

## TRIGONOMETRIC APPROXIMATION IN GENERALIZED LEBESGUE SPACES $L^{p(x)}$

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*Abstract.* The approximation properties of Nörlund  $(N_n)$  and Riesz  $(R_n)$  means of trigonometric Fourier series are investigated in generalized Lebesgue spaces  $L^{p(x)}$ . The deviations  $\|f - N_n(f)\|_{p(x)}$  and  $\|f - R_n(f)\|_{p(x)}$  are estimated by  $n^{-\alpha}$  for  $f \in Lip(\alpha, p(x))$  ( $0 < \alpha \leq 1$ ).

### 1. Introduction and main results

Let  $p : \mathbb{R} \rightarrow [1, \infty)$  be a measurable  $2\pi$ -periodic function, that is  $p(x + 2\pi) = p(x)$ . Denote by  $L^{p(x)} = L^{p(x)}([0, 2\pi])$  the set of all measurable  $2\pi$ -periodic functions  $f$  such that  $m_p(\lambda f) < \infty$  for some  $\lambda = \lambda(f) > 0$ , where

$$m_p(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx.$$

$L^{p(x)}$  becomes a Banach space with respect to the norm

$$\|f\|_{p(x)} = \inf \left\{ \lambda > 0 : m_p \left( \frac{f}{\lambda} \right) \leq 1 \right\}.$$

If  $p(x) \equiv p$  is constant ( $1 \leq p < \infty$ ), then the space  $L^{p(x)}$  is isometrically isomorphic to the Lebesgue space  $L^p$ .

If the function  $p$  satisfies

$$1 < p_- := \operatorname{ess\,inf}_{x \in [0, 2\pi]} p(x), \quad p_+ := \operatorname{ess\,sup}_{x \in [0, 2\pi]} p(x) < \infty \tag{1}$$

then the function

$$p'(x) := \frac{p(x)}{p(x) - 1}$$

is well defined and satisfies (1) itself.

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The space  $L^{p(x)}$  consists of all measurable  $2\pi$ -periodic functions  $f$  such that

$$\int_0^{2\pi} |f(x)g(x)| dx < \infty$$

for all measurable  $g$  with  $m_{p'}(g) \leq 1$  and

$$\|f\|_{p(x)}^* = \sup \left\{ \int_0^{2\pi} |f(x)g(x)| dx : m_{p'}(g) \leq 1 \right\}$$

is also a norm on  $L^{p(x)}$ . It is known that the inequalities

$$\|f\|_{p(x)} \leq \|f\|_{p(x)}^* \leq r_p \|f\|_{p(x)}$$

satisfied for all functions  $f \in L^{p(x)}$ , where

$$r_p := 1 + \frac{1}{p_-} - \frac{1}{p_+},$$

and hence the norms  $\|f\|_{p(x)}$  and  $\|f\|_{p(x)}^*$  are equivalent. We refer to [13], [9], [10] and [7] for properties above and for more general information about  $L^{p(x)}$  spaces.

Denote by  $M$  the Hardy-Littlewood maximal operator, defined for  $f \in L^1$  by

$$M(f)(x) = \sup_I \frac{1}{|I|} \int_I |f(t)| dt, \quad x \in [0, 2\pi],$$

where the supremum is taken over all intervals  $I$  with  $x \in I$ .

The boundedness problem of the operator  $M$  on the space  $L^{p(x)}$  was studied by many authors ([5], [8], [17], [18], etc.).

In [8] it was proved that if the function  $p$  satisfies (1) and the condition

$$|p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \quad 0 < |x-y| \leq \frac{1}{2}, \tag{2}$$

then the maximal operator  $M$  is bounded on  $L^{p(x)}$ , that is,

$$\|M(f)\|_{p(x)} \leq c \|f\|_{p(x)} \tag{3}$$

for all  $f \in L^{p(x)}$ , where  $c$  is a constant depends only on  $p$ .

The set of all measurable  $2\pi$ -periodic functions  $p : \mathbb{R} \rightarrow [1, \infty)$  satisfies the conditions (1) and (2) will be denoted by  $\mathcal{M}$ .

