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ELEMENTARY PROOF OF THE LEFT MULTIDIMENSIONAL HERMITE–HADAMARD INEQUALITY ON CERTAIN CONVEX SETS

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Abstract. The left Hermite-Hadamard inequality of several variables for convex functions on certain convex compact sets is proved via elementary approach, independently of Choquet theory.

1. Introduction

The Hermite-Hadamard inequality plays an important role in research on inequalities. The monographs [5] and [11] give a comprehensive review the literature. The classical Hermite-Hadamard inequality [7], [9] is:

If $f : \langle a,b\rangle \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

Among the various generalizations of this inequality, is the problem of extending Hermite-Hadamard inequality to convex function of several variables. C.P. Niculescu [12] gave the most general answer to this problem. From Choquet theory [8], [13], [10] it follows that if K is a compact convex set in a locally convex Hausdorff space, μ is a positive Borel measure, $f : K \to \mathbb{R}$ is a convex function, then

$$f\left(b_\mu\right) \leqslant \frac{1}{\mu(K)}\int_K f(x)d\mu,$$

where b_μ is the barycenter of K. Some particular cases for special convex sets have been investigated by S.S. Dragomir [2], [3], [4], B. Gavrea [6] and by Bessenyei [1]. The aim of the paper is to verify left Hermite-Hadamard inequality on "simple" convex compact sets via elementary approach, independently of Choquet theory.

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2. The main results

First, we define simple convex sets.

DEFINITION 2.1. Let $n \in \mathbb{N}$, $n > 1$, $a < b$, $a,b \in \mathbb{R}$, $M_1 = [a,b]$, $M_1^0 = (a,b)$. We say that M_n is a simple n dimensional convex set, if M_n is a convex set, $M_n = \{(x_1,...,x_n);\ a \leqslant x_1 \leqslant b,\ \varphi_1(x_1) \leqslant x_2 \leqslant \psi_1(x_1),\ \dots, \varphi_j(x_1,...,x_j) \leqslant x_{j+1} \leqslant \psi_j(x_1,...,x_j),\ \dots,\ \varphi_{n-1}(x_1,...,x_{n-1}) \leqslant x_n \leqslant \psi_{n-1}(x_1,...,x_{n-1})\}$ where $\varphi_j(x_1,...,x_j)$, $\psi_j(x_1,...,x_j)$ are functions with continuous first derivatives on M_j for $j = 1,...,n-1$, and $\varphi_j(x_1,...,x_j) < \psi_j(x_1,...,x_j)$ on the interior of M_j, $j = 1,...,n-1$.

The set of simple convex sets is "sufficiently" large set. For example, circles, ellipses, triangles, cones, balls,... are simple convex sets.

First, we prove an auxiliary lemma and the left two dimensional Hermite-Hadamard inequality.

LEMMA 2.1. *Let M_2 be a simple two dimensional convex set. Then*

$$\varphi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds \leqslant \frac{1}{2} \int_a^{x_1} \psi_1^2(s) - \varphi_1^2(s) ds \leqslant \psi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds, \tag{2.1}$$

$x_1 \in [a,b]$, *where*

$$\alpha(x_1) = \frac{\int_a^{x_1} s(\psi_1(s) - \varphi_1(s)) ds}{\int_a^{x_1} \psi_1(s) - \varphi_1(s) ds}, \quad x_1 \in (a,b] \text{ and } \alpha(a) = a. \tag{2.2}$$

Proof. Using the L'Hospital's rule we have

$$\lim_{x_1 \to a^+} \alpha(x_1) = \lim_{x_1 \to a^+} \frac{x_1(\psi_1(x_1) - \varphi_1(x_1))}{\psi_1(x_1) - \varphi_1(x_1)} = a = \alpha(a)$$

and thus $\alpha(x_1)$ is a continuous function on $[a,b]$. We also observe that $\alpha_1(x_1) < x_1$, $x_1 \in (a,b]$. It is sufficient to prove only the right side of (2.1). Indeed, from M_2 is a convex set we have $\varphi_1(x_1)$ is a convex function, $\psi_1(x_1)$ is a concave function, $-\varphi_1(x_1)$ is a concave function, $-\psi_1(x_1)$ is a convex function $x_1 \in [a,b]$. According to the assumption: $-\psi_1(x_1) < -\varphi_1(x_1)$, $x_1 \in (a,b)$ the inequality

$$\varphi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s) ds \leqslant \frac{1}{2} \int_a^{x_1} \psi_1^2(s) - \varphi_1^2(s) ds$$

is equivalent to

$$\frac{1}{2} \int_a^{x_1} (-\varphi_1(s))^2 - (-\psi_1(s))^2 ds \leqslant -\varphi_1(\alpha(x_1)) \int_a^{x_1} -\varphi_1(s) - (-\psi_1(s)) ds, \quad x_1 \in [a,b].$$

