ELENTARY PROOF OF THE LEFT MULTIDIMENSIONAL
HERMITE–HADAMARD INEQUALITY ON CERTAIN CONVEX SETS

LADISLAV MATEJÍČKA

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Abstract. The left Hermite-Hadamard inequality of several variables for convex functions on certain convex compact sets is proved via elementary approach, independently of Choquet theory.

1. Introduction

The Hermite-Hadamard inequality plays an important role in research on inequalities. The monographs [5] and [11] give a comprehensive review the literature. The classical Hermite-Hadamard inequality [7], [9] is:

If \( f : (a, b) \to \mathbb{R} \) is a convex function, then

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Among the various generalizations of this inequality, is the problem of extending Hermite-Hadamard inequality to convex function of several variables. C.P. Niculescu [12] gave the most general answer to this problem. From Choquet theory [8], [13], [10] it follows that if \( K \) is a compact convex set in a locally convex Hausdorff space, \( \mu \) is a positive Borel measure, \( f : K \to \mathbb{R} \) is a convex function, then

\[
f(b_{\mu}) \leq \frac{1}{\mu(K)} \int_{K} f(x) \, d\mu,
\]

where \( b_{\mu} \) is the barycenter of \( K \). Some particular cases for special convex sets have been investigated by S.S. Dragomir [2], [3], [4], B. Gavrea [6] and by Bessenyei [1]. The aim of the paper is to verify left Hermite-Hadamard inequality on “simple” convex compact sets via elementary approach, independently of Choquet theory.

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2. The main results

First, we define simple convex sets.

**Definition 2.1.** Let \( n \in \mathbb{N} \), \( n > 1 \), \( a < b \), \( a, b \in \mathbb{R} \), \( M_1 = [a, b] \), \( M_1^0 = (a, b) \). We say that \( M_n \) is a simple \( n \) dimensional convex set, if \( M_n \) is a convex set, \( M_n = \{(x_1, \ldots, x_n); a \leq x_1 \leq b, \phi_1(x_1) \leq x_2 \leq \psi_1(x_1), \ldots, \phi_j(x_1, \ldots, x_j) \leq x_{j+1} \leq \psi_j(x_1, \ldots, x_j), \ldots, \phi_{n-1}(x_1, \ldots, x_{n-1}) \leq x_n \leq \psi_{n-1}(x_1, \ldots, x_{n-1})\} \) where \( \phi_j(x_1, \ldots, x_j), \psi_j(x_1, \ldots, x_j) \) are functions with continuous first derivatives on \( M_j \) for \( j = 1, \ldots, n-1 \), and \( \phi_j(x_1, \ldots, x_j) < \psi_j(x_1, \ldots, x_j) \) on the interior of \( M_j \), \( j = 1, \ldots, n-1 \).

The set of simple convex sets is “sufficiently” large set. For example, circles, ellipses, triangles, cones, balls, ... are simple convex sets.

First, we prove an auxiliary lemma and the left two dimensional Hermite-Hadamard inequality.

**Lemma 2.1.** Let \( M_2 \) be a simple two dimensional convex set. Then

\[
\varphi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds \leq \frac{1}{2} \int_a^{x_1} \psi_1^2(s) - \varphi_1^2(s)ds \leq \varphi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds,
\]

\( x_1 \in [a, b], \) where

\[
\alpha(x_1) = \frac{\int_a^{x_1} s(\psi_1(s) - \varphi_1(s))ds}{\int_a^{x_1} \psi_1(s) - \varphi_1(s)ds}, \quad x_1 \in (a, b) \text{ and } \alpha(a) = a. \tag{2.2}
\]

