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NONLINEAR PARABOLIC INEQUALITIES ON A GENERAL CONVEX DOMAIN

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Abstract. The paper deals with the existence and uniqueness of solutions of some non linear parabolic inequalities in the Orlicz-Sobolev spaces framework.

1. Introduction

We consider boundary value problems of type

$$\begin{cases} u \in \mathscr{K} & \\ \dfrac{\partial u}{\partial t} + A(u) = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{P}$$

where

$$A(u) = -\mathrm{div}(a(.,t,\nabla u)),$$

$Q = \Omega \times [0,T]$, $T > 0$ and Ω is a bounded domain of $\mathbb{R}^N$, with the segment property. Further, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (measurable with respect to x in Ω for every (t,ξ) in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to ξ in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) such that for all $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$,

$$a(x,t,\xi)\xi \geqslant \alpha M(|\xi|), \tag{1.1}$$

$$[a(x,t,\xi) - a(x,t,\xi^*)][\xi - \xi^*] > 0, \tag{1.2}$$

$$|a(x,t,\xi)| \leqslant c(x,t) + k_1 \overline{M}^{-1} M(k_2|\xi|), \tag{1.3}$$

where $c(x,t) \in E_{\overline{M}}(Q), c \geqslant 0$, $k_i \in \mathbb{R}^+$, for $i = 1,2$ and $\alpha \in \mathbb{R}^+_*$.

$$f \in W^{-1,x}E_{\overline{M}}(Q), f \geqslant 0, \tag{1.4}$$

$$u_0 \in L^2(\Omega) \cap K, u_0 \geqslant 0, \tag{1.5}$$

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where K is a given closed convex set of $W_0^1 L_M(\Omega)$ and $\mathscr{K}$ is defined by:

$$\mathscr{K} := \{v \in W_0^{1,x} L_M(Q) : v(t) \in K\}.$$

During the last decades, the theory of variational inequalities and complementarity problems have been applied in different fields such as Mathematical Programming, Game Theory, Economics and Mathematical Finance. In the last case, the problem is used for the pricing of American options (see [1] and the references therein). One of the most interesting and important problems in the theory of variational inequalities is the development of efficient iterative algorithms to approximate their solutions.

It is well known that (P) admits at least one solution (see Lions [13] and Landes-Mustonen [15]). In the last papers, the function $a(x,t,\xi)$ was assumed to satisfy a polynomial growth condition with respect to ∇u. When trying to relax this restriction on the function $a(.,\xi)$, we are led to replace the space $L^p(0,T;W^{1,p}(\Omega))$ by an inhomogeneous Sobolev space $W^{1,x}L_M$ built from an Orlicz space L_M instead of L^p, where the N-function M which defines L_M is related to the actual growth of the Carathéodory's function. It is our purpose in this paper, to prove existence results and uniqueness for the problem (P) in the setting of the inhomogeneous Sobolev space $W^{1,x}L_M$. For the sake of simplicity, we suppose through this paper that $a(x,t,\nabla u) = \frac{m(|\nabla u|)}{|\nabla u|}\nabla u$, where m is the derivative of the N-function M having the representation $M(t) = \int_0^t m(s)\,ds$.

We refer the reader to [18, 19, 16, 6, 4] for additional recent and classical results for some parabolic inequalities problems.

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation: $M(t) = \int_0^t a(\tau)d\tau$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(\tau)d\tau$, where $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \leqslant t\}$ (see [2], [11] and [12]).

The N-function M is said to satisfy the Δ_2 condition if, for some $k > 0$:

$$M(2t) \leqslant kM(t) \quad \text{for all } t \geqslant 0. \tag{2.1}$$

In case this inequality holds only for $t \geqslant t_0 > 0$, M is said to satisfy the Δ_2 condition near infinity.