Denote

$$G(x_1) = \psi_1(\alpha(x_1)) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds - \frac{1}{2}\int_a^{x_1} \psi_1^2(s) - \varphi_1^2(s)ds, \quad x_1 \in [a,b].$$

If we show $G'(x_1) > 0$, $x_1 \in (a,b)$ the proof will be complete ($G(a) = 0$).

$$\begin{aligned} G'(x_1) &= \psi_1'(\alpha(x_1))\alpha'(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds + \psi_1(\alpha(x_1))(\psi_1(x_1) - \varphi_1(x_1)) \\ &\quad - \frac{1}{2}(\psi_1^2(x_1) - \varphi_1^2(x_1)) \\ &= \psi_1'(\alpha(x_1))\alpha'(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds + (\psi_1(\alpha(x_1) - \psi_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)) \\ &\quad + \frac{1}{2}(\psi_1(x_1) - \varphi_1(x_1))^2. \end{aligned}$$

From

$$\alpha(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds = \int_a^{x_1} s(\psi_1(s) - \varphi_1(s))ds$$

we get a formula for the derivative of α :

$$\alpha'(x_1) \int_a^{x_1} \psi_1(s) - \varphi_1(s)ds = (x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)).$$

The Mean Value Theorem gives that there is some $\xi(x_1)$ such that $a < \alpha(x_1) < \xi(x_1) < x_1$ and $\psi(x_1) - \psi_1(\alpha(x_1)) = \psi_1'(\xi(x_1))(x_1 - \alpha(x_1))$. $G'(x_1)$ can be rewritten as

$$\begin{aligned} G'(x_1) &= \psi_1'(\alpha(x_1))(x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)) \\ &\quad - \psi_1'(\xi(x_1))(x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)) + \frac{1}{2}(\psi_1(x_1) - \varphi_1(x_1))^2 \\ &= (\psi_1'(\alpha(x_1)) - \psi_1'(\xi(x_1)))(x_1 - \alpha_1(x_1))(\psi_1(x_1) - \varphi_1(x_1)) + \frac{1}{2}(\psi_1(x_1) - \varphi_1(x_1))^2. \end{aligned}$$

Since $\varphi_1(x_1) < \psi_1(x_1)$, $x_1 \in (a,b)$, ψ_1 is a concave function on $[a,b]$ we get $G'(x_1) > 0$, $x_1 \in (a,b)$. The proof is complete. □

THEOREM 2.1. *Let M_2 be a simple two dimensional convex set. Let $F(x,y)$ be a convex function on M_2. Then*

$$F(\alpha(b),\beta(b)) \int_a^b \psi_1(s) - \varphi_1(s)ds \leqslant \int_a^b \int_{\varphi_1(x)}^{\psi_1(x)} F(x,y)dydx, \tag{2.3}$$

where

$$\alpha(x)=\frac{\int\limits_a^x s(\psi_1(s)-\varphi_1(s))ds}{\int\limits_a^x \psi_1(s)-\varphi_1(s)ds},\quad \beta(x)=\frac{1}{2}\frac{\int\limits_a^x (\psi_1^2(s)-\varphi_1^2(s))ds}{\int\limits_a^x \psi_1(s)-\varphi_1(s)ds},\ x\in(a,b]$$

and

$$\alpha(a)=a,\ \beta(a)=\frac{1}{2}(\psi_1(a)+\varphi_1(a)).$$

Proof. Using the L'Hospital's rule we get

$$\lim_{x\to a^+}\beta(x)=\frac{1}{2}\lim_{x\to a^+}\frac{\psi_1^2(x)-\varphi_1^2(x)}{\psi_1(x)-\varphi_1(x)}=\frac{1}{2}(\psi_1(a)+\varphi_1(a))=\beta(a)$$

and thus $\beta(x)$ is a continuous function on $[a,b]$. Denote

$$H(x)=\int\limits_a^x\int\limits_{\varphi_1(s)}^{\psi_1(s)}F(s,y)dyds-F(\alpha(x),\beta(x))\int\limits_a^x\psi_1(s)-\varphi_1(s)ds,\ x\in[a,b].$$