**Proof.** Using the L’Hospital’s rule we have

\[
\lim_{x_1 \to a^+} \alpha(x_1) = \lim_{x_1 \to a^+} \frac{x_1(\psi_1(x_1) - \varphi_1(x_1))}{\psi_1(x_1) - \varphi_1(x_1)} = a = \alpha(a)
\]

and thus \( \alpha(x_1) \) is a continuous function on \([a, b]\). We also observe that \( \alpha_1(x_1) < x_1 \), \( x_1 \in (a, b] \). It is sufficient to prove only the right side of (2.1). Indeed, from \( M_2 \) is a convex set we have \( \varphi_1(x_1) \) is a convex function, \( \psi_1(x_1) \) is a concave function, \( -\varphi_1(x_1) \) is a concave function, \( -\psi_1(x_1) \) is a convex function \( x_1 \in [a, b] \). According to the assumption: \( -\psi_1(x_1) < -\varphi_1(x_1) \), \( x_1 \in (a, b) \) the inequality

\[
\varphi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds \leq \frac{1}{2} \int_a^{x_1} \psi_1^2(s) - \varphi_1^2(s)ds
\]

is equivalent to

\[
\frac{1}{2} \int_a^{x_1} (-\varphi_1(s))^2 - (-\psi_1(s))^2ds \leq -\varphi_1(\alpha(x_1)) \int_a^{x_1} -\varphi_1(s) - (-\psi_1(s))ds, \quad x_1 \in [a, b].
\]
Denote
\[ G(x_1) = \psi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds - \frac{1}{2} \int_a^{x_1} \psi_1^2(s) - \varphi_1^2(s) ds, \quad x_1 \in [a, b]. \]

If we show \( G'(x_1) > 0, \ x_1 \in (a, b) \) the proof will be complete (\( G(a) = 0 \)).

\[
G'(x_1) = \psi_1'(\alpha(x_1)) \alpha'(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds + \psi_1(\alpha(x_1))(\psi_1(x_1) - \varphi_1(x_1)) \]
\[
- \frac{1}{2}(\psi_1^2(x_1) - \varphi_1^2(x_1))
\]
\[
= \psi_1'(\alpha(x_1)) \alpha'(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds + (\psi_1(\alpha(x_1) - \psi_1(x_1))(\psi_1(x_1) - \varphi_1(x_1))
\]
\[
+ \frac{1}{2}(\psi_1(x_1) - \varphi_1(x_1))^2.
\]

From
\[
\alpha(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds = \int_a^{x_1} s(\psi_1(s) - \varphi_1(s)) ds
\]
we get a formula for the derivative of \( \alpha : \)
\[
\alpha'(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds = (x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)).
\]

The Mean Value Theorem gives that there is some \( \xi(x_1) \) such that \( a < \alpha(x_1) < \xi(x_1) < x_1 \) and \( \psi(x_1) - \psi_1(\alpha(x_1)) = \psi_1'(\xi(x_1))(x_1 - \alpha(x_1)) \). \( G'(x_1) \) can be rewritten as
\[
G'(x_1) = \psi_1'(\alpha(x_1))(x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)) \]
\[
- \psi_1'(\xi(x_1))(x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)) + \frac{1}{2}(\psi_1(x_1) - \varphi_1(x_1))^2
\]
\[
= (\psi_1'(\alpha(x_1)) - \psi_1'(\xi(x_1)))(x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)) + \frac{1}{2}(\psi_1(x_1) - \varphi_1(x_1))^2.
\]

Since \( \varphi_1(x_1) < \psi_1(x_1), \ x_1 \in (a, b), \ \psi_1 \) is a concave function on \([a, b]\) we get \( G'(x_1) > 0, \ x_1 \in (a, b) \). The proof is complete. \( \square \)

**Theorem 2.1.** Let \( M_2 \) be a simple two dimensional convex set. Let \( F(x, y) \) be a convex function on \( M_2 \). Then
\[
F(\alpha(b), \beta(b)) \int_a^b \psi_1(s) - \varphi_1(s) ds \leq \int_a^b \int_{\varphi_1(x)}^{\psi_1(x)} F(x, y) dy dx, \quad (2.3)
\]
where
\[
\alpha(x) = \frac{\int_a^x s(\psi_1(s) - \varphi_1(s))ds}{\int_a^x \psi_1(s) - \varphi_1(s)ds}, \quad \beta(x) = \frac{\int_a^x (\psi_1^2(s) - \varphi_1^2(s))ds}{\int_a^x \psi_1(s) - \varphi_1(s)ds}, \quad x \in (a,b)
\]
and
\[
\alpha(a) = a, \quad \beta(a) = \frac{1}{2}(\psi_1(a) + \varphi_1(a)).
\]