Let P and Q be two N-functions. PlQ means that P grows essentially less rapidly than Q, i.e., for each $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon t)} \to 0 \quad \text{as } t \to \infty.$$

This is the case if and only if

$$\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathscr{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))dx < +\infty \quad \left(\text{resp.} \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right).$$

Note that $L_M(\Omega)$ is a Banach space with the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leqslant 1\right\}$$

and $\mathscr{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

We now turn to the Orlicz-Sobolev space: $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space with the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leqslant 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified by subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwarz space $\mathscr{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathscr{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \to 0$ for all $|\alpha| \leqslant 1$. This implies the $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ convergence. If M satisfies the Δ_2 condition on $\mathbb{R}^+$ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) be the space of distributions on Ω which can be written as sums of derivatives of order $\leqslant 1$ of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space with the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathscr{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [8],

[9]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

For $k>0$, we define the truncation at height $k, T_k:\mathbb{R}\to\mathbb{R}$ by

$$T_k(s)=\begin{cases} s & \text{if } |s|\leqslant k,\\ k & \text{if } |s|>k.\end{cases}$$

The following abstract lemmas will be applied to the truncation operators.

LEMMA 2.1. (see [3]) *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly lipschitzian function such that $F(0)=0$. Let M be an N-function and $u\in W_0^1L_M(\Omega)$ (resp. $W_0^1E_M(\Omega)$). Then $F(u)\in W_0^1L_M(\Omega)$ (resp. $W_0^1E_M(\Omega)$). Moreover, if the set of discontinuity points of F' is finite then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{a.e. in } \{x\in\Omega: u(x)\notin D\}\\ 0 & \text{a.e. in } \{x\in\Omega: u(x)\in D\}.\end{cases}$$

Proof. By hypothesis, $F(u)\in W^1L_M(\Omega)$ for all $u\in W^1L_M(\Omega)$ and

$$\|F(u)\|_{1,M,\Omega}\leqslant C\,\|u\|_{1,M,\Omega},$$

which gives the result. □

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T>0$ and set $Q=\Omega\times]0,T[$. Let $m\geqslant 1$ be an integer and let M be an N-function. For each $\alpha\in\mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to $x\in\mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows: $W^{m,x}L_M(Q)=\{u\in L_M(Q): D_x^\alpha u\in L_M(Q)\ \forall|\alpha|\leqslant m\}$, $W^{m,x}E_M(Q)=\{u\in E_M(Q): D_x^\alpha u\in E_M(Q)\ \forall|\alpha|\leqslant m\}$. This second space is a subspace of the first one, and both are Banach spaces with the norm

$$\|u\|=\sum_{|\alpha|\leqslant m}\|D_x^\alpha u\|_{M,Q}.$$

These spaces constitute a complementary system since Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there is α-order derivatives, $|\alpha|\leqslant m$. We shall also consider the weak topologies $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ and $\sigma(\Pi L_M,\Pi L_{\overline{M}})$. If $u\in W^{m,x}L_M(Q)$ then the function $: t\longmapsto u(t)=u(t,.)$ is defined on $[0,T]$ with values in $W^mL_M(\Omega)$. If $u\in W^{m,x}E_M(Q)$ the concerned function is a $W^mE_M(\Omega)$-valued and is strongly measurable. Furthermore, the imbedding $W^{m,x}E_M(Q)\subset L^1(0,T;W^mE_M(\Omega))$ holds. The space $W^{m,x}L_M(Q)$ is not in general separable; for $u\in W^{m,x}L_M(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0,T]$. However, the scalar function $t\mapsto\|u(t)\|_{M,\Omega}\in L^1(0,T)$. The space $W_0^{m,x}E_M(Q)$ is defined as the (norm) closure in $W^{m,x}E_M(Q)$ of $\mathscr{D}(Q)$. We can easily show as in [9] that when Ω has the segment property then each element u of the closure of $\mathscr{D}(Q)$ with respect to the weak * topology $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ is limit in $W^{m,x}L_M(Q)$ of some subsequence $(u_i)\subset\mathscr{D}(Q)$ for the modular convergence, i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leqslant m$,

$$\int_Q M\left(\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}\right) dxdt \to 0 \text{ as } i \to \infty,$$

which gives that (u_i) converges to u in $W^{m,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently

$$\overline{\mathscr{D}(Q)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathscr{D}(Q)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