We show that $H'(x)\geqslant 0,\ x\in(a,b)$ which implies $H(x)\geqslant 0,\ x\in[a,b]$ and the proof will be complete.

$$H'(x)=\int\limits_{\varphi_1(x)}^{\psi_1(x)}F(x,y)dy-\left(\frac{\partial F(\alpha(x),\beta(x))}{\partial\alpha}\alpha'(x)+\frac{\partial F(\alpha(x),\beta(x))}{\partial\beta}\beta'(x)\right)$$

$$\times\int\limits_a^x\psi_1(s)-\varphi_1(s)ds-F\left(\alpha(x),\beta(x)\right)\left(\psi_1(x)-\varphi_1(x)\right).$$

The convexity of $F(x,y)$ in variable y on each $[\varphi_1(x),\psi_1(x)]$, $x\in[a,b]$ and the classical Hermite-Hadamard inequality imply

$$\int\limits_{\varphi_1(x)}^{\psi_1(x)}F(x,y)dy\geqslant F\left(x,\frac{\varphi_1(x)+\psi_1(x)}{2}\right)(\psi_1(x)-\varphi_1(x)),\ x\in[a,b].$$

So

$$H'(x)\geqslant\left(F\left(x,\frac{\varphi_1(x)+\psi_1(x)}{2}\right)-F(\alpha(x),\beta(x))\right)(\psi_1(x)-\varphi_1(x))$$

$$-\left(\frac{\partial F(\alpha(x),\beta(x))}{\partial\alpha}\alpha'(x)+\frac{\partial F(\alpha(x),\beta(x))}{\partial\beta}\beta'(x)\right)\int\limits_a^x\psi_1(s)-\varphi_1(s)ds,\ x\in(a,b).$$

According to Lemma 2.1 $\varphi_1(\alpha(x)) \leqslant \beta(x) \leqslant \psi_1(\alpha(x))$, $x \in [a,b]$. From the convexity of F we get

$$F(x,y) \geqslant F(x_0,y_0) + \frac{\partial F(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial F(x_0,y_0)}{\partial y}(y-y_0), \quad (x,y),(x_0,y_0) \in M_2.$$

This implies

$$F\left(x, \frac{\varphi_1(x)+\psi_1(x)}{2}\right) - F(\alpha(x),\beta(x))$$

$$\geqslant \frac{\partial F(\alpha(x),\beta(x))}{\partial \xi}(x-\alpha(x)) + \frac{\partial F(\alpha(x),\beta(x))}{\partial \eta}\left(\frac{\varphi_1(x)+\psi_1(x)}{2} - \beta(x)\right), \quad x \in [a,b].$$

So

$$H'(x) \geqslant \frac{\partial F(\alpha(x),\beta(x))}{\partial \alpha}\left((x-\alpha(x))(\psi_1(x)-\varphi_1(x)) - \alpha'(x)\int_a^x \psi_1(s)-\varphi_1(s)ds\right)$$
$$+ \frac{\partial F(\alpha(x),\beta(x))}{\partial \beta}\left(\left(\frac{\varphi_1(x)+\psi_1(x)}{2} - \beta(x)\right)(\psi_1(x)-\varphi_1(x))\right.$$
$$\left. -\beta'(x)\int_a^x \psi_1(s)-\varphi_1(s)ds\right)$$

for $x \in (a,b)$. From

$$\beta(x)\int_a^x \psi_1(s)-\varphi_1(s)ds = \frac{1}{2}\int_a^x (\psi_1^2(s)-\varphi_1^2(s))ds$$

we get a formula for the derivative of β :

$$\beta'(x)\int_a^x \psi_1(s)-\varphi_1(s)ds = \left(\frac{\varphi_1(x)+\psi_1(x)}{2} - \beta(x)\right)(\psi_1(x_1)-\varphi_1(x_1)).$$

Since $\alpha(x)$, $\beta(x)$ are solutions of differential equations

$$(x-\alpha(x))(\psi_1(x)-\varphi_1(x)) - \alpha'(x)\int_a^x \psi_1(s)-\varphi_1(s)ds = 0,$$

$$\left(\frac{\varphi_1(x)+\psi_1(x)}{2} - \beta(x)\right)(\psi_1(x)-\varphi_1(x)) - \beta'(x)\int_a^x \psi_1(s)-\varphi_1(s)ds = 0,$$

$x \in (a,b)$ we get $H'(x) \geqslant 0$ on (a,b). The proof is complete. $\square$

By the mathematical induction we prove the n dimensional version of the left hand side Hermite-Hadamard inequality. First, we prove the following auxiliary lemma.

