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THE CORRECTED TWO–POINT WEIGHTED QUADRATURE FORMULAE

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Dedicated to Professor Josip Pečarić
in honour of his 60^{th} *birthday*

(*Communicated by A. Guessab*)

Abstract. We derive a corrected version of the weighted two-point quadrature formula, which provides a better approximation accuracy than the ordinary two-point quadrature formulae. In the corrected two-point formula the integral is approximated both with the values of the integrand in nodes $-x$ and x, and the values of its first derivative at the endpoints of the interval $[-1,1]$. The error estimates under various regularity conditions for such formulae are established. As special cases, the corrected two-point formulae of Gauss type are obtained. Also, corrected version of weighted trapezoid, midpoint, two-point Maclaurin and two-point Newton-Cotes formulae are considered.

1. Introduction

The ordinary two-point quadrature formula states that

$$\int_{-1}^{1} f(t)w(t)dt \approx A_w\,[f(-x)+f(x)]\,. \tag{1.1}$$

Here, $x \in [0,1]$, f is an integrable function defined on $[-1,1]$, $w:[-1,1]\to\mathbb{R}_+$ is an even integrable function called weight and

$$A_w = \int_0^1 w(t)dt. \tag{1.2}$$

Recently, A. Guessab and G. Schmeisser ([1]) studied a class of two-point formulae for $w\equiv 1$. Some of the most famous quadrature rules belong to this group: trapezoid formula ($x=1$), Newton-Cotes two-point formula ($x=\frac{1}{3}$), Maclaurin two-point formula

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($x = \frac{1}{2}$) and midpoint formula ($x = 0$). These formulae are exact for all polynomials of order $\leqslant 1$. Guessab and Schmeisser established the sharp estimates for the remainder

$$E(f,x) := \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f(x) + f(a+b-x)\right] \tag{1.3}$$

under various regularity conditions. They proved the following theorem

THEOREM 1. *Let f be a function defined on $[a,b]$ and having there a piecewise continuous $n-$th derivative. Let Q_n be any monic polynomial of degree n such that $Q_n(t) \equiv (-1)^n Q_n(a+b-t)$. Define*

$$K_n(t) = \begin{cases} (t-a)^n, & \text{for } a \leqslant t \leqslant x \\ Q_n(t), & \text{for } x < t \leqslant a+b-x \\ (t-b)^n, & \text{for } a+b-x < t \leqslant b. \end{cases} \tag{1.4}$$

Then, for the remainder in (1.3), we have

$$\begin{aligned} E(f;x) &= \sum_{\nu=1}^{n-1}\left[\frac{(x-a)^{\nu+1}}{(\nu+1)!} - \frac{Q_n^{(n-\nu-1)}(x)}{n!}\right]\frac{f^{(\nu)}(a+b-x)+(-1)^\nu f^{(\nu)}(x)}{b-a} \\ &+ \frac{(-1)^n}{n!(b-a)}\int_a^b K_n(t) f^{(n)}(t)\mathrm{d}t. \end{aligned} \tag{1.5}$$

A number of error estimates for the identity (1.5) are obtained, and various examples of the general two-point quadrature formula are given in [2].

The goal of this paper is to establish two-point quadrature formulae with a higher degree of exactness. Such formulae will contain the first derivative at the endpoints of the interval, that is

$$\int_{-1}^1 f(t)w(t)dt \approx A_w\left[f(-x)+f(x)\right] + B_w(x)\left[f'(1) - f'(-1)\right]. \tag{1.6}$$

Here,

$$B_w(x) = \frac{1}{2}\int_0^1 (t^2 - x^2)w(t)dt. \tag{1.7}$$

Quadrature formulae of this form are usually called corrected.

The main tool used are the $w-$harmonic sequences of functions and related weighted integral identity obtained in [3].

DEFINITION 1. Let $w : [a,b] \to \mathbb{R}$ be an integrable weight function and $w_k : [a,b] \to \mathbb{R}$ are differentiable functions for $k \in \mathbb{N}$. We say that $\{w_k\}_{k\in\mathbb{N}}$ is *$w-$harmonic sequence of functions* if for $k \geqslant 2$, $w_k'(t) = w_{k-1}(t)$ and $w_1'(t) = w(t)$, for $t \in [a,b]$.

Given a subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ of the interval $[a,b]$, let us consider different $w-$harmonic sequences of functions $\{w_{jk}\}_{k\in\mathbb{N}}$ on each interval $[x_{j-1},x_j],\ j \in \{1,2,\ldots,m\}$. Define

$$W_{n,w}(t,\sigma) = \begin{cases} w_{1n}(t),\ t \in [a,x_1] \\ w_{2n}(t),\ t \in (x_1,x_2] \\ \vdots \\ w_{mn}(t),\ t \in (x_{m-1},b], \end{cases} \tag{1.8}$$

Then for every function $f : [a,b] \to \mathbb{R}$ such that $f^{(n)}$ is piecewise continuous on $[a,b]$ it is proved in [3] that

$$\begin{aligned} \int_a^b w(t)f(t)\mathrm{d}t &= \sum_{k=1}^{n} (-1)^{k-1} \Big[w_{mk}(b) f^{(k-1)}(b) \\ &+ \sum_{j=1}^{m-1} \left[w_{jk}(x_j) - w_{j+1,k}(x_j) \right] f^{(k-1)}(x_j) - w_{1k}(a) f^{(k-1)}(a) \Big] \\ &+ (-1)^n \int_a^b W_{n,w}(t,\sigma) f^{(n)}(t)\mathrm{d}t. \end{aligned} \tag{1.9}$$

Throughout the paper we use the convention $0^0 = 1$.

2. General weighted corrected two-point formula

Let $w : [-1,1] \to \mathbb{R}$ be an even weight function and $x \in [0,1]$. Consider a subdivision

$$\sigma = \{x_0 = -1, x_1 = -x, x_2 = x, x_3 = 1\} \tag{2.1}$$

of the interval $[-1,1]$. Define

$$\begin{aligned} C_{1,w}(x) &:= -\int_0^1 (x-s)w(s)ds - B_w(x) \\ C_{2,w}(x) &:= -\frac{1}{6}\int_0^1 (x^3-s^3)w(s)ds - \frac{1}{2}B_w(x) \\ C_{3,w}(x) &:= -\frac{1}{120}\int_0^1 (x^5-s^5)w(s)ds - \frac{1}{24}B_w(x). \end{aligned}$$