The space of functions satisfying such property will be denoted by $W_0^{m,x}L_M(Q)$. Furthermore, $W_0^{m,x}E_M(Q) = W_0^{m,x}L_M(Q) \cap \Pi E_M$. Poincaré's inequality also holds in $W_0^{m,x}L_M(Q)$ i.e. there exists a constant $C > 0$ such that for all $u \in W_0^{m,x}L_M(Q)$,

$$\sum_{|\alpha| \leqslant m} \|D_x^\alpha u\|_{M,Q} \leqslant C \sum_{|\alpha| = m} \|D_x^\alpha u\|_{M,Q}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{m,x}L_M(Q)$. We then have the following complementary system

$$\begin{pmatrix} W_0^{m,x}L_M(Q) & F \\ W_0^{m,x}E_M(Q) & F_0 \end{pmatrix}.$$

F states for the dual space of $W_0^{m,x}E_M(Q)$ and can be defined, except for an isomorphism, as the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{m,x}E_M(Q)^\perp$. It will be denoted by $F = W^{-m,x}L_{\overline{M}}(Q)$ with

$$W^{-m,x}L_{\overline{M}}(Q) = \Big\{ f = \sum_{|\alpha| \leqslant m} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q) \Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\|_F = \inf \sum_{|\alpha| \leqslant m} \|f_\alpha\|_{\overline{M},Q},$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha| \leqslant m} D_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q).$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha| \leqslant m} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q) \Big\}$$

and is denoted by $F_0 = W^{-m,x}E_{\overline{M}}(Q)$.

REMARK 2.1. Using lemma 4.4 of [9], we can check that each uniformly lipschitzian mapping F such that $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1, $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.

3. Main results

THEOREM 3.1. *Under the hypotheses (1.1)–(1.5), the problem (P) has at least one solution in the following sense:*

$$\begin{cases} u \in \mathscr{K} \cap L^2(Q) \\ \langle \dfrac{\partial v}{\partial t}, u-v\rangle + \displaystyle\int_Q a(.,\nabla u)(\nabla u - \nabla v)dxdt \leqslant \langle f, u-v\rangle \end{cases}$$

for all $v \in \mathscr{K} \cap L^\infty(Q) \cap D$, where $D := \{v \in W_0^{1,x}L_M(Q) \cap L^2(Q) : \dfrac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$ and $v(0) = u_0\}$.

Proof. In the sequel and throughout the paper, we will omit for simplicity the dependence on t in the function $a(x,t,\xi)$ and denote $\varepsilon(n,j,\mu,i,s)$ all quantities (possibly different) such that

$$\lim_{s\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon(n,j,\mu,i,s) = 0.$$

The order in which the parameters will tend to infinity, is, first n, then j,μ,i and finally s. Similarly, we will skip some parameters such as in $\varepsilon(n)$ or $\varepsilon(n,j)$, ...to mean that the limits are taken only on the specified parameters.

Let us define the indicator functional: $\Phi : W_0^{1,x}L_M(Q) \to \mathbb{R} \cup \{+\infty\}$ such that:

$$\Phi(v) := \begin{cases} 0 & \text{if } v(t) \in K \text{ a.e. (almost everywhere)}, \\ +\infty & \text{otherwise.} \end{cases}$$

Φ is weakly lower semicontinuous. Let us denote by $S_k(t) := \displaystyle\int_0^t T_k(s)ds$.

Step 1. Derivation of a priori estimate

Let us consider the following approximate problem:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} + A(u_n) + nT_n(\Phi(u_n)) = f & \text{in } Q, \\ u_n(.,0) = u_{0n} & \text{in } \Omega, \end{cases} \qquad (P_n)$$

where $(u_{0n}) \subset D(\Omega)$ such that $u_{0n} \to u_0$ strongly in $L^2(\Omega)$.