LEMMA 2.2. *Let $n \in \mathbb{N}$, $n > 1$, M_{n+1} be a simple $n+1$ dimensional convex set. Let*

$$\alpha_{n+1,j}(x) = \frac{1}{g_{n+1}(x)} \int_a^x \int_{\varphi_1(x_1)}^{\psi_1(x_1)} \cdots \int_{\varphi_n(x_1,\dots,x_n)}^{\psi_n(x_1,\dots,x_n)} x_j dx_{n+1} \dots dx_1, \; j = 1,\dots,n+1, \; x \in (a,b],$$

$$\alpha^*_{n,j}(x) = \frac{1}{g'_{n+1}(x)} \int_{\varphi_1(x)}^{\psi_1(x)} \cdots \int_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} x_{j+1} dx_{n+1} \dots dx_2, \; j = 1,\dots,n, \; x \in (a,b], \tag{2.4}$$

$$g_{n+1}(x) = \int_a^x \int_{\varphi_1(x_1)}^{\psi_1(x_1)} \cdots \int_{\varphi_n(x_1,\dots,x_n)}^{\psi_n(x_1,\dots,x_n)} 1 dx_{n+1} \dots dx_1, \quad x \in [a,b].$$

Then

$$(\alpha_{n+1,1}(x) g_{n+1}(x))' = x g'_{n+1}(x), \quad x \in (a,b), \tag{2.5}$$

$$(\alpha_{n+1,j}(x) g_{n+1}(x))' = \alpha^*_{n,j-1}(x) g'_{n+1}(x) \quad j = 2,\dots,n+1, \quad x \in (a,b), \tag{2.6}$$

$$a < \alpha_{n+1,1}(x) < x, \quad x \in (a,b], \quad \alpha_{n+1,1}(a) = a, \tag{2.7}$$

$$\varphi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)) \leqslant \alpha_{n+1,j+1}(x) \leqslant \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)), \tag{2.8}$$

$j = 1,\dots,n, \quad x \in (a,b]$.

Proof. First, we prove that $\alpha_{n+1,j}(x)$, $\alpha^*_{n,j}(x)$ are right continuous functions in the point a. Using the L'Hospital's rule we have $\lim\limits_{x \to a^+} \alpha_{n+1,1}(x) = a$, $\lim\limits_{x \to a^+} \alpha_{n+1,j}(x) = \lim\limits_{x \to a^+} \alpha^*_{n,j-1}(x) = t_{j-1}$ for $j = 2,\dots,n+1$, where t_i is a $i-$barycentric coordinate of $M^*_n = M_{n+1} \cap \{(x_1,\dots,x_{n+1});\ x_1 = a\}$. It follows from that $(x, \alpha^*_{n,1}(x),\dots,\alpha^*_{n,n}(x))$, $x \in (a,b)$ is a barycentre of $n-$dimensional convex set $M_{n+1} \cap \{(x_1,\dots,x_{n+1});\ x_1 = x\}$ (a cross section of M_{n+1}) and $\lim\limits_{x \to a^+} (x, \alpha^*_{n,1}(x),\dots,\alpha^*_{n,n}(x)) = (a, t_1,\dots,t_n)$. Next, using elementary calculations we obtain (2.5), (2.6), (2.7). By induction on j we get (2.8). We need prove only the ride side of (2.8), because the left side of (2.8) can be written in the form of the right side of (2.8). Indeed,

$$\varphi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)) \leqslant \alpha_{n+1,j+1}(x), \quad j = 1,\dots,n$$

is equivalent to

$$\int_a^x \int_{-\psi_1(s_1)}^{-\varphi_1(s_1)} \int_{-\psi_2(s_1,-s_2)}^{-\varphi_2(s_1,-s_2)} \cdots \int_{-\psi_n(s_1,-s_2,\dots,-s_n)}^{-\varphi_n(s_1,-s_2,\dots,-s_n)} s_j ds_{n+1} \dots ds_1 \leqslant -\varphi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)) g_{n+1}(x),$$

where $x \in [a,b]$, $-\varphi_j$ are concave functions, $-\psi_j$ are convex functions, $j = 1,\dots,n$ owing to the convexity of M_{n+1}. We prove (2.8) for $j = 1$. Let $j = 1$ then $\alpha_{n+1,2}(x) \leqslant$