Proof. Using the L’Hospital’s rule we get
\[
\lim_{x \to a^+} \beta(x) = \frac{1}{2} \lim_{x \to a^+} \frac{\psi_1^2(x) - \varphi_1^2(x)}{\psi_1(x) - \varphi_1(x)} = \frac{1}{2}(\psi_1(a) + \varphi_1(a)) = \beta(a)
\]
and thus \(\beta(x)\) is a continuous function on \([a,b]\). Denote
\[
H(x) = \int_a^x \int_a^y F(s,y)dsdy - F(\alpha(x),\beta(x)) \int_a^x \psi_1(s) - \varphi_1(s)ds, \quad x \in [a,b].
\]
We show that \(H'(x) \geq 0, \quad x \in (a,b)\) which implies \(H(x) \geq 0, \quad x \in [a,b]\) and the proof will be complete.

\[
H'(x) = \int_{\varphi_1(x)}^{\psi_1(x)} F(x,y)dy - \left( \frac{\partial F(\alpha(x),\beta(x))}{\partial \alpha} \alpha'(x) + \frac{\partial F(\alpha(x),\beta(x))}{\partial \beta} \beta'(x) \right)
\]
\[
\times \int_a^x \psi_1(s) - \varphi_1(s)ds - F(\alpha(x),\beta(x))(\psi_1(x) - \varphi_1(x)).
\]
The convexity of \(F(x,y)\) in variable \(y\) on each \([\varphi_1(x),\psi_1(x)]\), \(x \in [a,b]\) and the classical Hermite-Hadamard inequality imply
\[
\int_{\varphi_1(x)}^{\psi_1(x)} F(x,y)dy \geq F\left(x, \frac{\varphi_1(x) + \psi_1(x)}{2}\right)(\psi_1(x) - \varphi_1(x)), \quad x \in [a,b].
\]
So
\[
H'(x) \geq \left( F\left(x, \frac{\varphi_1(x) + \psi_1(x)}{2}\right) - F(\alpha(x),\beta(x)) \right)(\psi_1(x) - \varphi_1(x))
\]
\[
- \left( \frac{\partial F(\alpha(x),\beta(x))}{\partial \alpha} \alpha'(x) + \frac{\partial F(\alpha(x),\beta(x))}{\partial \beta} \beta'(x) \right) \int_a^x \psi_1(s) - \varphi_1(s)ds, \quad x \in (a,b).
\]
According to Lemma 2.1 $\varphi_1(\alpha(x)) \leq \beta(x) \leq \psi_1(\alpha(x))$, $x \in [a,b]$. From the convexity of $F$ we get

$$F(x,y) \geq F(x_0,y_0) + \frac{\partial F(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial F(x_0,y_0)}{\partial y}(y-y_0), \quad (x,y),(x_0,y_0) \in M_2.$$ 

This implies

$$F\left(x,\frac{\varphi_1(x) + \psi_1(x)}{2}\right) - F(\alpha(x),\beta(x)) \geq \frac{\partial F(\alpha(x),\beta(x))}{\partial \xi}(x-\alpha(x)) + \frac{\partial F(\alpha(x),\beta(x))}{\partial \eta} \left(\frac{\varphi_1(x) + \psi_1(x)}{2} - \beta(x)\right), \quad x \in [a,b].$$