Let  $p \in \mathcal{M}$  and  $f \in L^{p(x)}$ . The modulus of continuity of the function  $f$  is defined by

$$\Omega_{p(x)}(f, \delta) = \sup_{|h| \leq \delta} \|T_h(f)\|_{p(x)}, \quad \delta > 0, \tag{4}$$

where

$$T_h(f)(x) := \frac{1}{h} \int_0^h |f(x+t) - f(x)| dt. \tag{5}$$

The existence of  $\Omega_{p(x)}(f, \delta)$  follows from (3), and also the inequality

$$\Omega_{p(x)}(f, \delta) \leq c \|f\|_{p(x)}$$

satisfied for all  $\delta > 0$ .

The modulus  $\Omega_{p(x)}(f, \cdot)$  is nonnegative, continuous function such that

$$\lim_{\delta \rightarrow 0} \Omega_{p(x)}(f, \delta) = 0, \quad \Omega_{p(x)}(f_1 + f_2, \cdot) \leq \Omega_{p(x)}(f_1, \cdot) + \Omega_{p(x)}(f_2, \cdot).$$

In the Lebesgue spaces  $L^p$  ( $1 < p < \infty$ ), the classical modulus of continuity  $\omega_p(f, \cdot)$  is defined by

$$\omega_p(f, \delta) = \sup_{|h| \leq \delta} \|T'_h(f)\|_p, \quad \delta > 0, \tag{6}$$

where

$$T'_h(f)(x) := f(x+h) - f(x).$$

It is known that in the Lebesgue spaces  $L^p$  the moduli of continuity (4) and (6) are equivalent (see [14]).

We define in the spaces  $L^{p(x)}$  the modulus of continuity by using the shift (5), because the space  $L^{p(x)}$  is not translation invariant, in general (see, for example [13, Example 2.9]).

Let  $p \in \mathcal{M}$  and  $0 < \alpha \leq 1$ . We define the Lipschitz class  $Lip(\alpha, p(x))$  as

$$Lip(\alpha, p(x)) = \left\{ f \in L^{p(x)} : \Omega_{p(x)}(f, \delta) = O(\delta^\alpha), \delta > 0 \right\}.$$

Let  $f \in L^1$  has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{7}$$

Denote by  $S_n(f)(x)$ ,  $n = 0, 1, \dots$  the  $n$ th partial sums of the series (7) at the point  $x$ , that is,

$$S_n(f)(x) = \sum_{k=0}^n A_k(f)(x),$$

where

$$A_0(f)(x) = \frac{a_0}{2}, \quad A_k(f)(x) = a_k \cos kx + b_k \sin kx, \quad k = 1, 2, \dots$$

Let  $\{p_n\}_0^\infty$  be a sequence of positive real numbers. We consider two means of the series (7) defined by

$$N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^n p_{n-m} S_m(f)(x)$$

and

$$R_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^n p_m S_m(f)(x),$$

where  $P_n := \sum_{m=0}^n p_m$ ,  $p_{-1} = P_{-1} := 0$ . The means  $N_n(f)$  and  $R_n(f)$  are called the Nörlund and Riesz means of the series (7), respectively. In the case  $p_n = 1, n \geq 0$ , both of  $N_n(f)$  and  $R_n(f)$  are equal to the Cesàro mean

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{m=0}^n S_m(f)(x).$$

If we take  $p_n = A_n^{\beta-1}$  ( $\beta > 0$ ), where

$$A_0^\beta := 1, \quad A_k^\beta := \frac{\beta(\beta+1)\dots(\beta+k)}{k!}, \quad k \geq 1,$$

the mean  $N_n(f)$  be the generalized Cesàro mean  $\sigma_n^\beta(f)(x)$ , that is