$\psi_1(\alpha_{n+1,1}(x))$ is equivalent to

$$v_1(x) = g_{n+1}(x)\psi_1(\alpha_{n+1,1}(x)) - \int_a^x \int_{\varphi_1(x_1)}^{\psi_1(x_1)} \dots \int_{\varphi_n(x_1,\dots,x_n)}^{\psi_n(x_1,\dots,x_n)} x_2 dx_{n+1}\dots dx_1 \geqslant 0.$$

Since $v_1(a) = 0$ it is sufficient to show that $v_1'(x) \geqslant 0$, $x \in (a,b)$.

$$\begin{aligned}
v_1'(x) &\geqslant g_{n+1}'(x)\psi_1(\alpha_{n+1,1}(x)) + g_{n+1}(x)\psi_1'(\alpha_{n+1,1}(x))\alpha_{n+1,1}'(x) \\
&\quad - \psi_1(x) \int_{\varphi_1(x)}^{\psi_1(x)} \dots \int_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} 1 dx_{n+1}\dots dx_2 \\
&= (\psi_1(\alpha_{n+1,1}(x)) - \psi_1(x))g_{n+1}'(x) + g_{n+1}(x)\psi_1'(\alpha_{n+1,1}(x))\alpha_{n+1,1}'(x) \\
&= g_{n+1}(x)\psi_1'(\alpha_{n+1,1}(x))\alpha_{n+1,1}'(x) - g_{n+1}'(x)\psi_1'(\xi_1(x))(x - \alpha_{n+1,1}(x)),
\end{aligned}$$

where $\alpha_{n+1,1}(x) < \xi_1(x) < x$. We used the Mean Value Theorem. From this

$$v_1'(x) \geqslant (\psi_1'(\alpha_{n+1,1}(x)) - \psi_1'(\xi_1(x)))(x - \alpha_{n+1,1}(x))g_{n+1}'(x) \geqslant 0$$

because of ψ_1 is a concave function.

Assume that $\alpha_{n+1,i+1}(x) \leqslant \psi_i(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,i}(x))$ is valid for all i such that $1 \leqslant i < j$. We prove $\alpha_{n+1,j+1}(x) \leqslant \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))$ which is equivalent to

$$v_j(x) = g_{n+1}(x)\psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)) - \int_a^x \int_{\varphi_1(x_1)}^{\psi_1(x_1)} \dots \int_{\varphi_n(x_1,\dots,x_n)}^{\psi_n(x_1,\dots,x_n)} x_{j+1} dx_{n+1}\dots dx_1 \geqslant 0.$$

Since $v_j(a) = 0$ it is sufficient to show that $v_j'(x) \geqslant 0$, $x \in (a,b)$.

$$\begin{aligned}
v_j'(x) &= g_{n+1}'(x)\psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)) + g_{n+1}(x)\sum_{i=1}^{j} \frac{\partial \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \\
&\quad \times \alpha_{n+1,i}'(x) - \int_{\varphi_1(x)}^{\psi_1(x)} \dots \int_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} x_{j+1} dx_{n+1}\dots dx_2 \\
&\geqslant g_{n+1}'(x)\psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)) + g_{n+1}(x)\sum_{i=1}^{j} \frac{\partial \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \\
&\quad \times \alpha_{n+1,i}'(x) - \int_{\varphi_1(x)}^{\psi_1(x)} \dots \int_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} \psi_j(x,x_2,\dots,x_j) dx_{n+1}\dots dx_2 \\
&= g_{n+1}(x)\sum_{i=1}^{j} \frac{\partial \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \alpha_{n+1,i}'(x)
\end{aligned}$$

$$-\int_{\varphi_1(x)}^{\psi_1(x)} \dots \int_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} (\psi_j(x,x_2,\dots,x_j) - \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)))dx_{n+1}\dots dx_2.$$

The concavity of ψ_j implies

$$\begin{aligned} &\psi_j(x,x_2,\dots,x_j) - \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x)) \\ &\leqslant \sum_{i=1}^{j} \frac{\partial \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} (x_i - \alpha_{n+1,i}(x)) \end{aligned}$$

where $x_1 = x$.

