

A NOTE ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

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Abstract. By means of the Ruscheweyh derivative we define a new class $\mathcal{BR}_n(m, \mu, \alpha)$ involving functions $f \in \mathcal{A}_n$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

1. Introduction and definitions

Let \mathcal{A}_n denote the class of functions of the form

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U , $n \in \mathbb{N} = \{1, 2, \dots\}$.

Let \mathcal{S}_n denote the subclass of functions that are univalent in U .

By $\mathcal{S}_n^*(\alpha)$ we denote a subclass of \mathcal{A}_n consisting of starlike univalent functions of order α , $0 \leq \alpha < 1$ which satisfies

$$\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > \alpha, \quad z \in U. \tag{1.2}$$

Further, a function f belonging to \mathcal{S}_n is said to be convex of order α in U , if and only if

$$\operatorname{Re} \left(\frac{z f''(z)}{f'(z)} + 1 \right) > \alpha, \quad z \in U \tag{1.3}$$

for some α , ($0 \leq \alpha < 1$). We denote by $\mathcal{K}_n(\alpha)$ the class of functions in \mathcal{S}_n which are convex of order α in U and denote by $\mathcal{B}_n(\alpha)$ the class of functions in \mathcal{A}_n which satisfy

$$\operatorname{Re} f'(z) > \alpha, \quad z \in U. \tag{1.4}$$

It is well known that $\mathcal{K}_n(\alpha) \subset \mathcal{S}_n^*(\alpha) \subset \mathcal{S}_n$.

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If f and g are analytic functions in U , we say that f is subordinate to g , written $f \prec g$, if there is a function w analytic in U , with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In [3] Ruscheweyh has defined the operator $D^m : \mathcal{A}_n \rightarrow \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= z f'(z) \\ (m+1)D^{m+1} f(z) &= z [D^m f(z)]' + m D^m f(z), \quad z \in U. \end{aligned}$$

We note that if $f \in \mathcal{A}_n$, then

$$D^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j, \quad z \in U.$$

To prove our main theorem we shall need the following lemma.

LEMMA 1.1. [2] Let p be analytic in U with $p(0) = 1$ and suppose that

$$\operatorname{Re} \left(1 + \frac{z p'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \quad (1.5)$$

Then $\operatorname{Re} p(z) > \alpha$ for $z \in U$ and $1/2 \leq \alpha < 1$.

2. Main results

DEFINITION 2.1. We say that a function $f \in \mathcal{A}_n$ is in the class $\mathcal{BR}_n(m, \mu, \alpha)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $\alpha \in [0, 1)$ if

$$\left| \frac{D^{m+1} f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^\mu - 1 \right| < 1 - \alpha \quad z \in U. \quad (2.1)$$

REMARK 2.2. The family $\mathcal{BR}_n(m, \mu, \alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BR}_n(0, 1, \alpha) \equiv \mathcal{S}_n^*(\alpha)$, $\mathcal{BR}_n(1, 1, \alpha) \equiv \mathcal{K}_n(\alpha)$ and $\mathcal{BR}_n(0, 0, \alpha) \equiv \mathcal{R}_n(\alpha)$. Another interesting subclass is the special case $\mathcal{BR}_1(0, 2, \alpha) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [1] and also the class $\mathcal{BR}_1(0, \mu, \alpha) \equiv \mathcal{B}(\mu, \alpha)$ which has been introduced by Frasin and Jahangiri [2].

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BR}_n(m, \mu, \alpha)$. Consequently, as a special case, we show that convex functions of order $1/2$ are also members of the above defined family.

THEOREM 2.3. For the function $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $1/2 \leq \alpha < 1$ if

$$(m+2) \frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu(m+1) \frac{D^{m+1}f(z)}{D^m f(z)} + \mu(m+1) - (m+2) \prec 1 - \beta z, \quad z \in U \quad (2.2)$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha}$$

then $f \in \mathcal{BR}_n(m, \mu, \alpha)$.

Proof. If we consider

$$p(z) = \frac{D^{m+1}f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^\mu \quad (2.3)$$

then $p(z)$ is analytic in U with $p(0) = 1$. A simple differentiation yields

$$\frac{zp'(z)}{p(z)} = (m+2) \frac{D^{m+2}f(z)}{D^{m+1}f(z)} - \mu(m+1) \frac{D^{m+1}f(z)}{D^m f(z)} + \mu(m+1) - (m+2). \quad (2.4)$$

Using (2.2) we get

$$\operatorname{Re} \left(1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}.$$

Thus, from Lemma 1.1 we deduce that

$$\operatorname{Re} \left\{ \frac{D^{m+1}f(z)}{z} \left(\frac{z}{D^m f(z)} \right)^\mu \right\} > \alpha.$$

Therefore, $f \in \mathcal{BR}_n(m, \mu, \alpha)$, by Definition 2.1.

As a consequence of the above theorem we have the following interesting corollaries.

COROLLARY 2.4. If $f \in \mathcal{A}_n$ and

$$\operatorname{Re} \left\{ \frac{6zf'(z) + 6z^2f''(z) + z^3f'''(z)}{2zf'(z) + z^2f''(z)} - \frac{zf''(z)}{f'(z)} \right\} > \frac{3}{2}, \quad z \in U \quad (2.5)$$

then

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{3}{2}, \quad z \in U. \quad (2.6)$$

That is, f is convex of order $\frac{3}{2}$.

COROLLARY 2.5. If $f \in \mathcal{A}_n$ and

$$\operatorname{Re} \left\{ \frac{4zf'(z) + 5z^2f''(z) + z^3f'''(z)}{2zf'(z) + z^2f''(z)} \right\} > \frac{1}{2}, \quad z \in U \quad (2.7)$$

then

$$\operatorname{Re} \left\{ f'(z) + \frac{1}{2}zf''(z) \right\} > \frac{1}{2}, \quad z \in U. \quad (2.8)$$

COROLLARY 2.6. If $f \in \mathcal{A}_n$ and

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in U \quad (2.9)$$

then

$$\operatorname{Re} f'(z) > \frac{1}{2}, \quad z \in U. \quad (2.10)$$

In another words, if the function f is convex of order $\frac{1}{2}$ then $f \in \mathcal{BR}_n(0, 0, \frac{1}{2}) \equiv \mathcal{R}_n(\frac{1}{2})$.

COROLLARY 2.7. If $f \in \mathcal{A}_n$ and

$$\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} > -\frac{3}{2}, \quad z \in U \quad (2.11)$$

then f is starlike of order $\frac{1}{2}$ hence $f \in \mathcal{BR}_n(0, 1, \frac{1}{2})$.

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