$$N_n(f)(x) = \frac{1}{A_n^\beta} \sum_{m=0}^n A_{n-m}^{\beta-1} S_m(f)(x).$$

The approximation properties of the Cesàro means  $\sigma_n$  in Lipschitz classes  $Lip(\alpha, p)$ ,  $1 \leq p < \infty$ ,  $0 < \alpha \leq 1$  were investigated by Quade in [19]. The generalizations of Quade's results were studied by Mohapatra and Russell [16], Chandra ([1], [2], [3], [4]) and Leindler [15]. In [1], Chandra obtained estimates for  $\|f - N_n(f)\|_p$ , where  $1 < p < \infty$ . Chandra also gave estimates for the difference  $\|f - R_n(f)\|_p$ , where  $f \in Lip(\alpha, p)$ ,  $1 < p < \infty$ ,  $0 < \alpha \leq 1$  [2]. In the paper [4], Chandra gave some conditions on the sequence  $\{p_n\}_0^\infty$  and obtained very satisfactory results about approximation by the means  $N_n(f)$  and  $R_n(f)$  in  $Lip(\alpha, p)$ ,  $1 \leq p < \infty$ ,  $0 < \alpha \leq 1$ . Later, Leindler in [15] weakened the conditions given by Chandra on the sequence  $\{p_n\}_0^\infty$  and generalized his results. In [11], the analogues of Chandra's results was obtained for weighted Lebesgue spaces.

In the present paper we give  $L^{p(x)}$  analogues of the results obtained by Leindler in [15] and Chandra in [4].

We shall use the notations

$$\Delta g_n := g_n - g_{n+1}, \quad \Delta_m g(n, m) := g(n, m) - g(n, m+1).$$

A sequence of positive real numbers  $\{p_n\}_0^\infty$  is called almost monotone decreasing (increasing) if there exists a constant  $c$ , depending only on the sequence  $\{p_n\}_0^\infty$  such that for all  $n \geq m$  the inequality

$$p_n \leq c p_m \quad (c p_n \geq p_m)$$

holds. Such sequences will be denoted by  $\{p_n\}_0^\infty \in AMDS$  ( $\{p_n\}_0^\infty \in AMIS$ ).

Our main results are the following.

**THEOREM 1.** Let  $p \in \mathcal{M}$ ,  $0 < \alpha < 1$ ,  $f \in Lip(\alpha, p(x))$  and  $\{p_n\}_0^\infty$  be a sequence of positive numbers. If

$$\{p_n\}_0^\infty \in AMDS,$$

or

$$\{p_n\}_0^\infty \in AMIS \quad \text{and} \quad (n+1)p_n = O(P_n),$$

then

$$\|f - N_n(f)\|_{p(x)} = O(n^{-\alpha}).$$

**THEOREM 2.** Let  $p \in \mathcal{M}$ ,  $f \in Lip(1, p(x))$  and  $\{p_n\}_0^\infty$  be a sequence of positive numbers. If

$$\sum_{k=1}^{n-1} k |\Delta p_k| = O(P_n),$$

or

$$\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right),$$

then the estimate

$$\|f - N_n(f)\|_{p(x)} = O(n^{-1})$$

holds for  $n = 1, 2, \dots$

**THEOREM 3.** Let  $p \in \mathcal{M}$ ,  $0 < \alpha \leq 1$ ,  $f \in Lip(\alpha, p(x))$  and  $\{p_n\}_0^\infty$  be a sequence of positive numbers. If

$$\sum_{m=0}^{n-1} \left| \Delta \left( \frac{P_m}{m+1} \right) \right| = O\left(\frac{P_n}{n+1}\right), \tag{8}$$

then for  $n = 1, 2, \dots$  the estimate

$$\|f - R_n(f)\|_{p(x)} = O(n^{-\alpha})$$

holds.

In the classical Lebesgue spaces  $L^p$ , the analogues of Theorem 1 and Theorem 2 were proved by Leindler in [15], and Theorem 3 in  $L^p$  spaces was obtained by Chandra [4].

### 2. Some auxiliary results

In this section  $c$  will denote the constants (in general, different in different relations) depend only on quantities that are not important for the questions of interest.

Let  $p \in \mathcal{M}$ . Denote by  $E_n(f)_{p(x)}$  ( $n = 0, 1, \dots$ ) the best approximation of  $f \in L^{p(x)}$  in  $\Pi_n$  (the set of trigonometric polynomials of degree at most  $n$ ), that is

$$E_n(f)_{p(x)} = \inf \left\{ \|f - t_n\|_{p(x)} : t_n \in \Pi_n \right\}.$$

It follows that, for example from Theorem 1.1 in [6], there exists a trigonometric polynomial  $t_n^* \in \Pi_n$  such that

$$E_n(f)_{p(x)} = \|f - t_n^*\|_{p(x)}$$

for  $n = 0, 1, \dots$ .

By  $W^{p(x)} = W^{p(x)}([0, 2\pi])$  we denote the set of absolutely continuous functions  $f$  such that  $f' \in L^{p(x)}$ .