So

$$H'(x) \geq \frac{\partial F(\alpha(x),\beta(x))}{\partial \alpha} \left((x-\alpha(x))(\psi_1(x) - \varphi_1(x)) - \alpha'(x) \int_a^x \psi_1(s) - \varphi_1(s) ds\right)$$

$$+ \frac{\partial F(\alpha(x),\beta(x))}{\partial \beta} \left(\left(\frac{\varphi_1(x) + \psi_1(x)}{2} - \beta(x)\right)(\psi_1(x) - \varphi_1(x)) - \beta'(x) \int_a^x \psi_1(s) - \varphi_1(s) ds\right)$$

for $x \in (a,b)$. From

$$\beta(x) \int_a^x \psi_1(s) - \varphi_1(s) ds = \frac{1}{2} \int_a^x (\psi_1^2(s) - \varphi_1^2(s)) ds$$

we get a formula for the derivative of $\beta$:

$$\beta'(x) \int_a^x \psi_1(s) - \varphi_1(s) ds = \left(\frac{\varphi_1(x) + \psi_1(x)}{2} - \beta(x)\right)(\psi_1(x) - \varphi_1(x)).$$

Since $\alpha(x), \beta(x)$ are solutions of differential equations

$$(x-\alpha(x))(\psi_1(x) - \varphi_1(x)) - \alpha'(x) \int_a^x \psi_1(s) - \varphi_1(s) ds = 0,$$

$$\left(\frac{\varphi_1(x) + \psi_1(x)}{2} - \beta(x)\right)(\psi_1(x) - \varphi_1(x)) - \beta'(x) \int_a^x \psi_1(s) - \varphi_1(s) ds = 0,$$

$x \in (a,b)$ we get $H'(x) \geq 0$ on $(a,b)$. The proof is complete. \qed

By the mathematical induction we prove the $n$ dimensional version of the left hand side Hermite-Hadamard inequality. First, we prove the following auxiliary lemma.
Lemma 2.2. Let \( n \in \mathbb{N} \), \( n > 1 \), \( M_{n+1} \) be a simple \( n+1 \)-dimensional convex set.

Let

\[
\alpha_{n+1,j}(x) = \frac{1}{g_{n+1}(x)} \int_a^x \psi_1(x_1) \psi_n(x_1, \ldots, x_n) \int_a^x \cdots \int_a^x x_j dx_{n+1} \ldots dx_1, \quad j = 1, \ldots, n+1, \quad x \in (a,b],
\]

\[
(\alpha_{n+1,1}(x))' = g_{n+1}'(x), \quad x \in (a,b),
\]

\[
(\alpha_{n+1,j}(x))' = \alpha_{n,j-1}(x) g_{n+1}'(x) \quad j = 2, \ldots, n+1, \quad x \in (a,b),
\]

\[
a < \alpha_{n+1,1}(x) < b, \quad x \in (a,b], \quad \alpha_{n+1,1}(a) = a,
\]

\[
\phi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) \leq \alpha_{n+1,j+1}(x) \leq \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)),
\]

\[
\phi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) \leq \alpha_{n+1,j+1}(x) \leq \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)),
\]

\( j = 1, \ldots, n \), \( x \in (a,b] \).

Proof. First, we prove that \( \alpha_{n+1,j}(x) \), \( \alpha_{n,j}^*(x) \) are right continuous functions in the point \( a \). Using the L’Hospital’s rule we have \( \lim_{x \to a^+} \alpha_{n+1,1}(x) = a \), \( \lim_{x \to a^+} \alpha_{n+1,j}(x) = \lim_{x \to a^+} \alpha_{n,j-1}(x) = t_{j-1} \) for \( j = 2, \ldots, n+1 \), where \( t_i \) is a \( i \)-barycentric coordinate of \( M_n = M_{n+1} \cap \{(x_1, \ldots, x_{n+1}); x_1 = a\} \). It follows from that \( (x, \alpha_{n,1}^*(x), \ldots, \alpha_{n,n}^*(x)) \), \( x \in (a,b) \) is a barycentre of \( n \)-dimensional convex set \( M_{n+1} \cap \{(x_1, \ldots, x_{n+1}); x_1 = x\} \) (a cross section of \( M_{n+1} \)) and \( \lim_{x \to a^+} (x, \alpha_{n,1}^*(x), \ldots, \alpha_{n,n}^*(x)) = (a,t_1, \ldots, t_n) \). Next, using elementary calculations we obtain (2.5), (2.6), (2.7). By induction on \( j \) we get (2.8). We need prove only the ride side of (2.8), because the left side of (2.8) can be written in the form of the right side of (2.8). Indeed,