For $k \in \mathbb{N}$ define

$$\begin{aligned} w_{1k}(t) &= \frac{1}{(k-1)!}\int_{-1}^t (t-s)^{k-1}w(s)ds - B_w(x)\frac{(t+1)^{k-2}}{(k-2)!}\mathbf{1}_{\{k\geqslant 2\}} \\ w_{2k}(t) &= \frac{1}{(k-1)!}\int_0^t (t-s)^{k-1}w(s)ds + C_{1,w}(x)\frac{t^{k-2}}{(k-2)!}\mathbf{1}_{\{k\geqslant 2\}} \\ &\quad + C_{2,w}(x)\frac{t^{k-4}}{(k-4)!}\mathbf{1}_{\{k\geqslant 4\}} + C_{3,w}(x)\frac{t^{k-6}}{(k-6)!}\mathbf{1}_{\{k\geqslant 6\}} \\ w_{3k}(t) &= \frac{-1}{(k-1)!}\int_t^1 (t-s)^{k-1}w(s)ds - B_w(x)\frac{(t-1)^{k-2}}{(k-2)!}\mathbf{1}_{\{k\geqslant 2\}}. \end{aligned}$$

LEMMA 1. *The sequences* $\{w_{jk}\}_{k\in\mathbb{N}}$ *are* $w-$*harmonic sequences of functions on* $(-1,1)$, *i.e. for* $j=1,2,3$ *we have*

$$\begin{aligned} w_{jk}'(t) &= w_{j,k-1}(t), \quad \forall t\in(-1,1), \quad \forall k\geqslant 2\\ w_{j1}'(t) &= w(t), \quad \forall t\in(-1,1). \end{aligned}$$

Proof. The proof follows by direct differentiation of functions w_{jk}.

LEMMA 2. *We have*

$$\begin{aligned} w_{1k}(t) &= (-1)^k w_{3k}(-t), \quad \forall t\in[-1,-x],\\ w_{2k}(t) &= (-1)^k w_{2k}(-t), \quad \forall t\in[-x,x]. \end{aligned}$$

Further, $w_{1k}(-1)=0$, $\forall k\neq 2$ *and* $w_{12}(-1)=-B_w(x)$.

Proof. Let $t\in[-1,-x]$. Then $-t\in[x,1]$ hence we obtain

$$\begin{aligned} w_{3k}(-t) &= \frac{-1}{(k-1)!}\int_{-t}^{1}(-t-s)^{k-1}w(s)ds - B_w(x)\frac{(-t-1)^{k-2}}{(k-2)!}1_{\{k\geqslant 2\}}\\ &= (y:=-s) = \frac{1}{(k-1)!}\int_{t}^{-1}(-t+y)^{k-1}w(-y)dy - B_w(x)\frac{(-t-1)^{k-2}}{(k-2)!}1_{\{k\geqslant 2\}}\\ &= \frac{(-1)^k}{(k-1)!}\int_{-1}^{t}(t-y)^{k-1}w(y)dy - (-1)^k B_w(x)\frac{(t+1)^{k-2}}{(k-2)!}1_{\{k\geqslant 2\}} = (-1)^k w_{1k}(t). \end{aligned}$$

Now let $t\in[-x,x]$. We have

$$\begin{aligned} w_{2k}(-t) &= \frac{1}{(k-1)!}\int_{0}^{-t}(-t-s)^{k-1}w(s)ds + C_{1,w}(x)\frac{(-t)^{k-2}}{(k-2)!}1_{\{k\geqslant 2\}}\\ &\quad + C_{2,w}(x)\frac{(-t)^{k-4}}{(k-4)!}1_{\{k\geqslant 4\}} + C_{3,w}(x)\frac{(-t)^{k-6}}{(k-6)!}1_{\{k\geqslant 6\}}\\ &= (y:=-s) = \frac{-1}{(k-1)!}\int_{0}^{t}(-t+y)^{k-1}w(-y)dy + C_{1,w}(x)\frac{(-t)^{k-2}}{(k-2)!}1_{\{k\geqslant 2\}}\\ &\quad + C_{2,w}(x)\frac{(-t)^{k-4}}{(k-4)!}1_{\{k\geqslant 4\}} + C_{3,w}(x)\frac{(-t)^{k-6}}{(k-6)!}1_{\{k\geqslant 6\}} = (-1)^k w_{2k}(t). \end{aligned}$$

For $k\neq 2$ we obviously have $w_{1k}(-1)=0$. On the other hand, for $k=2$ we have

$$w_{12}(-1) = \int_{-1}^{-1}(t-s)w(s)ds - B_w(x) = -B_w(x).$$

Put $H_{k,w}(x) := (-1)^{k-1}\left[w_{1k}(-x)-w_{2k}(-x)\right]$, for $k\in\mathbb{N}$.

LEMMA 3. *The coefficients* $H_{k,w}(x)$ *satisfy the following identities:*

a) $H_{k,w}(x) = w_{2k}(x) - w_{3k}(x)$

b) $H_{1,w}(x) = A_w$

c) $H_{k,w}(x) = 0$, *for* $k = 2,3,4$

d) For $k \geqslant 5$ *we have*

$$H_{k,w}(x) = \frac{1}{(k-1)!}\int_0^1 (x-s)^{k-1} w(s)ds + \frac{C_{1,w}(x)x^{k-2} + B_w(x)(x-1)^{k-2}}{(k-2)!} + \frac{C_{2,w}(x)x^{k-4}}{(k-4)!} + \frac{C_{3,w}(x)x^{k-6}}{(k-6)!}\mathbf{1}_{\{k\geqslant 6\}}$$

Proof. In view of the definition of $H_{k,w}(x)$ the identities follow directly from Lemma 2.

Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[-1,1]$ for some $n \geqslant 1$. We introduce the following notation:

$$T_{n,w}(x) = 0, \quad \text{for} \quad n \in \{1,2,3,4\}$$

$$T_{n,w}(x) = \sum_{k=5}^{n} H_{k,w}(x)\left[f^{(k-1)}(-x) + (-1)^{k-1} f^{(k-1)}(x)\right], \quad \text{for} \quad n \geqslant 5.$$

and

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) & \text{for } t \in [-1,-x], \\ w_{2n}(t) & \text{for } t \in (-x,x], \\ w_{3n}(t) & \text{for } t \in (x,1]. \end{cases}$$

In the next theorem we establish the identity which plays the key role in this paper. We call it the corrected weighted two-point quadrature formula.