LEMMA 1. *Let  $p \in \mathcal{M}$  and  $f \in W^{p(x)}$ . Then the estimate*

$$E_n(f)_{p(x)} = O\left(\frac{1}{n} \|f'\|_{p(x)}\right) \tag{9}$$

holds for  $n = 1, 2, \dots$ .

*Proof.* It follows from Theorem 6.1 of [21] that

$$\|f - S_n(f)\|_{p(x)} \rightarrow 0, \quad n \rightarrow \infty.$$

It is easy to see that

$$A_k(\tilde{f}') (x) = kA_k(f) (x), \quad k = 1, 2, \dots,$$

where  $\tilde{f}'$  is the conjugate function of  $f'$ . By considering the uniform boundedness of  $\{S_n\}_0^\infty$  and the boundedness of the conjugation operator in the space  $L^{p(x)}$  (see [21]), we get

$$\begin{aligned} \|f - S_n(f)\|_{p(x)} &= \left\| \sum_{k=n+1}^\infty A_k(f) \right\|_{p(x)} = \left\| \sum_{k=n+1}^\infty \frac{1}{k} A_k(\tilde{f}') \right\|_{p(x)} \\ &= \left\| \sum_{k=n+1}^\infty \left( \frac{1}{k} - \frac{1}{k+1} \right) (S_k(\tilde{f}') - \tilde{f}') + \frac{1}{n+1} (S_n(\tilde{f}') - \tilde{f}') \right\|_{p(x)} \\ &\leq \sum_{k=n+1}^\infty \left( \frac{1}{k} - \frac{1}{k+1} \right) \|S_k(\tilde{f}') - \tilde{f}'\|_{p(x)} + \frac{1}{n+1} \|S_n(\tilde{f}') - \tilde{f}'\|_{p(x)} \\ &\leq c \left\{ \sum_{k=n+1}^\infty \left( \frac{1}{k} - \frac{1}{k+1} \right) \right\} \|f'\|_{p(x)} + \frac{1}{n+1} \|f'\|_{p(x)} \\ &\leq \frac{c}{n} \|f'\|_{p(x)}, \end{aligned}$$

and hence (9) follows.  $\square$

LEMMA 2. *If  $p \in \mathcal{M}$ , the Jackson type inequality*

$$E_n(f)_{p(x)} = O\left(\Omega_{p(x)}\left(f, \frac{1}{n}\right)\right), \quad n = 1, 2, \dots$$

holds for  $f \in L^{p(x)}$ .

*Proof.* Let  $f \in L^{p(x)}$ . Consider the transform

$$U_\delta(f)(x) := \frac{2}{\delta} \int_{\delta/2}^{\delta} \left( \frac{1}{h} \int_0^h f(x+t) dt \right) dh, \quad \delta > 0.$$

It is clear that  $U_\delta(f) \in W^{p(x)}$  for each  $\delta > 0$  and

$$(U_\delta(f))'(x) = \frac{2}{\delta} \int_{\delta/2}^{\delta} \frac{1}{h} (f(x+h) - f(x)) dh$$

for almost all  $x$ . Since

$$|(U_\delta(f))'(x)| \leq \frac{4}{\delta} \left( \frac{1}{\delta} \int_0^{\delta} |f(x+h) - f(x)| dh \right),$$

it follows from definition of  $\Omega_{p(x)}(f, \delta)$  that

$$\begin{aligned} \|(U_\delta(f))'\|_{p(x)} &\leq \frac{4}{\delta} \left\| \frac{1}{\delta} \int_0^{\delta} |f(\cdot + h) - f| dh \right\|_{p(x)} \\ &\leq \frac{4}{\delta} \Omega_{p(x)}(f, \delta). \end{aligned}$$