From this

$$\begin{aligned} v_j'(x) \geqslant &\sum_{i=1}^{j} \frac{\partial \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \\ &\times \left(\alpha_{n+1,i}'(x) g_{n+1}(x) - \int_{\varphi_1(x)}^{\psi_1(x)} \dots \int_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} (x_i - \alpha_{n+1,i}(x)) dx_{n+1}\dots dx_2 \right) \\ = &\frac{\partial \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))}{\partial \alpha_{n+1,1}} \Big(\alpha_{n+1,1}'(x) g_{n+1}(x) - (x - \alpha_{n+1,1}(x)) g_{n+1}'(x) \Big) \\ &+ \sum_{i=2}^{j} \frac{\partial \psi_j(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,j}(x))}{\partial \alpha_{n+1,i}} \\ &\times \Big(\alpha_{n+1,i}'(x) g_{n+1}(x) - (\alpha_{n,i-1}^*(x) - \alpha_{n+1,i}(x)) g_{n+1}'(x) \Big). \end{aligned}$$

From (2.5), (2.6) we get

$$\alpha_{n+1,1}'(x) g_{n+1}(x) - (x - \alpha_{n+1,1}(x)) g_{n+1}'(x) = 0, \tag{2.9}$$

$$\alpha_{n+1,i}'(x) g_{n+1}(x) - (\alpha_{n,i-1}^*(x) - \alpha_{n+1,i}(x)) g_{n+1}'(x) = 0, \quad i = 2,\dots,j. \tag{2.10}$$

It implies $v_j'(x) \geqslant 0$. The proof is complete. □

THEOREM 2.2. *Let $n \in \mathbb{N}$, M_{n+1} be a simple $n+1$ dimensional convex set. Let $F(x_1,\dots,x_{n+1})$ be a convex function on M_{n+1}. Then*

$$F(\alpha_{n+1,1}(b),\dots,\alpha_{n+1,n+1}(b))\mu(M_{n+1}) \leqslant \int_{M_{n+1}} F(x_1,\dots,x_{n+1}) d(M_{n+1}),$$

where $\alpha_{n+1,j}(x)$, $j = 1,\dots,n+1$ are defined in Lemma 2.2,

$$\mu(M_{n+1}) = \int_a^b \int_{\varphi_1(x_1)}^{\psi_1(x_1)} \dots \int_{\varphi_n(x_1,\dots,x_n)}^{\psi_n(x_1,\dots,x_n)} 1 dx_{n+1}\dots dx_1,$$

$$\int\limits_{M_{n+1}} F(x_1,\dots,x_{n+1})d(M_{n+1}) = \int\limits_a^b \int\limits_{\varphi_1(x_1)}^{\psi_1(x_1)} \dots \int\limits_{\varphi_n(x_1,\dots,x_n)}^{\psi_n(x_1,\dots,x_n)} F(x_1,\dots,x_{n+1})dx_{n+1}\dots dx_1.$$

Proof. We use the mathematical induction on n. Theorem 2.2 is valid for $n=1$ (Theorem 2.1). Suppose that Theorem 2.2 is valid for all i such that $1 \leqslant i < n$. We prove that Theorem 2.2 is valid for $i=n$. Denote

$$\begin{aligned}H_{n+1}(x) = &\int\limits_a^x \int\limits_{\varphi_1(x_1)}^{\psi_1(x_1)} \dots \int\limits_{\varphi_n(x_1,\dots,x_n)}^{\psi_n(x_1,\dots,x_n)} F(x_1,\dots,x_{n+1})dx_{n+1}\dots dx_1\\ &- F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x))g_{n+1}(x).\end{aligned}$$

Since $H_{n+1}(a)=0$ it is sufficient to show $H'_{n+1}(x) \geqslant 0$, $x \in (a,b)$.

$$\begin{aligned}H'_{n+1}(x) = &\int\limits_{\varphi_1(x)}^{\psi_1(x)} \dots \int\limits_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} F(x,x_2,\dots,x_{n+1})dx_{n+1}\dots dx_2\\ &- g_{n+1}(x)\sum_{i=1}^{n+1} \frac{\partial F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,i}}\alpha'_{n+1,i}(x)\\ &- g'_{n+1}(x)F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x)).\end{aligned}$$