THEOREM 2. *Let* $f : [-1,1] \to \mathbb{R}$ *be such that* $f^{(n)}$ *is piecewise continuous on* $[-1,1]$*, for some* $n \in \mathbb{N}$*. Then*

$$\int_{-1}^{1} w(t)f(t)dt = A_w[f(-x) + f(x)] + B_w(x)\left[f'(1) - f'(-1)\right] + T_{n,w}(x) + (-1)^n \int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t)dt. \tag{2.2}$$

Proof. We apply the general $(m+1)$-point formula (1.9) to the special case $m = 3$ with the subdivision (2.1). It follows

$$\int_{-1}^{1} w(t)f(t)dt = \sum_{k=1}^{n} (-1)^{k-1}\Big[w_{3k}(1)f^{(k-1)}(1) + [w_{1k}(-x) - w_{2k}(-x)]\,f^{(k-1)}(-x) + [w_{2k}(x) - w_{3k}(x)]\,f^{(k-1)}(x) - w_{1k}(-1)f^{(k-1)}(-1)\Big] + (-1)^n \int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t)dt.$$

According to Lemma 2, the terms in front of $f^{(k-1)}(1)$ and $f^{(k-1)}(-1)$ vanish for $k \neq 2$. On the other hand, the term in front of $f'(1)$ and $-f'(-1)$ equals $B_w(x)$. By Lemmas 2 and 3 the terms in front of $f^{(k-1)}(-x)$ and $(-1)^{k-1} f^{(k-1)}(x)$ equal $H_{k,w}(x)$. Now the assertion follows directly.

THEOREM 3. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[-1,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[-1,1]$ for some $n \in \mathbb{N}$. Then we have*

$$\left| \int_{-1}^{1} f(t)w(t)dt \; - A_w\left[f(-x)+f(x)\right] - B_w(x)\left[f'(1)-f'(-1)\right] - T_{n,w}(x) \right| \leqslant C_w(n,x,q) \cdot \|f^{(n)}\|_p, \tag{2.3}$$

where

$$C_w(n,x,q) = \begin{cases} 2 \cdot \left[\int_{-1}^{-x} |w_{1n}(t)|^q dt + \int_{-x}^{0} |w_{2n}(t)|^q dt\right]^{\frac{1}{q}}, 1 \leqslant q < \infty, \\ \max\{\sup_{t\in[-1,-x]} |w_{1n}(t)|, \sup_{t\in[0,x]} |w_{2n}(t)|\}, q = \infty. \end{cases}$$

The inequality is the best possible for $p=1$ and sharp for $1 < p \leqslant \infty$. The equality is attained for every function f such that

$$f(t) = M \cdot f_*(t) + p_{n-1}(t), \tag{2.4}$$

where $M \in \mathbb{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n-1$, and f_* is the function on $[a,b]$ defined by

$$f_*(t) = \int_{-1}^{t} \frac{(t-\xi)^{n-1}}{(n-1)!} \operatorname{sgn} W_{n,w}(\xi,x) \cdot |W_{n,w}(\xi,x)|^{\frac{1}{p-1}} d\xi, \tag{2.5}$$

for $1 < p < \infty$, and

$$f_*(t) = \int_{-1}^{t} \frac{(t-\xi)^{n-1}}{(n-1)!} \operatorname{sgn} W_{n,w}(\xi,x) d\xi, \tag{2.6}$$

for $p = \infty$.

Proof. Applying Hölder inequality to the integral $(-1)^n \int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t)dt$ we get

$$\left|(-1)^n \int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t)dt\right| \leqslant \|W_{n,w}(\cdot,x)\|_q \cdot \|f^{(n)}\|_p = C_w(n,x,q) \cdot \|f^{(n)}\|_p,$$

so the inequality (2.3) follows. In order to prove the sharpness, we need to find function f such that

$$\left| \int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t)dt \right| = C_w(n,x,q) \cdot \|f^{(n)}\|_p,$$

for $1 < p \leqslant \infty$ and $1 \leqslant q < \infty$ such that $\frac{1}{p}+\frac{1}{q}=1$. The function f_* defined by (2.5) and (2.6) is n times differentiable, and its $n-$th derivative is a piecewise continuous function. Further, we have

$$f_*^{(n)}(t) = \begin{cases} \operatorname{sgn} W_{n,w}(t,x), & p=\infty \\ |W_{n,w}(t,x)|^{\frac{1}{p-1}} \operatorname{sgn} W_{n,w}(t,x), & 1<p<\infty, \end{cases}$$

The function $f:[a,b]\to\mathbb{R}$ defined by (2.4) is also n times differentiable and satisfies $f^{(n)}\in L_p[a,b]$ and $f^{(n)}(t)=Mf_*^{(n)}(t)$.
Obviously, for $p=\infty$ and $q=1$ there is $\|f^{(n)}\|_\infty=|M|$, so we have

$$\begin{aligned} \left|\int_{-1}^{1} W_{n,w}(t,x)f^{(n)}(t)dt\right| &= \left|M\int_{-1}^{1} W_{n,w}(t,x)f_*^{(n)}(t)dt\right| \\ &= \left|M\int_{-1}^{1} W_{n,w}(t,x)\operatorname{sgn} W_{n,w}(t,x)dt\right| \\ &= |M|\int_{-1}^{1} |W_{n,w}(t,x)|\,dt = C_w(n,x,1)\|f^{(n)}\|_\infty. \end{aligned}$$

On the other hand if $1<p<\infty$ and $1<q<\infty$ we get

$$\|f^{(n)}\|_p = |M|\left[\int_{-1}^{1}|W_{n,w}(t,x)|^{\frac{p}{p-1}}\,dt\right]^{\frac{1}{p}} = |M|\left[\int_{-1}^{1}|W_{n,w}(t,x)|^{q}\,dt\right]^{\frac{1}{p}},$$

which implies

$$\begin{aligned} &\left|\int_{-1}^{1} W_{n,w}(t,x)f^{(n)}(t)dt\right| = \left|M\int_{-1}^{1} W_{n,w}(t,x)f_*^{(n)}(t)dt\right| \\ &= \left|M\int_{-1}^{1} W_{n,w}(t,x)\,|W_{n,w}(t,x)|^{\frac{1}{p-1}}\operatorname{sgn} W_{n,w}(t,x)dt\right| \\ &= |M|\int_{-1}^{1}|W_{n,w}(t,x)|^{\frac{p}{p-1}}\,dt = |M|\int_{-1}^{1}|W_{n,w}(t,x)|^{q}\,dt = C_w(n,x,q)\|f^{(n)}\|_p, \end{aligned}$$

so we proved the equality in (2.3). For $p=1$ and $q=\infty$ we shall prove that

$$\left|\int_{-1}^{1} W_{n,w}(t,x)f^{(n)}(t)dt\right| \leqslant \sup_{t\in[-1,1]}|W_{n,w}(t,x)|\cdot\int_{-1}^{1}|f^{(n)}(t)|dt \tag{2.7}$$

is the best possible inequality. Suppose that $|W_{n,w}(t,x)|$ attains its supremum at the point $t_0\in[-1,1]$ and let $\sup_{t\in[-1,1]}|W_{n,w}(t,x)|=|w_{kn}(t_0)|$, for some $k=1,2,3$. First, let us assume that $w_{kn}(t_0)>0$. For ε small enough define $f_\varepsilon^{(n-1)}(t)$ by

