Journal of Mathematical Inequalities
Volume 4, Number 2 (2010), 307–313

RESTRICTED CURVATURE IN THE MINKOWSKI PLANE

MOSTAFA GHANDEHARI

(*Communicated by S. Gomis*)

Abstract. A geometric proof for the following problem is given: Let E be the unit circle in a Minkowski plane. Let C be any continuously differentiable closed curve with length $l(C)$ (measured in Minkowski metric). Assume $|\kappa_e(C,.)| \leqslant k\kappa_e(E,.)$ and $\kappa_e(E,.)$ denote Euclidean curvatures. Then C can be contained in a similar copy of the unit disk translated and magnified by a factor of

$$\frac{l(C)}{4} - \frac{1}{4k}(l(E)-4).$$

1. Introduction

Our main purpose in this article is to generalize the following theorem from the Euclidean plane to Minkowski planes. Minkowski spaces are simply finite dimensional normal linear spaces.

THEOREM 1. *A closed curve in R^2 of length L and curvature bounded by K can be contained in a circle of radius* $\frac{L}{4} - \frac{(\pi-2)}{2K}$.

Johnson [17] used optimal control theory to prove Theorem 1. Chakerian, Johnson, and Vogt [6] gave a geometric proof of Theorem 1. Melzak [21] gave an interesting treatment of plane motion with curvature limitations. Isaacs [16] discusses pursuer evader games where the radius of curvature of the pursuer is bounded.

Preliminary definitions and concepts are discussed in Section 2. A generalization of Theorem 1 is proved in Section 3.

2. Preliminaries

By a plane convex body we shall mean a compact, convex subset of the Euclidean plane having a non-empty interior. We shall take a "unit circle" E for the Minkowski plane to be a centrally symmetric convex body with its center at the origin in the Euclidean plane. The *Minkowski distance* from x to y is defined by

$$\|x-y\| = \frac{\|x-y\|_e}{r}$$

Mathematics subject classification (2010): 52A38.
Keywords and phrases: geometric inequalities; Minkowskian geometry; restricted curvature.

Paper JMI-04-27

where $\|x-y\|_e$ is the Euclidean length from x to y, and r is the Euclidean radius of E in the direction of the vector $y-x$.

Minkowski distance defined by means of a convex body was developed by Minkowski [22]. The articles by Busemann [5] and Petty [24] contain basic concepts for the study of Minkowski geometry, as does Chapter 6 of Benson's book [1] and Chapter 4 of Valentine's book [30]. These last two references also contain useful background material from the theory of convex sets.

We use techniques from integral geometry. Santaló [27] is a good reference for integral geometry in the Euclidean spaces. Given a curve C in the Euclidean plane, let L denote the length of C. Crofton's simplest formula (see Santaló [27]) is

$$\iint n\, dp\, d\theta = 2L,$$

where the integral is taken over all lines intersecting the curve. Assume (p, θ) is the polar coordinate representation of the foot of the perpendicular from the origin to the line, and n is the number of intersections of a line with coordinates (p, θ) with C. The differential element $dG = dp\,d\theta$ is the integral geometric density for lines.

Chakerian [9] treats integral geometry in the Minkowski plane. We sketch the definitions he uses to develop Crofton's simplest formula in the Minkowski plane. Assume E is "sufficiently" differentiable and has positive finite curvature everywhere. Parameterize E by twice its sectional area ϕ and write the equation of E as

$$t = t(\phi), \qquad 0 \leqslant \phi \leqslant 2\pi \qquad \|t\| = \|t-0\| = 1.$$

E is called the *indicatrix*. Define the *isoperimetrix* T by the parametric representation

$$n(\phi) = \frac{dt(\varphi)}{d(\varphi)}, \qquad 0 \leqslant \phi \leqslant 2\pi.$$

Define $\lambda(\varphi)$ by $\frac{dn(\varphi)}{d(\varphi)} = -\lambda^{-1}(\phi)t(\phi)$. Then the density for lines in two-dimensional Minkowski spaces is defined as follows. Let $G = G(p, \varphi)$ be parallel to the direction $t(\varphi)$.

The equation of G is

$$[t(\varphi), x] = p.$$

where $[x-y] = x_1y_2 - x_2y_1$. Then the density dG for lines is

$$dG = \lambda^{-1}(\phi)\, dp\, d\phi.$$

It is then shown in Chakerian [9] that the simplest formula of Crofton holds:

$$\int n\, dG = 2l$$

where n is the number of intersections of a line G with a curve C, integration is taken over all lines intersecting C and l is the Minkowskian length of C.