For the existence of a weak solution $u_n \in W_0^{1,x}L_M(Q) \cap L^2(Q)$, $u_n \geqslant 0$ of the above problem, see [7], also (u_n) satisfies $\dfrac{\partial u_n}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$. Let $v = u_n$ be test function in (P_n), then

$$\langle \frac{\partial u_n}{\partial t}, u_n\rangle + \int_Q a(.,\nabla u_n)\nabla u_n dxdt + \int_Q nT_n(\Phi(u_n))u_n dxdt \leqslant \langle f, u_n\rangle.$$

We can deduce that:

$$(u_n) \quad \text{is bounded in } W_0^{1,x}L_M(Q),$$

$$\int_Q a(.,\nabla u_n)\nabla u_n dxdt \leqslant C,$$

$$\int_Q nT_n(\Phi(u_n))u_n dxdt \leqslant C.$$

Therefore there exists a subsequence (also denoted (u_n)) and a measurable function u such that:

$$u_n \rightharpoonup u,$$

$$\text{weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$

$$\text{strongly in } E_M(Q) \text{ and a.e in } Q.$$

Moreover there exists a measurable function $h \in (L_{\overline{M}}(Q))^N$ such that:

$$a(.,\nabla u_n) \rightharpoonup h \text{ in } (L_{\overline{M}}(Q))^N \text{ weakly}.$$

Let us consider now $v = T_k(u_n) \in W_0^{1,x}L_M(Q)$ as test function in (P_n), which gives

$$\langle\frac{\partial u_n}{\partial t}, T_k(u_n)\rangle + \int_Q a(.,\nabla u_n)\nabla T_k(u_n)dxdt + \int_Q nT_n(\Phi(u_n))T_k(u_n)dxdt \leqslant \langle f, T_k(u_n)\rangle \leqslant Ck$$

Since $\langle\frac{\partial u_n}{\partial t}, T_k(u_n)\rangle = \int_\Omega S_k(u_n(T)) - \int_\Omega S_k(u_n(0))$,

$$\int_Q nT_n(\Phi(u_n))T_k(u_n)dxdt \leqslant Ck.$$

By letting k tend to zero and using Fatou lemma, one has:

$$\int_Q nT_n(\Phi(u_n))dxdt \leqslant Ck,$$

and since $(T_n)_n$ is a continuous increasing sequence, we can deduce

$$\Phi(u) = 0,$$

which ensures that $u \in \mathscr{K}$.

Step 2. Almost everywhere convergence of the gradients

We intend to prove that

$$\lim_{n\to\infty} \int_Q (a(.,\nabla u_n) - a(.,\nabla u))\,(\nabla u_n - \nabla u)\,dx\,dt = 0.$$

Let us set

$$\omega_{\mu,j}^i = (v_j)_\mu + e^{-\mu t}\psi_i,$$

where $v_j \in D(Q)$ such that $v_j \to u$ with the modular convergence in $W_0^{1,x}L_M(Q)$, ψ_i a smooth function such that $\psi_i \to u_0$ strongly in $L^2(\Omega)$ and ω_μ is the mollifier function defined in [14], and the function $\omega^i_{\mu,j}$ have the following properties:

$$\begin{cases} \dfrac{\partial \omega^i_{\mu,j}}{\partial t} = \mu(v_j - \omega^i_{\mu,j}), \omega^i_{\mu,j}(0) = \psi_i, \\ \omega^i_{\mu,j} \to u_\mu + e^{-\mu t}\psi_i \text{in} W_0^{1,x}L_M(Q) \quad \text{for the modular convergence with respect to } j, \\ u_\mu + e^{-\mu t}\psi_i \to u \text{ in } W_0^{1,x}L_M(Q) \quad \text{for the modular convergence with respect to } \mu. \end{cases}$$