On the other hand, since

$$U_\delta(f)(x) - f(x) = \frac{2}{\delta} \int_{\delta/2}^{\delta} \left( \frac{1}{h} \int_0^h (f(x+t) - f(x)) dt \right) dh,$$

we get

$$\begin{aligned} \|U_\delta(f) - f\|_{p(x)} &\leq \frac{2}{\delta} \int_{\delta/2}^{\delta} \left\| \frac{1}{h} \int_0^h |f(\cdot + t) - f| dt \right\|_{p(x)} dh \\ &\leq \sup_{\delta/2 \leq h \leq \delta} \frac{2}{\delta} \int_{\delta/2}^{\delta} \left\| \frac{1}{h} \int_0^h |f(\cdot + t) - f| dt \right\|_{p(x)} dh \\ &\leq \frac{2}{\delta} \int_{\delta/2}^{\delta} \left( \sup_{\delta/2 \leq h \leq \delta} \left\| \frac{1}{h} \int_0^h |f(\cdot + t) - f| dt \right\|_{p(x)} \right) dh \end{aligned}$$

$$= \sup_{\delta/2 \leq h \leq \delta} \left\| \frac{1}{h} \int_0^h |f(\cdot + t) - f| dt \right\|_{p(x)} \leq \Omega_{p(x)}(f, \delta).$$

Hence, by subadditivity of the best approximation and (9) we obtain

$$\begin{aligned} E_n(f)_{p(x)} &\leq E_n(f - U_{1/n}(f))_{p(x)} + E_n(U_{1/n}(f))_{p(x)} \\ &\leq \|f - U_{1/n}(f)\|_{p(x)} + \frac{c}{n} \left\| (U_{1/n}(f))' \right\|_{p(x)} \\ &\leq \Omega_{p(x)}\left(f, \frac{1}{n}\right) + \frac{c}{n} 4n \Omega_{p(x)}\left(f, \frac{1}{n}\right), \end{aligned}$$

which finishes the proof.  $\square$

LEMMA 3. Let  $p \in \mathcal{M}$  and  $0 < \alpha \leq 1$ . Then for every  $f \in Lip(\alpha, p(x))$  the estimate

$$\|f - S_n(f)\|_{p(x)} = O(n^{-\alpha}), \quad n = 1, 2, \dots$$

holds.

*Proof.* Let  $t_n^*$  ( $n = 0, 1, \dots$ ) be the trigonometric polynomial of best approximation to  $f \in Lip(\alpha, p(x))$ . By Lemma 2

$$\|f - t_n^*\|_{p(x)} = O(\Omega_{p(x)}(f, 1/n)),$$

and hence

$$\|f - t_n^*\|_{p(x)} = O(n^{-\alpha}).$$

By the uniform boundedness of the partial sums  $S_n(f)$  in the space  $L^{p(x)}$  (see [21]), we get

$$\begin{aligned} \|f - S_n(f)\|_{p(x)} &\leq \|f - t_n^*\|_{p(x)} + \|t_n^* - S_n(f)\|_{p(x)} \\ &= \|f - t_n^*\|_{p(x)} + \|S_n(t_n^* - f)\|_{p(x)} \\ &= O\left(\|f - t_n^*\|_{p(x)}\right) \\ &= O(n^{-\alpha}). \quad \square \end{aligned}$$

LEMMA 4. Let  $p \in \mathcal{M}$ . If  $f \in Lip(1, p(x))$ , then  $f$  is absolutely continuous and  $f' \in L^{p(x)}$ , that is  $f \in W^{p(x)}$ .

*Proof.* Let  $f \in Lip(1, p(x))$  and  $\delta > 0$ . Since  $p_- \leq p(x)$  almost everywhere, by Theorem 2.8 of [13] the space  $L^{p(x)}$  is continuously embedded in  $L^{p_-}$ . Hence we have

$$\|T_h(f)\|_{p_-} \leq c \|T_h(f)\|_{p(x)}$$

for every  $h$  with  $|h| \leq \delta$ . This inequality and equivalence of  $\omega_{p_-}(f, \cdot)$  and  $\Omega_{p_-}(f, \cdot)$  yield

$$\omega_{p_-}(f, \delta) \leq c \Omega_{p(x)}(f, \delta).$$



Hence,  $f \in Lip(1, p(x))$  implies  $\omega_{p_-}(f, \delta) = O(\delta)$ , and this implies that  $f$  is absolutely continuous and  $f' \in L^{p_-}$  ([6, pp. 51–54]).