\[
\phi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) \leq \alpha_{n+1,j+1}(x), \quad j = 1, \ldots, n
\]

is equivalent to

\[
\int_a^{s_1} \int_a^{s_2} \cdots \int_a^{s_n} s_j ds_{n+1} \ldots ds_1 \leq -\phi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) g_{n+1}(x),
\]

where \( x \in [a,b], -\phi_j \) are concave functions, \( -\psi_j \) are convex functions, \( j = 1, \ldots, n \) owing to the convexity of \( M_{n+1} \). We prove (2.8) for \( j = 1 \). Let \( j = 1 \) then \( \alpha_{n+1,2}(x) \leq
Assume that $\alpha_{n+1,i+1}(x) \leq \psi_i(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,i}(x))$ is valid for all $i$ such that $1 \leq i < j$. We prove $\alpha_{n+1,j+1}(x) \leq \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x))$ which is equivalent to

$$v_j(x) = g_{n+1}(x) \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) - \int \int \ldots \int x_j dx_{n+1} \ldots dx_1 \geq 0.$$ 

Since $v_j(a) = 0$ it is sufficient to show that $v_j'(x) \geq 0$, $x \in (a, b)$.

$$v_j'(x) \geq g_{n+1}'(x) \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) + g_{n+1}(x) \left( \sum_{i=1}^j \frac{\partial \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \right)$$

$$\times \alpha_{n+1,j}'(x) - \int \int \ldots \int \psi_j(x, x_2, \ldots, x_j) dx_{n+1} \ldots dx_2$$

$$\geq g_{n+1}'(x) \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) + g_{n+1}(x) \left( \sum_{i=1}^j \frac{\partial \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \right)$$

$$\times \alpha_{n+1,j}'(x) - \int \int \ldots \int \psi_j(x, x_2, \ldots, x_j) dx_{n+1} \ldots dx_2$$

$$= g_{n+1}(x) \sum_{i=1}^j \frac{\partial \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \alpha_{n+1,i}'(x)$$
\[ \psi_1(x) \quad \psi_2(x, x_2, \ldots, x_n) \]

\[ - \int_{\varphi_1(x)}^{\varphi_2(x, x_2, \ldots, x_n)} (\psi_1(x, x_2, \ldots, x_j) - \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x))) \, dx_{n+1} \ldots dx_2. \]

The concavity of \( \psi_j \) implies

\[ \psi_j(x, x_2, \ldots, x_j) - \psi_j(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) \leq \sum_{i=1}^{j} \frac{\partial \psi_j}{\partial \alpha_{n+1,i}}(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) (x_i - \alpha_{n+1,i}(x)) \]

where \( x_1 = x \).

From this

\[ v'_j(x) \geq \sum_{i=1}^{j} \frac{\partial \psi_j}{\partial \alpha_{n+1,i}}(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) \]

\[ \times \left( \alpha'_{n+1,i}(x) g_{n+1}(x) - \int_{\varphi_1(x)}^{\varphi_2(x, x_2, \ldots, x_n)} (x_i - \alpha_{n+1,i}(x)) \, dx_{n+1} \ldots dx_2 \right) \]

\[ = \frac{\partial \psi_j}{\partial \alpha_{n+1,1}}(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) \left( \alpha'_{n+1,1}(x) g_{n+1}(x) - (x - \alpha_{n+1,1}(x)) g'_{n+1}(x) \right) \]

\[ + \sum_{i=2}^{j} \frac{\partial \psi_j}{\partial \alpha_{n+1,i}}(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,j}(x)) \]

\[ \times \left( \alpha'_{n+1,i}(x) g_{n+1}(x) - (\alpha'_{n,i-1}(x) - \alpha_{n+1,i}(x)) g'_{n+1}(x) \right). \]