Consider now $v = u_n - \omega^i_{\mu,j}$ as test function in (P_n),

$$\langle \frac{\partial u_n}{\partial t}, u_n - \omega^i_{\mu,j}\rangle + \int_Q a(.,\nabla u_n)(\nabla u_n - \nabla\omega^i_{\mu,j})dxdt + \int_Q nT_n(\Phi(u_n))(u_n - \omega^i_{\mu,j})dxdt \\ = \langle f, u_n - \omega^i_{\mu,j}\rangle. \tag{3.1}$$

Since $u_n \in W_0^{1,x}L_M(Q)$, there exists a smooth function $u_{n\sigma}$ (see [7]) such that:

$$u_{n\sigma} \to u_n \text{ for the modular convergence in } W_0^{1,x}L_M(Q),$$

$$\frac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,x}L_{\overline{M}}(Q) + L^2(Q).$$

Then,

$$\begin{aligned} \langle \frac{\partial u_n}{\partial t}, u_n - \omega^i_{\mu,j}\rangle &= \lim_{\sigma\to 0^+}\int_Q (u_{n\sigma})'(u_{n\sigma} - \omega^i_{\mu,j})dxdt \\ &= \lim_{\sigma\to 0^+}\left(\int_Q (u_{n\sigma}-\omega^i_{\mu,j})'(u_{n\sigma}-\omega^i_{\mu,j})dxdt + \int_Q (\omega^i_{\mu,j})'(u_{n\sigma}-\omega^i_{\mu,j})dxdt\right) \\ &= \lim_{\sigma\to 0^+}\left(\left[\frac{1}{2}\int_\Omega (u_{n\sigma} - \omega^i_{\mu,j})^2\right]_0^T + \mu\int_Q (v_j - \omega^i_{\mu,j})(u_{n\sigma} - \omega^i_{\mu,j})dxdt\right) \\ &= \lim_{\sigma\to 0^+}(I_1(\sigma) + I_2(\sigma)). \end{aligned}$$

We have,

$$\begin{aligned} I_1(\sigma) &= \frac{1}{2}\int_\Omega (u_{n\sigma} - \omega^i_{\mu,j})^2(T)dx - \frac{1}{2}\int_\Omega (u_{n\sigma}(0) - \omega^i_{\mu,j}(0))^2 dx \\ &\geqslant -\frac{1}{2}\int_\Omega (u_{n\sigma}(0) - \omega^i_{\mu,j}(0))^2 dx. \end{aligned}$$

So,

$$\lim\sup_{\sigma\to 0^+} I_1(\sigma) \geqslant \varepsilon(n,j,\mu,i).$$

Similarly, we have

$$\lim\sup_{\sigma\to 0^+} I_2(\sigma) = \varepsilon(n,j,\mu,i),$$

hence

$$\langle \frac{\partial u_n}{\partial t}, u_n - \omega^i_{\mu,j}\rangle \geqslant \varepsilon(n,j,\mu,i).$$

Now let us set for $s > 0, Q_s = \{(x,t) \in Q : |\nabla u| \leqslant s\}$ and $Q^s_j = \{(x,t) \in Q : |\nabla v_j| \leqslant s\}$. Then

$$\begin{aligned}\int_Q a(.,\nabla u_n)(\nabla u_n - \nabla\omega^i_{\mu,j})dxdt &= \int_Q \left(a(.,\nabla u_n) - a(.,\nabla v_j\chi^s_j)\right)(\nabla u_n - \nabla v_j\chi^s_j)dxdt \\ &\quad + \int_Q a(.,\nabla v_j\chi^s_j)(\nabla u_n - \nabla v_j\chi^s_j)dxdt \\ &\quad + \int_Q a(.,\nabla u_n)\nabla v_j\chi^s_j dxdt - \int_Q a(.,\nabla u_n)\nabla\omega^i_{\mu,j}dxdt \\ &=: J_1 + J_2 + J_3 + J_4.\end{aligned}$$