Since the relation

$$\frac{f(x+t) - f(x)}{t} \rightarrow f'(x), \quad t \rightarrow 0$$

holds almost everywhere, for almost all  $x$  we get

$$\frac{2}{\delta} \int_{\delta/2}^{\delta} \frac{|f(x+t) - f(x)|}{t} dt \rightarrow |f'(x)|, \quad \delta \rightarrow 0^+.$$

By Fatou Lemma, for every measurable function  $g$  with  $m_{p'}(g) \leq 1$ ,

$$\begin{aligned} \int_0^{2\pi} |f'(x)| |g(x)| dx &= \int_0^{2\pi} \left( \lim_{\delta \rightarrow 0^+} \frac{2}{\delta} \int_{\delta/2}^{\delta} \frac{|f(x+t) - f(x)|}{t} dt \right) |g(x)| dx \\ &\leq \liminf_{\delta \rightarrow 0^+} \int_0^{2\pi} \left( \frac{2}{\delta} \int_{\delta/2}^{\delta} \frac{|f(x+t) - f(x)|}{t} dt \right) |g(x)| dx \\ &\leq \liminf_{\delta \rightarrow 0^+} \frac{4}{\delta} \int_0^{2\pi} \left( \frac{1}{\delta} \int_0^{\delta} |f(x+t) - f(x)| dt \right) |g(x)| dx \\ &= \liminf_{\delta \rightarrow 0^+} \frac{4}{\delta} \int_0^{2\pi} T_{\delta}(f)(x) |g(x)| dx \\ &\leq \liminf_{\delta \rightarrow 0^+} \frac{4}{\delta} \|T_{\delta}(f)\|_{p(x)} \leq \liminf_{\delta \rightarrow 0^+} \frac{4}{\delta} \Omega_{p(x)}(f, \delta) \\ &= \liminf_{\delta \rightarrow 0^+} \frac{4}{\delta} O(\delta) = O(1), \end{aligned}$$

and this means that  $f' \in L^{p(x)}$ .  $\square$

LEMMA 5. Let  $p \in \mathcal{M}$  and  $f \in Lip(1, p(x))$ . Then for  $n = 1, 2, \dots$  the estimate

$$\|S_n(f) - \sigma_n(f)\|_{p(x)} = O(n^{-1})$$

holds.

*Proof.* By Lemma 4,  $f \in W^{p(x)}$ . If  $f$  has the Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} A_k(f)(x),$$

then the Fourier series of the conjugate function  $\tilde{f}'$  be

$$\tilde{f}'(x) \sim \sum_{k=1}^{\infty} kA_k(f)(x).$$

On the other hand,

$$\begin{aligned} S_n(f)(x) - \sigma_n(f)(x) &= \sum_{k=1}^n \frac{k}{n+1} A_k(f)(x) \\ &= \frac{1}{n+1} S_n(\tilde{f}')(x). \end{aligned}$$

Hence, by considering the uniform boundedness of the partial sums and the conjugation operator in the space  $L^{p(x)}$  (see [21]), we obtain

$$\|S_n(f) - \sigma_n(f)\|_{p(x)} = O(n^{-1})$$

for  $n = 1, 2, \dots$   $\square$

The following Lemma was proved in [15].

LEMMA 6. Let  $\{p_n\}_0^\infty$  be a sequence of positive numbers. If  $\{p_n\}_0^\infty \in AMDS$ , or  $\{p_n\}_0^\infty \in AMIS$  and  $(n+1)p_n = O(P_n)$ , then

$$\sum_{m=1}^n m^{-\alpha} p_{n-m} = O(n^{-\alpha} P_n)$$

for  $0 < \alpha < 1$ .

### 3. Proofs of the main results

*Proof of Theorem 1.* Since

$$f(x) = \frac{1}{P_n} \sum_{m=0}^n p_{n-m} f(x),$$

we have

$$f(x) - N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^n p_{n-m} \{f(x) - S_m(f)(x)\}.$$

By Lemma 3 and Lemma 6 we obtain

$$\begin{aligned} \|f - N_n(f)\|_{p(x)} &\leq \frac{1}{P_n} \sum_{m=0}^n p_{n-m} \|f - S_m(f)\|_{p(x)} \\ &= \frac{1}{P_n} \sum_{m=1}^n p_{n-m} O(m^{-\alpha}) + \frac{p_n}{P_n} \|f - S_0(f)\|_{p(x)} \\ &= \frac{1}{P_n} O(n^{-\alpha} P_n) + O\left(\frac{1}{n+1}\right) \\ &= O(n^{-\alpha}). \quad \square \end{aligned}$$