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t\leqslant t_0-\varepsilon \\ \frac{t-t_0+\varepsilon}{\varepsilon}, & t\in[t_0-\varepsilon,t_0] \\ 1, & t\geqslant t_0, \end{cases}$$

if $t_0 \in (x_{k-1}, x_k]$. Then, for ε small enough,

$$\left|\int_{-1}^{1} W_{n,w}(t,x) f_\varepsilon^{(n)} dt\right| = \left|\int_{t_0-\varepsilon}^{t_0} w_{kn}(t)\frac{1}{\varepsilon} dt\right| = \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} w_{kn}(t) dt. \tag{2.8}$$

Now, relation (2.7) implies

$$\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} w_{kn}(t) dt \leqslant w_{kn}(t_0)\int_{t_0-\varepsilon}^{t_0} \frac{1}{\varepsilon} dt = w_{kn}(t_0). \tag{2.9}$$

Since

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} w_{kn}(t) dt = w_{kn}(t_0),$$

the statement follows.
If $t_0 = x_{k-1}$, then we define, for $\varepsilon > 0$ small enough, the function $f_\varepsilon^{(n-1)}(t)$ by

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leqslant t_0 \\ \frac{t-t_0}{\varepsilon}, & t \in [t_0, t_0+\varepsilon] \\ 1, & t \geqslant t_0+\varepsilon, \end{cases}$$

and we argue as above.
For the case $w_{kn}(t_0) < 0$ the proof is similar.

THEOREM 4. *Assume that for some even n, $f^{(n)}$ is a continuous function on $[-1,1]$ and $W_{n,w}(\cdot,x)$ has a constant sign on $[-1,0]$. Then there exists $\eta \in (-1,1)$ such that the following identity holds:*

$$\begin{aligned}\int_{-1}^{1} w(t) f(t) dt &= A_w [f(-x)+f(x)] + B_w(x)\left[f'(1)-f'(-1)\right] \\ &\quad + T_{n,w}(x) + 2H_{n+1,w}(x)\cdot f^{(n)}(\eta).\end{aligned} \tag{2.10}$$

Proof. According to the relation (2.2), we have to prove the identity

$$\int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t) dt = 2H_{n+1,w}(x)\cdot f^{(n)}(\eta).$$

Observe that $W_{n,w}(\cdot,x)$ is an even function. Since $W_{n,w}(\cdot,x)$ does not change the sign, then by the mean value theorem there exists $\eta \in (-1,1)$ such that

$$\begin{aligned}\int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t) dt &= f^{(n)}(\eta)\cdot\int_{-1}^{1} W_{n,w}(t,x) dt \\ &= f^{(n)}(\eta)\cdot 2\cdot\int_{0}^{1} W_{n,w}(t,x) dt = 2f^{(n)}(\eta)(w_{2,n+1}(x)-w_{3,n+1}(x)) \\ &= 2H_{n+1,w}(x)\cdot f^{(n)}(\eta).\end{aligned}$$

REMARK 1. In particular, for $n=4$ we get

$$\int_{-1}^{1} w(t)f(t)dt = A_w\left[f(-x)+f(x)\right]+B_w(x)\left[f'(1)-f'(-1)\right]+2H_{5,w}(x)\cdot f^{(4)}(\eta). \tag{2.11}$$

Imposing $x=0,\frac{1}{3},\frac{1}{2},1$ in (2.11) we get the corrected version of midpoint, Newton-Cotes two-point formula, Maclaurin two-point formula and trapezoid formula, respectively. By choosing $x\in[0,1]$ such that $B_w(x)=0$, the two-point quadrature formula of Gauss type is obtained. These formulae are exact for all polynomials of order $\leqslant 3$.

REMARK 2. In particular, for $n=6$, the following formula is obtained:

$$\begin{aligned}\int_{-1}^{1} w(t)f(t)dt &= A_w\left[f(-x)+f(x)\right]+B_w(x)\left[f'(1)-f'(-1)\right]\\ &\quad+H_{5,w}(x)\left[f^{(4)}(-x)+f^{(4)}(x)\right]+H_{6,w}(x)\left[f^{(5)}(-x)-f^{(5)}(x)\right]\\ &\quad+2H_{7,w}(x)\cdot f^{(6)}(\eta).\end{aligned} \tag{2.12}$$

From the condition $H_{5,w}(x)=0$, a unique solution $x\in[0,1]$ is obtained. For that x formula (2.12) becomes the corrected version of Gauss type quadrature formula. That formula is more accurate than the ordinary Gauss formula. In fact, it is exact for all polynomials of order $\leqslant 5$.

3. Special cases

In this section we apply the results of the second section to the special cases of weights: $w(t)=1$, $w(t)=\frac{1}{\sqrt{1-t^2}}$ and $w(t)=\sqrt{1-t^2}$, and we establish the corrected version of quadrature formulae of Gauss type. All the computations were done using the *Wolfram Mathematica* software.

3.1. The case $w(t)=1$

In this case we compute

$$\begin{aligned}&A_w=1,\quad B_w(x)=\frac{1}{6}-\frac{x^2}{2},\quad C_{1,w}(x)=\frac{x^2}{2}-x+\frac{1}{3},\\ &C_{2,w}(x)=-\frac{x^3}{6}+\frac{x^2}{4}-\frac{1}{24},\quad C_{3,w}(x)=-\frac{x^5}{120}+\frac{x^2}{48}-\frac{1}{180},\\ &H_{5,w}(x)=-\frac{x^4}{24}+\frac{x^2}{12}-\frac{7}{360}.\end{aligned}$$

COROLLARY 1. *Let* $f:[-1,1]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is piecewise continuous on* $[-1,1]$, *for some* $n\in\mathbb{N}$. *Then*

$$\int_{-1}^{1} f(t)dt = f(-x)+f(x)+\left(\frac{1}{6}-\frac{x^2}{2}\right)\left[f'(1)-f'(-1)\right] + T_{n,w}(x)+(-1)^n\int_{-1}^{1} W_{n,w}(t,x)f^{(n)}(t)dt.$$

Proof. Apply Theorem 2 for the case $w(t)=1$.
Let us consider special cases for $x\in[0,1]$:

a) For $x=0$ we get the corrected midpoint formula. In this case we compute $B_w(0)=\frac{1}{6}$ and $H_{5,w}(0)=-\frac{7}{360}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)dt = 2f(0)+\frac{1}{6}\left[f'(1)-f'(-1)\right]-\frac{7}{180}\cdot f^{(4)}(\eta).$$

If $f^{(n)}\in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)dt-2f(0)-\frac{1}{6}\left[f'(1)-f'(-1)\right]-T_{n,w}(x)\right|\leqslant C_w(n,0,q)\|f^{(n)}\|_p.$$