We first consider the term

$$J_2 = \int_Q a(.,\nabla v_j\chi^s_j)(\nabla u_n - \nabla v_j\chi^s_j)dxdt = \int_Q a(.,\nabla v_j\chi^s_j)(\nabla u - \nabla v_j\chi^s_j)dxdt + \varepsilon(n).$$

Since $a(.,\nabla v_j\chi^s_j) \to a(.,\nabla u\chi^s)$ strongly in $(E_{\overline{M}}(Q))^N$ and $\nabla v_j\chi^s_j \to \nabla u\chi^s$ strongly in $(L_M(Q))^N$, one has

$$J_2 = \varepsilon(n,j).$$

The same technique as in J_2 gives,

$$J_3 = \int_Q h\nabla u dxdt + \varepsilon(n,j,s),$$

and

$$J_4 = -\int_Q h\nabla\omega^i_{\mu,j}dxdt + \varepsilon(n) = -\int_Q h\nabla u dxdt + \varepsilon(n,j,\mu,i).$$

Then,

$$\begin{aligned}\int_Q a(.,\nabla u_n)(\nabla u_n - \nabla\omega^i_{\mu,j})dxdt &= \int_Q \left(a(.,\nabla u_n) - a(.,\nabla v_j\chi^s_j)\right)(\nabla u_n - \nabla v_j\chi^s_j)dxdt \\ &\quad + \varepsilon(n,j,\mu,i,s).\end{aligned}$$

Since terms $\int_Q nT_n(\Phi(u_n))(u_n - \omega^i_{\mu,j})dxdt$ and $= \langle f, u_n - \omega^i_{\mu,j}\rangle$ in (3.1) are of the form $\varepsilon(n)$, we obtain:

$$\int_Q \left(a(.,\nabla u_n) - a(.,\nabla v_j\chi^s_j)\right)(\nabla u_n - \nabla v_j\chi^s_j)dxdt \leqslant \varepsilon(n,j,\mu,i). \qquad (3.2)$$

On the other hand,

$$\begin{aligned}&\int_Q (a(.,\nabla u_n) - a(.,\nabla u\chi^s))(\nabla u_n - \nabla u\chi^s)dxdt \\ &\qquad - \int_Q \left(a(.,\nabla u_n) - a(.,\nabla v_j\chi^s_j)\right)(\nabla u_n - \nabla v_j\chi^s_j)dxdt\end{aligned}$$

$$= \int_Q a(.,\nabla u_n)(\nabla v_j\chi_j^s - \nabla u\chi^s)dx\,dt - \int_Q a(.,\nabla u)(\nabla v_j\chi_j^s - \nabla u_n\chi^s)dx\,dt$$
$$+ \int_Q a(.,\nabla v_j\chi_j^s)(\nabla u_n - \nabla v_j\chi_j^s)dx\,dt.$$

Since all terms are of the last sum are $\varepsilon(n,j,s)$, then

$$\int_Q (a(.,\nabla u_n) - a(.,\nabla u\chi^s))(\nabla u_n - \nabla u\chi^s)dx\,dt$$
$$= \int_Q \left(a(.,\nabla u_n) - a(.,\nabla v_j\chi_j^s)\right)(\nabla u_n - \nabla v_j\chi_j^s)dx\,dt + \varepsilon(n,j,s) \tag{3.3}$$

Finally, for $r < s$, we get:

$$\lim_{n\to\infty}\int_{Q_r} (a(.,\nabla u_n) - a(.,\nabla u))(\nabla u_n - \nabla u)\,dx\,dt = 0,$$

which gives by the same argument as in [3],

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q.$$

Step 3. The passage to the limit

Let us consider $v \in \mathscr{K} \cap L^\infty(Q) \cap D$ and $0 < \theta < 1$. Using $u_n - \theta v$ as test function in (P_n), the fact that

$$\langle\frac{\partial u_n}{\partial t}, u_n - \theta v\rangle = \langle\frac{\partial(u_n - \theta v)}{\partial t}, u_n - \theta v\rangle + \theta\langle\frac{\partial v}{\partial t}, u_n - \theta v\rangle$$

and letting n tend to infinity and θ to 1, we obtain

$$\langle\frac{\partial v}{\partial t}, u - \theta v\rangle + \int_Q a(.,\nabla u)(\nabla u - \nabla v)dxdt \leqslant \langle f, u - v\rangle.$$