*Proof of Theorem 2.* It is clear that

$$N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^n P_{n-m} A_m(f)(x).$$

By Abel transform,

$$\begin{aligned} S_n(f)(x) - N_n(f)(x) &= \frac{1}{P_n} \sum_{m=1}^n (P_n - P_{n-m}) A_m(f)(x) \\ &= \frac{1}{P_n} \sum_{m=1}^n \Delta_m \left( \frac{P_n - P_{n-m}}{m} \right) \left( \sum_{k=1}^m k A_k(f)(x) \right) + \frac{1}{n+1} \sum_{k=1}^n k A_k(f)(x), \end{aligned}$$

and hence

$$\begin{aligned} \|S_n(f) - N_n(f)\|_{p(x)} &\leq \frac{1}{P_n} \sum_{m=1}^n \left| \Delta_m \left( \frac{P_n - P_{n-m}}{m} \right) \right| \left\| \sum_{k=1}^m k A_k(f) \right\|_{p(x)} \\ &\quad + \frac{1}{n+1} \left\| \sum_{k=1}^n k A_k(f) \right\|_{p(x)}. \end{aligned}$$

Since

$$S_n(f)(x) - \sigma_n(f)(x) = \frac{1}{n+1} \sum_{k=1}^n k A_k(f)(x),$$

by Lemma 5 we get

$$\frac{1}{n+1} \left\| \sum_{k=1}^n k A_k(f) \right\|_{p(x)} = \|S_n(f) - \sigma_n(f)\|_{p(x)} = O(n^{-1}).$$

Hence,

$$\|S_n(f) - N_n(f)\|_{p(x)} = O\left(\frac{1}{P_n}\right) \sum_{m=1}^n \left| \Delta_m \left( \frac{P_n - P_{n-m}}{m} \right) \right| + O(n^{-1}). \tag{10}$$

Suppose the condition

$$\sum_{k=1}^{n-1} k |\Delta p_k| = O(P_n)$$

holds. This implies that (see [15])

$$\sum_{m=1}^n \left| \Delta_m \left( \frac{P_n - P_{n-m}}{m} \right) \right| = O\left(\frac{P_n}{n}\right),$$

and hence by (10) we have

$$\|S_n(f) - N_n(f)\|_{p(x)} = O(n^{-1}).$$

This relation and Lemma 3 yield

$$\|f - N_n(f)\|_{p(x)} = O(n^{-1}).$$

Now let

$$\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right). \tag{11}$$

A simple calculation yields

$$\Delta_m \left(\frac{P_n - P_{n-m}}{m}\right) = \frac{1}{m(m+1)} \left(\sum_{k=n-m}^n p_k - (m+1)p_{n-m}\right),$$

and by induction one can easily get

$$\left|\sum_{k=n-m}^n p_k - (m+1)p_{n-m}\right| \leq \sum_{k=1}^m k |p_{n-k+1} - p_{n-k}|.$$

Thus,

$$\begin{aligned} \sum_{m=1}^n \left|\Delta_m \left(\frac{P_n - P_{n-m}}{m}\right)\right| &\leq \sum_{m=1}^n \frac{1}{m(m+1)} \left(\sum_{k=1}^m k |p_{n-k+1} - p_{n-k}|\right) \\ &= \sum_{k=1}^n k |p_{n-k+1} - p_{n-k}| \left(\sum_{m=k}^n \frac{1}{m(m+1)}\right) \\ &\leq \sum_{k=1}^n k |p_{n-k+1} - p_{n-k}| \left(\sum_{m=k}^{\infty} \frac{1}{m(m+1)}\right) \\ &= \sum_{k=1}^n |p_{n-k+1} - p_{n-k}| = \sum_{k=0}^{n-1} |\Delta p_k|. \end{aligned}$$

Combining this, the assumption (11) and (10) we get

$$\|S_n(f) - N_n(f)\|_{p(x)} = O(n^{-1}),$$

and considering Lemma 3 again we obtain the desired result.  $\square$

*Proof of Theorem 3.* Let  $0 < \alpha < 1$ . By definition of  $R_n(f)(x)$ ,

$$f(x) - R_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^n p_m \{f(x) - S_m(f)(x)\}.$$