In particular, for $n=2,3,4$ there is $T_{n,w}(x)=0$, so we have

$$\begin{aligned}
C_w(2,0,1) &= \frac{4}{9\sqrt{3}}\approx 0.25660011\\
C_w(2,0,\infty) &= \frac{1}{3}\approx 0.33333333\\
C_w(3,0,1) &= \frac{1}{12}\approx 0.08333333\\
C_w(3,0,\infty) &= \frac{1}{9\sqrt{3}}\approx 0.06415003\\
C_w(4,0,1) &= \frac{7}{180}\approx 0.03888888\\
C_w(4,0,\infty) &= \frac{1}{24}\approx 0.04166667.
\end{aligned}$$

REMARK 3. For $n\in\{2,3,4\}$ and $q\in\{1,\infty\}$ the same constants $C_w(n,0,q)$ have been obtained in [6].

b) For $x=\frac{1}{3}$ the corrected Newton-Cotes two-point formula is established. In this case we compute $B_w(\frac{1}{3})=\frac{1}{9}$ and $H_{5,w}(\frac{1}{3})=-\frac{13}{1215}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{1}{3}\right)+f\left(\frac{1}{3}\right)+\frac{1}{9}\left[f'(1)-f'(-1)\right]-\frac{26}{1215}\cdot f^{(4)}(\eta).$$

If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)dt - f\left(-\frac{1}{3}\right) - f\left(\frac{1}{3}\right) - \frac{1}{9}\left[f'(1) - f'(-1)\right] - T_{n,w}(x)\right| \leqslant C_w(n,\frac{1}{3},q)\|f^{(n)}\|_p.$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$\begin{aligned}
C_w(2,\frac{1}{3},1) &= \frac{8\sqrt{2}}{81} \approx 0.13967541 \\
C_w(2,\frac{1}{3},\infty) &= \frac{1}{9} \approx 0,11111111 \\
C_w(3,\frac{1}{3},1) &= \frac{13}{324} \approx 0.04012345 \\
C_w(3,\frac{1}{3},\infty) &= \frac{2\sqrt{2}}{81} \approx 0.03491885 \\
C_w(4,\frac{1}{3},1) &= \frac{26}{1215} \approx 0.02139917 \\
C_w(4,\frac{1}{3},\infty) &= \frac{13}{648} \approx 0.02006172.
\end{aligned}$$

REMARK 4. For $n \in \{3,4\}$ and $q \in \{1,\infty\}$ the same constants $C_w(n,\frac{1}{3},q)$ have been obtained in [7].

c) For $x = \frac{1}{2}$ we get the corrected Maclaurin two-point formula. In this case we compute $B_w(\frac{1}{2}) = \frac{1}{24}$ and $H_{5,w}(\frac{1}{2}) = -\frac{7}{5760}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + \frac{1}{24}\left[f'(1) - f'(-1)\right] - \frac{7}{2880}\cdot f^{(4)}(\eta).$$

If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)dt - f\left(-\frac{1}{2}\right) - f\left(\frac{1}{2}\right) - \frac{1}{24}\left[f'(1) - f'(-1)\right] - T_{n,w}(x)\right| \leqslant C_w(n,\frac{1}{2},q)\|f^{(n)}\|_p.$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$
\begin{aligned}
C_w(2,\frac{1}{2},1) &= \frac{1}{9\sqrt{3}} \approx 0.06415002 \\
C_w(2,\frac{1}{2},\infty) &= \frac{1}{12} \approx 0.08333333 \\
C_w(3,\frac{1}{2},1) &= \frac{1}{96} \approx 0.01041666 \\
C_w(3,\frac{1}{2},\infty) &= \frac{1}{72\sqrt{3}} \approx 0.00801875 \\
C_w(4,\frac{1}{2},1) &= \frac{7}{2880} \approx 0.00243055 \\
C_w(4,\frac{1}{2},\infty) &= \frac{1}{384} \approx 0.00260416.
\end{aligned}
$$

REMARK 5. The constants $C_w(2,\frac{1}{2},\infty)$, $C_w(3,\frac{1}{2},\infty)$ and $C_w(2,\frac{1}{2},1)$ are better than the constants obtained for Maclaurin two-point formula in [2], while the constant $C_w(4,\frac{1}{2},1)$ is weaker than the constant obtained for Maclaurin two-point formula. The constants $C_w(4,\frac{1}{2},\infty)$ and $C_w(3,\frac{1}{2},1)$ are the same as the appropriate constants obtained for Maclaurin formula in [2].

d) For $x = 1$ we get the corrected trapezoid formula. In this case we compute $B_w(1) = -\frac{1}{3}$ and $H_{5,w}(1) = \frac{1}{45}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$
\int_{-1}^{1} f(t)dt = f(-1) + f(1) - \frac{1}{3}\left[f'(1) - f'(-1)\right] + \frac{2}{45} \cdot f^{(4)}(\eta).
$$

If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$
\left| \int_{-1}^{1} f(t)dt - f(-1) - f(1) + \frac{1}{3}\left[f'(1) - f'(-1)\right] - T_{n,w}(x) \right| \leqslant C_w(n,1,q)\|f^{(n)}\|_p.
$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$
\begin{aligned}
C_w(2,1,1) &= \frac{4}{9\sqrt{3}} \approx 0.25660011 \\
C_w(2,1,\infty) &= \frac{1}{3} \approx 0.33333333 \\
C_w(3,1,1) &= \frac{1}{12} \approx 0.08333333 \\
C_w(3,1,\infty) &= \frac{1}{9\sqrt{3}} \approx 0.06415002 \\
C_w(4,1,1) &= \frac{2}{45} \approx 0.04444444 \\
C_w(4,1,\infty) &= \frac{1}{24} \approx 0.04166666.
\end{aligned}
$$

REMARK 6. For $n \in \{2,3,4\}$ and $q \in \{1,\infty\}$ the same constants $C_w(n,1,q)$ have been obtained in [5].

$e)$ The condition $H_{5,w}(x)=0$ implies $x=\sqrt{1-\frac{2\sqrt{30}}{15}}$. In this case we compute $B_w(x)=\frac{\sqrt{30}-5}{15}$, $H_{6,w}(x)=0$ and $H_{7,w}(x)=\frac{7\sqrt{30}-45}{70875}$. If $f^{(6)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)dt = f\left(-\sqrt{1-\frac{2\sqrt{30}}{15}}\right)+f\left(\sqrt{1-\frac{2\sqrt{30}}{15}}\right)$$
$$+\frac{\sqrt{30}-5}{15}\left[f'(1)-f'(-1)\right]-\frac{90-14\sqrt{30}}{70875}\cdot f^{(6)}(\eta),$$

which we call a corrected Gauss-Legendre two-point formula. If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)dt - f\left(-\sqrt{1-\frac{2\sqrt{30}}{15}}\right)-f\left(\sqrt{1-\frac{2\sqrt{30}}{15}}\right)\right.$$
$$\left.-\frac{\sqrt{30}-5}{15}\left[f'(1)-f'(-1)\right]-T_{n,w}(x)\right| \leqslant C_w(n,x,q)\|f^{(n)}\|_p.$$