So u is a weak solution of the problem (P). □

THEOREM 3.2. *The solution $u \in \mathscr{K} \cap L^2(Q)$ of the problem (P) obtained as limit of approximations of solutions (u_n) of the problem (P_n) is unique.*

Proof.

Step1: The modular convergence of the gradients

We have to prove that

$$\nabla u_n \to \nabla u \quad \text{in} \quad (L_M(Q))^N \text{ for the modular convergence.}$$

Let us recall that

$$\int_Q \left(a(.,\nabla u_n) - a(.,\nabla v_j\chi_j^s)\right)(\nabla u_n - \nabla v_j\chi_j^s)dx\,dt \leqslant \varepsilon(n,j,\mu,i)$$

and

$$\int_Q (a(.,\nabla u_n) - a(.,\nabla u\chi^s))\,(\nabla u_n - \nabla u\chi^s)dx\,dt$$
$$= \int_Q \big(a(.,\nabla u_n) - a(.,\nabla v_j\chi_j^s)\big)\,(\nabla u_n - \nabla v_j\chi_j^s)dx\,dt + \varepsilon(n,j,s).$$

Then,

$$\int_Q a(.,\nabla u_n)\nabla u_n dxdt \leqslant \int_Q a(.,\nabla u_n)\nabla u\chi^s dxdt + \int_Q a(.,\nabla u\chi^s)(\nabla u_n - \nabla u\chi^s)dx\,dt$$
$$+\,\varepsilon(n,j,\mu,i),$$

and,

$$\limsup_n \int_Q a(.,\nabla u_n)\nabla u_n dxdt \leqslant \int_Q a(.,\nabla u)\nabla u\chi^s dxdt \leqslant \liminf_n \int_Q a(.,\nabla u_n)\nabla u_n dxdt.$$

Then,

$$a(.,\nabla u_n)\nabla u_n \to a(.,\nabla u)\nabla u.\chi^s \quad \text{strongly in} \quad L^1(Q),$$

and

$$a(.,\nabla u_n)\nabla u_n \to a(.,\nabla u)\nabla u \quad \text{strongly in} \quad L^1(Q).$$

By a Vitali argument, we deduce

$$\nabla u_n \to \nabla u \quad \text{in} \quad (L_M(Q))^N \text{ for the modular convergence.} \tag{3.4}$$

Step 2: Uniqueness

Suppose there exist two solutions u_1,u_2 of the problem (P) obtained as limit of approximations of solutions of (P_n) such that $u_1(0)=u_2(0)=u_0$. Let u_1^n and u_2^n be the sequences associated respectively to u_1 and u_2. If we consider $v=(u_1^n-u_2^n)\chi_{(0,\tau)}$ as test function in the approximate problem (where we omit the index τ), we can deduce that:

$$\langle\frac{\partial(u_1^n-u_2^n)}{\partial t}, u_1^n-u_2^n\rangle + \int_Q (a(.,\nabla u_1^n) - a(.,\nabla u_2^n))(\nabla u_1^n - \nabla u_2^n)dxdt$$
$$+\int_Q n(T_n(\Phi(u_1^n)) - T_n(\Phi(u_2^n))(u_1^n-u_2^n) = 0. \tag{3.5}$$

Four situations may occur in the treatment of

$$J(n) := \int_Q n(T_n(\Phi(u_1^n)) - T_n(\Phi(u_2^n))(u_1^n-u_2^n) :$$

i) There exist two subsequences u_1^n, u_2^n belonging to $\mathscr{K}$.
ii) All subsequences u_1^n, u_2^n are not in $\mathscr{K}$.
iii) There exist two subsequences u_1^n, u_2^n such that $u_1^n \notin \mathscr{K}$ and $u_2^n \in \mathscr{K}$.
iv) There exist two subsequences u_1^n, u_2^n such that $u_1^n \in \mathscr{K}$ and $u_2^n \notin \mathscr{K}$.