Lemma 2.2 and the induction assumption imply that

$$\int\limits_{\varphi_1(x)}^{\psi_1(x)} \dots \int\limits_{\varphi_n(x,x_2,\dots,x_n)}^{\psi_n(x,x_2,\dots,x_n)} F(x,x_2,\dots,x_{n+1})dx_{n+1}\dots dx_2 \geqslant F(x,\alpha^*_{n,1}(x),\dots,\alpha^*_{n,n}(x))g'_{n+1}(x),$$

$x \in (a,b]$. From this we have

$$\begin{aligned}H'_{n+1}(x) \geqslant &(F(x,\alpha^*_{n,1}(x),\dots,\alpha^*_{n,n}(x)) - F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x))g'_{n+1}(x)\\ &- g_{n+1}(x)\sum_{i=1}^{n+1} \frac{\partial F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,i}}\alpha'_{n+1,i}(x).\end{aligned}$$

The convexity of F implies

$$\begin{aligned}&F(x,\alpha^*_{n,1}(x),\dots,\alpha^*_{n,n}(x)) - F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x))\\ &\quad\geqslant \frac{\partial F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,1}}(x-\alpha_{n+1,1}(x))\\ &\qquad+\sum_{i=2}^{n+1} \frac{\partial F(\alpha_{n+1,1}(x),\dots,\alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,i}}(\alpha^*_{n,i-1}(x)-\alpha_{n+1,i}(x)).\end{aligned}$$

So, we have

$$H'_{n+1}(x) \geqslant \frac{\partial F(\alpha_{n+1,1}(x),\ldots,\alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,1}}\Big((x-\alpha_{n+1,1}(x))g'_{n+1}(x)-\alpha'_{n+1,1}(x)g_{n+1}(x)\Big)$$
$$+\sum_{i=2}^{n+1}\frac{\partial F(\alpha_{n+1,1}(x),\ldots,\alpha_{n+1,n+1}(x))}{\partial \alpha_{n+1,i}}$$
$$\times\Big((\alpha^*_{n,i-1}(x)-\alpha_{n+1,i}(x))g'_{n+1}(x)-\alpha'_{n+1,i}(x)g_{n+1}(x)\Big).$$

From (2.9), (2.10) we get $H'_{n+1}(x)\geqslant 0$, $x\in(a,b)$. The proof is complete. $\square$

3. Examples

EXAMPLE 3.1. Let $M_2=\{(x_1,x_2);0\leqslant x_1\leqslant b,\ lx_1\leqslant x_2\leqslant kx_1\}$ where $l<k$, $0<b$. Then M_2 is a triangle.

$$\alpha(x)=\frac{\int\limits_0^x x_1(kx_1-lx_1)dx_1}{\int\limits_0^x (kx_1-lx_1)dx_1}=\frac{2}{3}x,$$

$$\beta(x)=\frac{1}{2}\frac{\int\limits_a^x (k^2x_1^2-l^2x_1^2)dx_1}{\int\limits_a^x (kx_1-lx_1)dx_1}=\frac{1}{3}(k+l)x.$$

So $\alpha(b)=\frac{2}{3}b$, $\beta(b)=\frac{1}{3}(k+l)b$. If $F(x,y)$ is a convex function on M_2 then

$$F\left(\frac{2}{3}b,\frac{1}{3}(k+l)b\right)\leqslant\frac{2}{(k-l)b^2}\int\limits_0^b\int\limits_{lx_1}^{kx_1}F(x_1,x_2)dx_2dx_1.$$

EXAMPLE 3.2. Let $M_2=\Big\{(x_1,x_2);\ -m\leqslant x_1\leqslant m,\ -\frac{n}{m}\sqrt{m^2-x_1^2}\leqslant x_2\leqslant\frac{n}{m}\sqrt{m^2-x_1^2}\Big\}$ where $0<m$, $0<n$. Then M_2 is a ellipse.

$$\alpha(x)=\frac{\int\limits_{-m}^x x_1\sqrt{m^2-{x_1}^2}dx_1}{\int\limits_{-m}^x \sqrt{m^2-{x_1}^2}dx_1}=-\frac{2}{3}\left(\frac{(m^2-x^2)\sqrt{(m^2-x^2)}}{x\sqrt{(m^2-x^2)}-m^2\arccos\left(\frac{x}{m}\right)+m^2\pi}\right),\ \beta(x)=0.$$