In particular, for $n=2,3,4,5,6$ there is $T_{n,w}(x)=0$, so we have

$$\begin{aligned}
&C_w(2,x,1)\approx 0.06502076\\
&C_w(2,x,\infty)\approx 0.08370696\\
&C_w(3,x,1)\approx 0.010903178\\
&C_w(3,x,\infty)\approx 0.01090496\\
&C_w(4,x,1)\approx 0.00209576\\
&C_w(4,x,\infty)\approx 0.00241499\\
&C_w(5,x,1)\approx 0.00050307\\
&C_w(5,x,\infty)\approx 0.00052394\\
&C_w(6,x,1)=\frac{90-14\sqrt{30}}{70875}\approx 0.00018792\\
&C_w(6,x,\infty)\approx 0.00025153.
\end{aligned}$$

REMARK 7. The same constants for the corrected Gauss-Legendre formula have been obtained in [4].

3.2. The case $w(t)=\frac{1}{\sqrt{1-t^2}}$

In this case we compute

$$A_w=\frac{\pi}{2},\quad B_w(x)=\frac{\pi}{8}-\frac{\pi x^2}{4},\quad C_{1,w}(x)=\frac{\pi x^2}{4}-\frac{\pi x}{2}+1-\frac{\pi}{8},$$
$$C_{2,w}(x)=-\frac{\pi x^3}{12}+\frac{\pi x^2}{8}+\frac{1}{9}-\frac{\pi}{16},\quad C_{3,w}(x)=-\frac{\pi x^5}{240}+\frac{\pi x^2}{96}-\frac{\pi}{192}+\frac{1}{225}$$
$$H_{5,x}(x)=-\frac{\pi x^4}{48}+\frac{\pi x^2}{24}-\frac{5\pi}{384}.$$

COROLLARY 2. *Let* $f:[-1,1]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is piecewise continuous on* $[-1,1]$, *for some* $n\in\mathbb{N}$. *Then*

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\frac{\pi}{2}[f(-x)+f(x)]+\left(\frac{\pi}{8}-\frac{\pi x^2}{4}\right)\left[f'(1)-f'(-1)\right]$$
$$+\,T_{n,w}(x)+(-1)^n\int_{-1}^{1}W_{n,w}(t,x)f^{(n)}(t)dt.$$

Proof. Apply Theorem 2 for the case $w(t)=\frac{1}{\sqrt{1-t^2}}$.

Let us consider special cases for $x\in[0,1]$: $a)$ For $x=0$ we get the corrected midpoint formula. In this case we compute $B_w(0)=\frac{\pi}{8}$ and $H_{5,w}(0)=-\frac{5\pi}{384}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\pi f(0)+\frac{\pi}{8}\left[f'(1)-f'(-1)\right]-\frac{5\pi}{192}\cdot f^{(4)}(\eta).$$

If $f^{(n)}\in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt-\pi f(0)-\frac{\pi}{8}\left[f'(1)-f'(-1)\right]-T_{n,w}(x)\right|\leqslant C_w(n,0,q)\|f^{(n)}\|_p.$$

In particular, for $n=2,3,4$ there is $T_{n,w}(x)=0$, so we have

$$C_w(2,0,1)\approx 0.51838540$$
$$C_w(2,0,\infty)=1-\frac{\pi}{8}\approx 0.60730091$$
$$C_w(3,0,1)=\frac{\pi}{8}-\frac{2}{9}\approx 0.17047685$$
$$C_w(3,0,\infty)\approx 0.12959635$$
$$C_w(4,0,1)=\frac{5\pi}{192}\approx 0.08181230$$
$$C_w(4,0,\infty)=\frac{1}{9}-\frac{\pi}{16}\approx 0.08523842.$$

$b)$ For $x=\frac{1}{3}$ we established the corrected Newton-Cotes two-point formula. In this case we compute $B_w(\frac{1}{3})=\frac{7\pi}{72}$ and $H_{5,w}(\frac{1}{3})=\frac{-269\pi}{31104}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\left(f\left(-\frac{1}{3}\right)+f\left(\frac{1}{3}\right)\right)+\frac{7\pi}{72}\left[f'(1)-f'(-1)\right]-\frac{269\pi}{15552}\cdot f^{(4)}(\eta).$$

If $f^{(n)}\in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt-\frac{\pi}{2}\left(f\left(-\frac{1}{3}\right)+f\left(\frac{1}{3}\right)\right)-\frac{7\pi}{72}\left[f'(1)-f'(-1)\right]-T_{n,w}(x)\right| \leqslant C_w(n,\frac{1}{3},q)\|f^{(n)}\|_p.$$

In particular, for $n=2,3,4$ there is $T_{n,w}(x)=0$, so we have

$$\begin{aligned}
&C_w(2,\frac{1}{3},1)\approx 0.34175596\\
&C_w(2,\frac{1}{3},\infty)=\frac{7\pi}{92}\approx 0.30543261\\
&C_w(3,\frac{1}{3},1)\approx 0.10260294\\
&C_w(3,\frac{1}{3},\infty)\approx 0.08543899\\
&C_w(4,\frac{1}{3},1)=\frac{269\pi}{15552}\approx 0.05433953\\
&C_w(4,\frac{1}{3},\infty)=\frac{67\pi-144}{1296}\approx 0.05130147.
\end{aligned}$$

$c)$ For $x=\frac{1}{2}$ we get the corrected Maclaurin two-point formula. In this case we compute $B_w(\frac{1}{2})=\frac{\pi}{16}$ and $H_{5,w}(\frac{1}{2})=-\frac{\pi}{256}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\left(f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right)+\frac{\pi}{16}\left[f'(1)-f'(-1)\right]-\frac{\pi}{128}\cdot f^{(4)}(\eta).$$

If $f^{(n)}\in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt-\frac{\pi}{2}\left(f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right)-\frac{\pi}{16}\left[f'(1)-f'(-1)\right]-T_{n,w}(x)\right| \leqslant C_w(n,\frac{1}{2},q)\|f^{(n)}\|_p.$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$C_w(2, \frac{1}{2}, 1) \approx 0.16414959$$
$$C_w(2, \frac{1}{2}, \infty) = \frac{\pi}{16} \approx 0.19634954$$
$$C_w(3, \frac{1}{2}, 1) \approx 0.03957716$$
$$C_w(3, \frac{1}{2}, \infty) \approx 0.04103739$$
$$C_w(4, \frac{1}{2}, 1) = \frac{\pi}{128} \approx 0.02454369$$
$$C_w(4, \frac{1}{2}, \infty) = \frac{8-3\pi}{72} \approx 0.01978858.$$

d) For $x = 1$ we get the corrected trapezoid formula. In this case we compute $B_w(1) = -\frac{\pi}{8}$ and $H_{5,w}(1) = \frac{\pi}{128}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{2}\left(f(-1)+f(1)\right) - \frac{\pi}{8}\left[f'(1)-f'(-1)\right] + \frac{\pi}{64}\cdot f^{(4)}(\eta).$$