The cases i) and ii) are simple since $J(n)=0$ and

$$\langle\frac{\partial(u_1^n-u_2^n)}{\partial t},u_1^n-u_2^n\rangle+\int_Q(a(.,\nabla u_1^n)-a(.,\nabla u_2^n))(\nabla u_1^n-\nabla u_2^n)dxdt=0.$$

Then,

$$\frac{1}{2}\int_\Omega|u_1^n(\tau)-u_2^n(\tau)|^2dx+\int_Q(a(.,\nabla u_1^n)-a(.,\nabla u_2^n))(\nabla u_1^n-\nabla u_2^n)dxdt$$
$$=\frac{1}{2}\int_\Omega|u_1^n(0)-u_2^n(0)|^2dx.$$

Letting n tend to infinity and using (3.4), we obtain:

$$\frac{1}{2}\int_\Omega|u_1(\tau)-u_2(\tau)|^2dx+\int_Q(a(.,\nabla u_1)-a(.,\nabla u_2))(\nabla u_1-\nabla u_2)dxdt=0,$$

which gives $u_1(\tau)=u_2(\tau)$ for all $\tau\in(0,T)$, and using (1.2) , $\nabla u_1=\nabla u_2$ a.e. in Q. Then

$$u_1=u_2 \text{ a.e. in } \quad Q.$$

The cases iii) and iv) are similar. Let us consider case iii).
We have

$$\frac{1}{2}\int_\Omega|u_1(\tau)-u_2(\tau)|^2dx+\int_Q(a(.,\nabla u_1)-a(.,\nabla u_2))(\nabla u_1-\nabla u_2)dxdt$$
$$+\lim_{n\to\infty}\left[n^2\int_Q(u_1^n-u_2^n)\right]=0,$$

which gives as previously $u_1=u_2$ a.e. in Q. □

REMARK 3.1. The existence result of theorem 3.1 remains true if a depends on $x,t,u,\nabla u$ and condition (1.3) is replaced by the following one

$$|a(x,t,s,\xi)|\leqslant c(x,t)+k_1\overline{P}^{-1}M(k_2|s|)+k_3\overline{M}^{-1}M(k_4|\xi|),$$

where $c(x,t)\in E_{\overline{M}}(Q),c\geqslant 0$ and $k_i\in\mathbb{R}^+,i=1,2,3,4$.

REMARK 3.2. The technique used in the proof of theorem 3.1 can be adapted to prove an existence result for solutions of the following parabolic inequalities:

$$\begin{cases} u\in\mathscr{K}\cap L^2(Q), \\ \displaystyle\int_0^T\langle\frac{\partial v}{\partial t},u-v\rangle dt+\int_Q a(x,t,\nabla u)\nabla(u-v)\,dx\,dt+\int_Q H(x,t,u,\nabla u)(u-v)\,dx\,dt\leqslant\langle f,u-v\rangle, \\ \text{for all } v\in\mathscr{K}\cap\mathscr{D}\cap\mathscr{L}^\infty(\mathscr{Q}), \end{cases}$$

where H is a given Carathéodory function satisfying, for all $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $(x,t) \in Q$, the following conditions

$$|H(x,t,s,\zeta)| \leqslant \lambda(|s|)(\delta(x,t) + |\zeta|^p),$$

and

$$H(x,t,s,\zeta)s \geqslant 0;$$

with $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $\delta(x,t)$ is a given positive function in $L^1(Q)$.

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