Thus $\alpha(m)=0$, $\beta(m)=0$. If $F(x_1,x_2)$ is a convex function on M_2 then

$$F(0,0)\leqslant\frac{1}{mn\pi}\int\limits_{M_2}F(x_1,x_2)dx_1dx_2.$$

EXAMPLE 3.3. Let $M_3=\Big\{(x_1,x_2,x_3); 0\leqslant x_1\leqslant b,\ -kx_1\leqslant x_2\leqslant kx_1,\ -\sqrt{k^2x_1^2-x_2^2}\leqslant x_3\leqslant \sqrt{k^2x_1^2-x_2^2}\Big\}$ where $0<k$, $0<b$. Then M_3 is a cone.

$$\alpha_{3,1}(x)=\frac{1}{g_3(x)}\int_0^x\int_{-kx_1}^{kx_1}\int_{-\sqrt{k^2x_1^2-x_2^2}}^{\sqrt{k^2x_1^2-x_2^2}} x_1dx_3dx_2dx_1=\frac{3}{4}x,$$

$$\alpha_{3,2}(x)=\frac{1}{g_3(x)}\int_0^x\int_{-kx_1}^{kx_1}\int_{-\sqrt{k^2x_1^2-x_2^2}}^{\sqrt{k^2x_1^2-x_2^2}} x_2dx_3dx_2dx_1=0,$$

$$\alpha_{3,3}(x)=\frac{1}{g_3(x)}\int_0^x\int_{-kx_1}^{kx_1}\int_{-\sqrt{k^2x_1^2-x_2^2}}^{\sqrt{k^2x_1^2-x_2^2}} x_3dx_3dx_2dx_1=0$$

$$g_3(x)=\int_0^x\int_{-kx_1}^{kx_1}\int_{-\sqrt{k^2x_1^2-x_2^2}}^{\sqrt{k^2x_1^2-x_2^2}} 1dx_3dx_2dx_1=\frac{\pi k^2x^3}{3}.$$

Thus $\alpha_{3,1}(b)=\frac{3}{4}b$, $\alpha_{3,2}(b)=0$, $\alpha_{3,3}(b)=0$. If $F(x_1,x_2,x_3)$ is a convex function on M_3 then

$$F\left(\frac{3}{4}b,0,0\right)\leqslant\frac{3}{\pi k^2b^3}\int_{M_3}F(x_1,x_2,x_3)dx_1dx_2dx_3.$$

EXAMPLE 3.4. Let $M_3=\Big\{(x_1,x_2,x_3); -R\leqslant x_1\leqslant R,\ -\sqrt{R^2-x_1^2}\leqslant x_2\leqslant\sqrt{R^2-x_1^2},\ -\sqrt{R^2-x_1^2-x_2^2}\leqslant x_3\leqslant\sqrt{R^2-x_1^2-x_2^2}\Big\}$ where $0<R$. Then M_3 is a sphere.

$$\alpha_{3,1}(x)=\frac{1}{g_3(x)}\int_{-R}^x\int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}}\int_{-\sqrt{R^2-x_1^2-x_2^2}}^{\sqrt{R^2-x_1^2-x_2^2}} x_1dx_3dx_2dx_1=-\frac{3}{4}\frac{(R-x)^2}{(2R-x)},$$

$$\alpha_{3,2}(x)=\frac{1}{g_3(x)}\int_{-R}^x\int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}}\int_{-\sqrt{R^2-x_1^2-x_2^2}}^{\sqrt{R^2-x_1^2-x_2^2}} x_2dx_3dx_2dx_1=0,$$

$$\alpha_{3,3}(x)=\frac{1}{g_3(x)}\int_{-R}^x\int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}}\int_{-\sqrt{R^2-x_1^2-x_2^2}}^{\sqrt{R^2-x_1^2-x_2^2}} x_3dx_3dx_2dx_1=0,$$

$$g_3(x) = \int_{-R}^{x} \int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}} \int_{-\sqrt{R^2-x_1^2-x_2^2}}^{\sqrt{R^2-x_1^2-x_2^2}} 1 dx_3 dx_2 dx_1 = \frac{\pi}{3}(R+x)^2(2R-x).$$

Thus $\alpha_{3,1}(R) = 0$, $\alpha_{3,2}(R) = 0$, $\alpha_{3,3}(R) = 0$. If $F(x_1, x_2, x_3)$ is a convex function on M_3 then

$$F(0,0,0) \leqslant \frac{3}{4\pi R^3} \int_{M_3} F(x_1, x_2, x_3) dx_1 dx_2 dx_3.$$

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