR}_n(m,\mu,\alpha)$.

Proof. If we consider

$$p(z) = \frac{D^{m+1}f(z)}{z}\left(\frac{z}{D^m f(z)}\right)^{\mu} \tag{2.3}$$

then $p(z)$ is analytic in U with $p(0) = 1$. A simple differentiation yields

$$\frac{zp'(z)}{p(z)} = (m+2)\frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu(m+1)\frac{D^{m+1}f(z)}{D^m f(z)} + \mu(m+1) - (m+2). \tag{2.4}$$

Using (2.2) we get

$$\operatorname{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}.$$

Thus, from Lemma 1.1 we deduce that

$$\operatorname{Re}\left\{\frac{D^{m+1}f(z)}{z}\left(\frac{z}{D^m f(z)}\right)^{\mu}\right\} > \alpha.$$

Therefore, $f \in \mathscr{BR}_n(m,\mu,\alpha)$, by Definition 2.1.

As a consequence of the above theorem we have the following interesting corollaries.

COROLLARY 2.4. *If* $f \in \mathscr{A}_n$ *and*

$$\operatorname{Re}\left\{\frac{6zf'(z) + 6z^2 f''(z) + z^3 f'''(z)}{2zf'(z) + z^2 f''(z)} - \frac{zf''(z)}{f'(z)}\right\} > \frac{3}{2}, \quad z \in U \tag{2.5}$$

then

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3}{2}, \quad z \in U. \tag{2.6}$$

That is, f *is convex of order* $\frac{3}{2}$.

COROLLARY 2.5. *If* $f \in \mathscr{A}_n$ *and*

$$\operatorname{Re}\left\{\frac{4zf'(z)+5z^2f''(z)+z^3f'''(z)}{2zf'(z)+z^2f''(z)}\right\} > \frac{1}{2}, \quad z \in U \tag{2.7}$$

then

$$\operatorname{Re}\left\{f'(z)+\frac{1}{2}zf''(z)\right\} > \frac{1}{2}, \quad z \in U. \tag{2.8}$$

COROLLARY 2.6. *If* $f \in \mathscr{A}_n$ *and*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U \tag{2.9}$$

then

$$\operatorname{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{2.10}$$

In another words, if the function f *is convex of order* $\frac{1}{2}$ *then* $f \in \mathscr{BR}_n(0,0,\frac{1}{2}) \equiv \mathscr{R}_n\left(\frac{1}{2}\right)$.

COROLLARY 2.7. *If* $f \in \mathscr{A}_n$ *and*

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U \tag{2.11}$$

then f *is starlike of order* $\frac{1}{2}$ *hence* $f \in \mathscr{BR}_n(0,1,\frac{1}{2})$.

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Alina Alb Lupaş
Department of Mathematics and Computer Science
University of Oradea
1 Universitatii Street
410087 Oradea
Romania
e-mail: dalb@uoradea.ro

Adriana Cătaş
Department of Mathematics and Computer Science
University of Oradea
1 Universitatii Street
410087 Oradea
Romania
e-mail: acatas@uoradea.ro