If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left| \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt - \frac{\pi}{2}\left(f(-1)+f(1)\right) + \frac{\pi}{8}\left[f'(1)-f'(-1)\right] - T_{n,w}(x) \right| \leqslant C_w(n,1,q)\|f^{(n)}\|_p.$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$C_w(2,1,1) \approx 0.28081219$$
$$C_w(2,1,\infty) = \frac{\pi}{8} \approx 0.39269908$$
$$C_w(3,1,1) = \frac{2}{9} - \frac{\pi}{24} \approx 0.09132252$$
$$C_w(3,1,\infty) \approx 0.07020304$$
$$C_w(4,1,1) = \frac{\pi}{64} \approx 0.04908738$$
$$C_w(4,1,\infty) = \frac{16-3\pi}{144} \approx 0.04566126.$$

e) The condition $H_{5,w}(x) = 0$ implies $x = \sqrt{1-\frac{\sqrt{6}}{4}}$. In this case we compute $B_w(x) = \frac{\pi(\sqrt{6}-2)}{16}$, $H_{6,w}(x) = 0$ and $H_{7,w}(x) = \frac{(3\sqrt{6}-10)\pi}{46080}$. If $f^{(6)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{2}\left(f\left(-\sqrt{1-\frac{\sqrt{6}}{4}}\right) + f\left(\sqrt{1-\frac{\sqrt{6}}{4}}\right)\right)$$
$$+ \frac{\pi(\sqrt{6}-2)}{16}\left[f'(1)-f'(-1)\right] + \frac{(3\sqrt{6}-10)\pi}{23040}\cdot f^{(6)}(\eta),$$

which we call a corrected Gauss-Chebyshev two-point formula. If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left| \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt - \frac{\pi}{2} \left(f\left(-\sqrt{1-\frac{\sqrt{6}}{4}} \right) + f\left(\sqrt{1-\frac{\sqrt{6}}{4}} \right) \right) \right.$$
$$\left. - \frac{\pi(\sqrt{6}-2)}{16} \left[f'(1) - f'(-1) \right] - T_{n,w}(x) \right| \leqslant C_w(n,x,q) \| f^{(n)} \|_p .$$

In particular, for $n = 2,3,4,5,6$ there is $T_{n,w}(x) = 0$, so we have

$$\begin{aligned}
C_w(2,x,1) &\approx 0.10751580 \\
C_w(2,x,\infty) &\approx 0.13473273 \\
C_w(3,x,1) &\approx 0.01835509 \\
C_w(3,x,\infty) &\approx 0.01601592 \\
C_w(4,x,1) &\approx 0.00362114 \\
C_w(4,x,\infty) &\approx 0.00380096 \\
C_w(5,x,1) &\approx 0.00091496 \\
C_w(5,x,\infty) &\approx 0.00090528 \\
C_w(6,x,1) &= \frac{(3\sqrt{6}-10)\pi}{23040} \approx 0.00036154 \\
C_w(6,x,\infty) &\approx 0.00045748.
\end{aligned}$$

3.3. The case $w(t) = \sqrt{1-t^2}$

In this case we compute

$$A_w = \frac{\pi}{4}, \quad B_w(x) = \frac{\pi}{32} - \frac{\pi x^2}{8}, \quad C_{1,w}(x) = \frac{\pi x^2}{8} - \frac{\pi x}{4} + \frac{1}{3} - \frac{\pi}{32}$$
$$C_{2,w}(x) = -\frac{\pi x^3}{24} + \frac{\pi x^2}{16} - \frac{\pi}{64} + \frac{1}{45} \quad C_{3,w}(x) = -\frac{\pi x^5}{480} + \frac{\pi x^2}{192} + \frac{1}{1575} - \frac{\pi}{768}$$
$$H_{5,w}(x) = -\frac{\pi x^4}{96} + \frac{\pi x^2}{48} - \frac{\pi}{256}.$$

COROLLARY 3. *Let $f:[-1,1] \to \mathbb{R}$ be such that $f^{(n)}$ is piecewise continuous on $[-1,1]$, for some $n \in \mathbb{N}$. Then*

$$\int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt = \frac{\pi}{4}\left[f(-x)+f(x)\right] + \left(\frac{\pi}{32} - \frac{\pi x^2}{8}\right)\left[f'(1)-f'(-1)\right]$$
$$+ T_{n,w}(x) + (-1)^n \int_{-1}^{1} W_{n,w}(t,x) f^{(n)}(t)\,dt.$$

Proof. Apply Theorem 2 for the case $w(t)=\sqrt{1-t^2}$.

Let us consider special cases for $x\in[0,1]$:

a) For $x=0$ we get the corrected midpoint formula. In this case we compute $B_w(0)=\frac{\pi}{32}$ and $H_{5,w}(0)=-\frac{\pi}{256}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)\sqrt{1-t^2}dt=\frac{\pi}{2}f(0)+\frac{\pi}{32}\left[f'(1)-f'(-1)\right]-\frac{\pi}{128}\cdot f^{(4)}(\eta).$$

If $f^{(n)}\in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)\sqrt{1-t^2}dt-\frac{\pi}{2}f(0)-\frac{\pi}{32}\left[f'(1)-f'(-1)\right]\right|\leqslant C_w(n,0,q)\|f^{(n)}\|_p.$$

In particular, for $n=2,3,4$ there is $T_{n,w}(x)=0$, so we have

$$\begin{aligned}
&C_w(2,0,1)\approx 0.16724762\\
&C_w(2,0,\infty)\approx 0.23515856\\
&C_w(3,0,1)\approx 0.05373032\\
&C_w(3,0,\infty)\approx 0.04181190\\
&C_w(4,0,1)=\frac{\pi}{128}\approx 0.02454369\\
&C_w(4,0,\infty)\approx 0.02686516.
\end{aligned}$$

b) For $x=\frac{1}{3}$ we established the corrected Newton-Cotes two-point formula. In this case we compute $B_w(\frac{1}{3})=\frac{5\pi}{288}$ and $H_{5,w}(\frac{1}{3})=-\frac{107\pi}{62208}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\begin{aligned}
\int_{-1}^{1} f(t)\sqrt{1-t^2}dt=\frac{\pi}{4}\left(f\left(-\frac{1}{3}\right)+f\left(\frac{1}{3}\right)\right)&+\frac{5\pi}{288}\left[f'(1)-f'(-1)\right]\\
&-\frac{107\pi}{31104}\cdot f^{(4)}(\eta).
\end{aligned}$$