From (2.5), (2.6) we get

\[ \alpha'_{n+1,1}(x) g_{n+1}(x) - (x - \alpha_{n+1,1}(x)) g'_{n+1}(x) = 0, \quad (2.9) \]

\[ \alpha'_{n+1,i}(x) g_{n+1}(x) - (\alpha'_{n,i-1}(x) - \alpha_{n+1,i}(x)) g'_{n+1}(x) = 0, \quad i = 2, \ldots, j. \quad (2.10) \]

It implies \( v'_j(x) \geq 0 \). The proof is complete. \( \square \)

**Theorem 2.2.** Let \( n \in \mathbb{N}, \, M_{n+1} \) be a simple \( n + 1 \) dimensional convex set. Let \( F(x_1, \ldots, x_{n+1}) \) be a convex function on \( M_{n+1} \). Then

\[ F(\alpha_{n+1,1}(b), \ldots, \alpha_{n+1,n+1}(b)) \mu(M_{n+1}) \leq \int_{M_{n+1}} F(x_1, \ldots, x_{n+1}) d(M_{n+1}), \]

where \( \alpha_{n+1,j}(x), \, j = 1, \ldots, n + 1 \) are defined in Lemma 2.2,

\[ \mu(M_{n+1}) = \int_{a}^{b} \int_{\varphi_1(x_1)}^{\varphi_2(x_1, \ldots, x_n)} \ldots \int_{\varphi_n(x_1, \ldots, x_n)} \psi_1(x_1) \psi_2(x_1, \ldots, x_n) \, dx_{n+1} \ldots dx_1. \]
\[
\int_{M_{n+1}} F(x_1, \ldots, x_{n+1}) d(M_{n+1}) = \int_a^b \int_{\mathcal{D}} \cdots \int_{\mathcal{D}} F(x_1, \ldots, x_{n+1}) dx_{n+1} \cdots dx_1.
\]

**Proof.** We use the mathematical induction on \( n \). Theorem 2.2 is valid for \( n = 1 \) (Theorem 2.1). Suppose that Theorem 2.2 is valid for all \( i \) such that \( 1 \leq i < n \). We prove that Theorem 2.2 is valid for \( i = n \). Denote

\[
H_{n+1}(x) = \int_a^b \int_{\mathcal{D}} \cdots \int_{\mathcal{D}} F(x_1, \ldots, x_{n+1}) dx_{n+1} \cdots dx_1 - F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x)) g_{n+1}(x).
\]

Since \( H_{n+1}(a) = 0 \) it is sufficient to show \( H_{n+1}^\prime(x) \geq 0, \ x \in (a,b) \).

\[
H_{n+1}^\prime(x) = \int_a^b \int_{\mathcal{D}} \cdots \int_{\mathcal{D}} F(x, x_2, \ldots, x_{n+1}) dx_{n+1} \cdots dx_2 \geq F(x, \alpha_{n,1}^\ast(x), \ldots, \alpha_{n,n}^\ast(x)) g_{n+1}^\prime(x),
\]

\( x \in (a,b) \). From this we have

\[
H_{n+1}^\prime(x) \geq \left( F(x, \alpha_{n,1}^\ast(x), \ldots, \alpha_{n,n}^\ast(x)) - F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x)) g_{n+1}^\prime(x) \right)
\]

\[
= g_{n+1}(x) \sum_{i=1}^{n+1} \frac{\partial F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,i}} \alpha_{n+1,i}^\ast(x).
\]