From Lemma 3, we get

$$\begin{aligned} \|f - R_n(f)\|_{p(x)} &\leq \frac{1}{P_n} \sum_{m=0}^n p_m \|f - S_m(f)\|_{p(x)} \\ &= O\left(\frac{1}{P_n}\right) \sum_{m=1}^n p_m m^{-\alpha} + \frac{p_0}{P_n} \|f - S_0(f)\|_{p(x)} \\ &= O\left(\frac{1}{P_n}\right) \sum_{m=1}^n p_m m^{-\alpha}. \end{aligned} \tag{12}$$

By Abel transform,

$$\begin{aligned} \sum_{m=1}^n p_m m^{-\alpha} &= \sum_{m=1}^{n-1} P_m \{m^{-\alpha} - (m+1)^{-\alpha}\} + n^{-\alpha} P_n \\ &\leq \sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} + n^{-\alpha} P_n, \end{aligned}$$

and

$$\begin{aligned} \sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} &= \sum_{m=1}^{n-1} \Delta \left( \frac{P_m}{m+1} \right) \left( \sum_{k=1}^m k^{-\alpha} \right) + \frac{P_n}{n+1} \sum_{m=1}^{n-1} m^{-\alpha} \\ &= O(n^{-\alpha} P_n) \end{aligned}$$

by condition (8). This yields

$$\sum_{m=1}^n p_m m^{-\alpha} = O(n^{-\alpha} P_n)$$

and from this and (12) we get

$$\|f - R_n(f)\|_{p(x)} = O(n^{-\alpha}).$$

Let's consider the case  $\alpha = 1$ .

By Abel transform,

$$\begin{aligned} R_n(f)(x) &= \frac{1}{P_n} \sum_{m=0}^{n-1} \{P_m (S_m(f)(x) - S_{m+1}(f)(x)) + P_n S_n(f)(x)\} \\ &= \frac{1}{P_n} \sum_{m=0}^{n-1} P_m (-A_{m+1}(f)(x)) + S_n(f)(x), \end{aligned}$$

and hence

$$R_n(f)(x) - S_n(f)(x) = -\frac{1}{P_n} \sum_{m=0}^{n-1} P_m A_{m+1}(f)(x).$$

Using Abel transform again yields

$$\begin{aligned} \sum_{m=0}^{n-1} P_m A_{m+1}(f)(x) &= \sum_{m=0}^{n-1} \frac{P_m}{m+1} (m+1) A_{m+1}(f)(x) \\ &= \sum_{m=0}^{n-1} \Delta \left( \frac{P_m}{m+1} \right) \left( \sum_{k=0}^m (k+1) A_{k+1}(f)(x) \right) \\ &\quad + \frac{P_n}{n+1} \sum_{k=0}^{n-1} (k+1) A_{k+1}(f)(x). \end{aligned}$$

Thus, by considering Lemma 5 and (8) we obtain

$$\begin{aligned}
 \left\| \sum_{m=0}^{n-1} P_m A_{m+1}(f) \right\|_{p(x)} &\leq \sum_{m=0}^{n-1} \left| \Delta \left( \frac{P_m}{m+1} \right) \right| \left\| \sum_{k=0}^m (k+1) A_{k+1}(f) \right\|_{p(x)} \\
 &\quad + \frac{P_n}{n+1} \left\| \sum_{k=0}^{n-1} (k+1) A_{k+1}(f) \right\|_{p(x)} \\
 &= \sum_{m=0}^{n-1} \left| \Delta \left( \frac{P_m}{m+1} \right) \right| (m+2) \|S_{m+1}(f) - \sigma_{m+1}(f)\|_{p(x)} \\
 &\quad + P_n \|S_n(f) - \sigma_n(f)\|_{p(x)} \\
 &= O(1) \sum_{m=0}^{n-1} \left| \Delta \left( \frac{P_m}{m+1} \right) \right| + O\left(\frac{P_n}{n}\right).
 \end{aligned}$$

This gives

$$\begin{aligned}
 \|R_n(f) - S_n(f)\|_{p(x)} &= \frac{1}{P_n} \left\| \sum_{m=0}^{n-1} P_m A_{m+1}(f) \right\|_{p(x)} \\
 &= \frac{1}{P_n} O\left(\frac{P_n}{n}\right) = O\left(\frac{1}{n}\right).
 \end{aligned}$$

Combining this estimate with Lemma 3 yields

$$\|f - R_n(f)\|_{p(x)} = O(n^{-1}). \quad \square$$

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