If $f^{(n)}\in L_p[-1,1]$, then Theorem 3 implies:

$$\begin{aligned}
\left|\int_{-1}^{1} f(t)\sqrt{1-t^2}dt-\frac{\pi}{4}\left(f\left(-\frac{1}{3}\right)+f\left(\frac{1}{3}\right)\right)-\frac{5\pi}{288}\left[f'(1)-f'(-1)\right]-T_{n,w}(x)\right|&\\
\leqslant C_w(n,\frac{1}{3},q)\|f^{(n)}\|_p.&
\end{aligned}$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$C_w(2,\frac{1}{3},1) \approx 0.07309774$$
$$C_w(2,\frac{1}{3},\infty) \approx 0.07202776$$
$$C_w(3,\frac{1}{3},1) \approx 0.01979336$$
$$C_w(3,\frac{1}{3},\infty) \approx 0.01827443$$
$$C_w(4,\frac{1}{3},1) = \frac{107\pi}{31104} \approx 0.01080730$$
$$C_w(4,\frac{1}{3},\infty) = \frac{576-265\pi}{25920} \approx 0.00989668.$$

$c)$ For $x = \frac{1}{2}$ we get $B_w(\frac{1}{2}) = 0$ and $H_{5,w}(\frac{1}{2}) = \frac{\pi}{1536}$, so the term associated with the first derivative disappears. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt = \frac{\pi}{4}\left(f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right)+\frac{\pi}{768}\cdot f^{(4)}(\eta),$$

which is well-known Chebyshev-Gauss two-point quadrature formula of the second kind. If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)\sqrt{1-t^2}dt - \frac{\pi}{4}\left(f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right) - T_{n,w}(x)\right| \leqslant C_w(n,\frac{1}{2},q)\|f^{(n)}\|_p.$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$C_w(2,\frac{1}{2},1) \approx 0.05471452$$
$$C_w(2,\frac{1}{2},\infty) \approx 0.06296013$$
$$C_w(3,\frac{1}{2},1) \approx 0.01171952$$
$$C_w(3,\frac{1}{2},\infty) \approx 0.01367863$$
$$C_w(4,\frac{1}{2},1) = \frac{\pi}{768} \approx 0.00409061$$
$$C_w(4,\frac{1}{2},\infty) = \frac{64-15\pi}{2880} \approx 0.00585976.$$

$d)$ For $x = 1$ we get the corrected trapezoid formula. In this case we compute $B_w(1) = -\frac{3\pi}{32}$ and $H_{5,w}(1) = \frac{5\pi}{768}$. If $f^{(4)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)\sqrt{1-t^2}dt = \frac{\pi}{4}\left(f(-1)+f(1)\right) - \frac{3\pi}{32}\left[f'(1)-f'(-1)\right]+\frac{5\pi}{768}\cdot f^{(4)}(\eta).$$

If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)\sqrt{1-t^2}dt - \frac{\pi}{4}\left(f(-1)+f(1)\right) + \frac{3\pi}{32}\left[f'(1)-f'(-1)\right] - T_{n,w}(x)\right| \leqslant C_w(n,1,q)\|f^{(n)}\|_p.$$

In particular, for $n = 2,3,4$ there is $T_{n,w}(x) = 0$, so we have

$$\begin{aligned}
C_w(2,1,1) &\approx 0.23778656\\
C_w(2,1,\infty) &= \frac{3\pi}{32} \approx 0.29452431\\
C_w(3,1,1) &= \frac{2}{45} + \frac{\pi}{96} \approx 0.07716936\\
C_w(3,1,\infty) &\approx 0.05944664\\
C_w(4,1,1) &= \frac{5\pi}{384} \approx 0.04090615\\
C_w(4,1,\infty) &= \frac{64+4\pi}{2880} \approx 0.03858468.
\end{aligned}$$

e) The condition $H_{5,w}(x) = 0$ implies $x = \sqrt{1-\frac{\sqrt{10}}{4}}$. In this case we compute $B_w(x) = \frac{\pi(\sqrt{10}-3)}{32}$, $H_{6,w}(x) = 0$ and $H_{7,w}(x) = \frac{(2\sqrt{10}-7)\pi}{36864}$, If $f^{(6)}$ is continuous, then we have by Theorem 4

$$\int_{-1}^{1} f(t)\sqrt{1-t^2}dt = \frac{\pi}{4}\left(f\left(-\sqrt{1-\frac{\sqrt{10}}{4}}\right) + f\left(\sqrt{1-\frac{\sqrt{10}}{4}}\right)\right) + \frac{\pi(\sqrt{10}-4)}{16}\left[f'(1)-f'(-1)\right] + \frac{(2\sqrt{10}-7)\pi}{18432}\cdot f^{(6)}(\eta).$$

which we call a corrected Gauss-Chebyshev two-point formula of the second kind. If $f^{(n)} \in L_p[-1,1]$, then Theorem 3 implies:

$$\left|\int_{-1}^{1} f(t)\sqrt{1-t^2}dt - \frac{\pi}{4}\left(f\left(-\sqrt{1-\frac{\sqrt{10}}{4}}\right) + f\left(\sqrt{1-\frac{\sqrt{10}}{4}}\right)\right) - \frac{\pi(\sqrt{10}-4)}{16}\left[f'(1)-f'(-1)\right]\right| \leqslant C_w(n,x,q)\|f^{(n)}\|_p.$$

In particular, for $n = 2,3,4,5,6$ there is $T_{n,w}(x) = 0$, so we have

$$\begin{aligned}
C_w(2,x,1) &\approx 0.04552286 \\
C_w(2,x,\infty) &\approx 0.06082252 \\
C_w(3,x,1) &\approx 0.00736171 \\
C_w(3,x,\infty) &\approx 0.00813962 \\
C_w(4,x,1) &\approx 0.00138313 \\
C_w(4,x,\infty) &\approx 0.00171062 \\
C_w(5,x,1) &\approx 0.00032053 \\
C_w(5,x,\infty) &\approx 0.00034578 \\
C_w(6,x,1) &= \frac{(7-2\sqrt{10})\pi}{18432} \approx 0.00011512 \\
C_w(6,x,\infty) &\approx 0.00016026.
\end{aligned}$$

REMARK 8. The results introduced in Example 3.2 and Example 3.3 are new, and they could be helpful in approximation of a wider class of integrals where an integrand is a product of two functions: $n-$times differentiable function f and weight w with possible discontinuities. In both cases, the maximum degree of exactness is achieved when applying a node x which is a solution of the equation $H_{5,w}(x) = 0$. Also, by comparing the constants $C_w(n,x,q)$, it is obvious that the best constants are reached in examples 3.2 e) and 3.3 e), that is for the corrected Gauss quadrature formulae.

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