The convexity of \( F \) implies

\[
F(x, \alpha_{n,1}^\ast(x), \ldots, \alpha_{n,n}^\ast(x)) - F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x))
\]

\[
\geq \frac{\partial F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,1}} (x - \alpha_{n+1,1}(x))
\]

\[
+ \sum_{i=2}^{n+1} \frac{\partial F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,i}} (\alpha_{n+1,i-1}(x) - \alpha_{n+1,i}(x)).
\]
So, we have
\[
H_{n+1}'(x) \geq \frac{\partial F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,1}} \left( (x - \alpha_{n+1,1}(x))g_{n+1}'(x) - \alpha_{n+1,1}'(x)g_{n+1}(x) \right)
+ \sum_{i=2}^{n+1} \frac{\partial F(\alpha_{n+1,1}(x), \ldots, \alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,i}} \times \left( (\alpha_{n,i-1}'(x) - \alpha_{n+1,i}(x))g_{n+1}'(x) - \alpha_{n+1,i}'(x)g_{n+1}(x) \right).
\]
From (2.9), (2.10) we get \( H_{n+1}'(x) \geq 0, \ x \in (a, b) \). The proof is complete. \( \square \)

3. Examples

Example 3.1. Let \( M_2 = \{(x_1, x_2); 0 \leq x_1 \leq b, \ l x_1 \leq x_2 \leq k x_1 \} \) where \( l < k, 0 < b \). Then \( M_2 \) is a triangle.

\[
\alpha(x) = \frac{\int_0^x (k x_1 - l x_1)dx_1}{\int_0^x (k x_1 - l x_1)dx_1} = \frac{2}{3} x,
\]

\[
\beta(x) = \frac{1}{2} \frac{\int_a^x (k x_1^2 - l x_1^2)dx_1}{\int_a^x (k x_1 - l x_1)dx_1} = \frac{1}{3} (k + l) x.
\]

So \( \alpha(b) = \frac{2}{3} b, \ \beta(b) = \frac{1}{3} (k + l) b \). If \( F(x, y) \) is a convex function on \( M_2 \) then

\[
F \left( \frac{2}{3} b, \frac{1}{3} (k + l) b \right) \leq \frac{2}{(k - l)b^2} \int_0^b \int_{lx_1}^{kx_1} F(x_1, x_2)dx_2dx_1.
\]

Example 3.2. Let \( M_2 = \{(x_1, x_2); -m \leq x_1 \leq m, -\frac{n}{m} \sqrt{m^2 - x_1^2} \leq x_2 \leq \frac{n}{m} \sqrt{m^2 - x_1^2} \} \) where \( 0 < m, 0 < n \). Then \( M_2 \) is a ellipse.

\[
\alpha(x) = \frac{\int_{-m}^x \sqrt{m^2 - x_1^2}dx_1}{\int_{-m}^x \sqrt{m^2 - x_1^2}dx_1} = -\frac{2}{3} \left( \frac{(m^2 - x^2) \sqrt{m^2 - x^2}}{x \sqrt{(m^2 - x^2)} - m^2 \arccos \left( \frac{x}{m} \right) + m^2 \pi} \right), \ \beta(x) = 0.
\]

Thus \( \alpha(m) = 0, \ \beta(m) = 0 \). If \( F(x_1, x_2) \) is a convex function on \( M_2 \) then

\[
F(0, 0) \leq \frac{1}{mn \pi} \int_{M_2} F(x_1, x_2)dx_1dx_2.
\]
EXAMPLE 3.3. Let $M_3 = \{(x_1, x_2, x_3); 0 \leq x_1 \leq b, -kx_1 \leq x_2 \leq kx_1, -\sqrt{k^2 x_1^2 - x_2^2} \leq x_3 \leq \sqrt{k^2 x_1^2 - x_2^2}\}$ where $0 < k, 0 < b$. Then $M_3$ is a cone.

$$
\alpha_{3,1}(x) = \frac{1}{g_3(x)} \int_{0}^{x} \int_{-kx_1}^{kx_1} \int_{-\sqrt{k^2 x_1^2 - x_2^2}}^{\sqrt{k^2 x_1^2 - x_2^2}} x_1 dx_3 dx_2 dx_1 = \frac{3}{4} x,
$$

$$
\alpha_{3,2}(x) = \frac{1}{g_3(x)} \int_{0}^{x} \int_{-kx_1}^{kx_1} \int_{-\sqrt{k^2 x_1^2 - x_2^2}}^{\sqrt{k^2 x_1^2 - x_2^2}} x_2 dx_3 dx_2 dx_1 = 0,
$$

$$
\alpha_{3,3}(x) = \frac{1}{g_3(x)} \int_{0}^{x} \int_{-kx_1}^{kx_1} \int_{-\sqrt{k^2 x_1^2 - x_2^2}}^{\sqrt{k^2 x_1^2 - x_2^2}} x_3 dx_3 dx_2 dx_1 = 0
$$

$$
g_3(x) = \int_{0}^{x} \int_{-kx_1}^{kx_1} \int_{-\sqrt{k^2 x_1^2 - x_2^2}}^{\sqrt{k^2 x_1^2 - x_2^2}} 1 dx_3 dx_2 dx_1 = \frac{\pi k^2 x^3}{3}.
$$

Thus $\alpha_{3,1}(b) = \frac{3}{4} b$, $\alpha_{3,2}(b) = 0$, $\alpha_{3,3}(b) = 0$. If $F(x_1, x_2, x_3)$ is a convex function on $M_3$ then

$$
F\left(\frac{3}{4} b, 0, 0\right) \leq \frac{3}{\pi k^2 b^3} \int_{M_3} F(x_1, x_2, x_3) dx_1 dx_2 dx_3.
$$

EXAMPLE 3.4. Let $M_3 = \{(x_1, x_2, x_3); -R \leq x_1 \leq R, -\sqrt{R^2 - x_1^2} \leq x_2 \leq \sqrt{R^2 - x_1^2}, -\sqrt{R^2 - x_1^2 - x_2^2} \leq x_3 \leq \sqrt{R^2 - x_1^2 - x_2^2}\}$ where $0 < R$. Then $M_3$ is a sphere.

$$
\alpha_{3,1}(x) = \frac{1}{g_3(x)} \int_{-R}^{x} \int_{-\sqrt{R^2 - x_1^2}}^{\sqrt{R^2 - x_1^2}} \int_{-\sqrt{R^2 - x_1^2 - x_2^2}}^{\sqrt{R^2 - x_1^2 - x_2^2}} x_1 dx_3 dx_2 dx_1 = -\frac{3}{4} \frac{(R-x)^2}{(2R-x)},
$$

$$
\alpha_{3,2}(x) = \frac{1}{g_3(x)} \int_{-R}^{x} \int_{-\sqrt{R^2 - x_1^2}}^{\sqrt{R^2 - x_1^2}} \int_{-\sqrt{R^2 - x_1^2 - x_2^2}}^{\sqrt{R^2 - x_1^2 - x_2^2}} x_2 dx_3 dx_2 dx_1 = 0,
$$

$$
\alpha_{3,3}(x) = \frac{1}{g_3(x)} \int_{-R}^{x} \int_{-\sqrt{R^2 - x_1^2}}^{\sqrt{R^2 - x_1^2}} \int_{-\sqrt{R^2 - x_1^2 - x_2^2}}^{\sqrt{R^2 - x_1^2 - x_2^2}} x_3 dx_3 dx_2 dx_1 = 0,
$$

where $0 < k, 0 < b$. Then $M_3$ is a cone.
\[ g_3(x) = \int_{-R}^{R} \int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}} \int_{-\sqrt{R^2-x_1^2-x_2^2}}^{\sqrt{R^2-x_1^2-x_2^2}} 1 \, dx_3 \, dx_2 \, dx_1 = \frac{\pi}{3} (R^2 + x)^2 (2R - x). \]

Thus \( \alpha_{3,1}(R) = 0 \), \( \alpha_{3,2}(R) = 0 \), \( \alpha_{3,3}(R) = 0 \). If \( F(x_1, x_2, x_3) \) is a convex function on \( M_3 \) then

\[ F(0, 0, 0) \leq \frac{3}{4\pi R^3} \int_{M_3} F(x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3. \]

REFERENCES


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Ladislav Matejíčka
Institute of Information, Automation and Mathematics
Faculty of Chemical and Food-processing Technology
Radlinského 9
81237 Bratislava
Slovakia
e-mail: matejicka@tnuni.sk
URL: http://www.myhomepage.edu