3. The Main Result

The following Theorem 2 is the generalization of Theorem 1 to the Minkowski plane. The author has used the theory of optimal control to prove Theorem 2 in [13].

THEOREM 2. *Let E be the unit circle in a Minkowski plane. Let C be any continuously differentiable closed curve with length $l(C)$ (measured in the Minkowski metric). Assume $|\kappa_e(C,.)| \leqslant k\kappa_e(E,.)$ where $\kappa_e(C,.)$ and $\kappa_e(E,.)$ denote Euclidean curvatures of the respective curves. Then C can be contained in a similar copy of the unit disk translated and magnified by a factor $\mu \geqslant \frac{l(C)}{4} - \frac{1}{4k}(l(E)-4)$*

In order to prove Theorem 2, we need the case with no restriction on curvature and two lemmas. Theorem 3 gives a bound on the size of the unit ball containing a closed curve of Minkowski length less than or equal to $\frac{l}{4}$.

THEOREM 3. *Any continuous closed curve C of Minkowski length l in an n-dimensional Minkowski space can be enclosed by a similar copy of the unit ball magnified by a factor less than or equal to $\frac{l}{4}$.*

The case of equality is discussed after the proof of Theorem 3. The Euclidean version of Theorem 3 was first proved in a more general form by Segre [28] and independently by Rustishauser and Samelson [26]. Nitsche [23] gives an elementary proof in the Euclidean 3-space. The proof given here is the same as the proof given in Chakerian and Klamkin [8] where they prove Theorem 3 in Euclidean space, and they give complete references related to Theorem 3. They also consider minimal covers other than the ball.

Proof of Theorem 3. Let $u \neq v$ be two points on C dividing it into two arcs each of Minkowskian length $\frac{l}{2}$. Let p be the midpoint of the segment joining u and v. For $\omega \in C$, we have

$$\|\omega - p\| \leqslant \frac{1}{2}[\|u-\omega\| + \|v-\omega\|] \leqslant \frac{l}{4}. \tag{1}$$

To see the first inequality consider the central reflection of ω through p to a point ω^* with $\|\omega^* - p\| = \|\omega - p\|$. The first inequality is the a consequence of the triangle inequality applied to the triangle $\omega^* u \omega$. The second inequality follows since the straight line segments joining u and v to ω have lengths less than or equal to the length from u to v along the curve. □

In Euclidean spaces, equality in (1) holds if and only if ω is collinear with u and v in which case C is a "needle," i.e., a line segment of length $\frac{l}{2}$ traversed twice. If the unit ball of the Minkowski space does not contain a line segment, then the same argument applies. However, if the unit ball contains the line segments then we do not necessarily have the case of a needle. E.

Theorem 3 implies that a triangle inscribed in the unite circle of a Minkowski plane and having the center as an interior point has perimeter greater than 4. This result is due to Laugwitz [19]. We now proceed to prove two lemmas needed to prove Theorem 2. We also require a generalization of Blasche's Rolling Theorem [3, pp. 114–119] due to Koutroufiotis [18]. Theorem 4 is Koutroufiotis' generation of Lemma 2 which will follow.

Assuming the boundary of the unit circle E is smooth, we use integral geometry to prove the following Lemma 1. The proof given here is the same as the proof given in Chakerian [7] extended to Minkowski planes.

LEMMA 1. *Consider a Minkowski plane with smooth unit circle E. Let $\bar{C}$ be the convex hull of a closed curve C. Then*

$$l(\bar{C}) \leqslant l(C).$$

Proof. Using Crofton's formula for Minkowski planes (see Chakerian [9]), we can compute $l(\bar{C})$ by the measure of lines intersecting $\bar{C}$. Any line G meeting $\bar{C}$ in two points must C in at least two points. If not, C would be contained in the closed half plane H and determined by G and hence in the proper convex subset of $\bar{C}$ determined by H and the boundary of $\bar{C}$. This yields a contradiction.

Schaer [29] gives a proof of Lemma 1 for the Euclidian plane. A continuously differentiable curve X in R^n, parameterized by arc length s, is called a K-*curve* if and only if $\|X'(s_2) - X'(s_1)\| \leqslant K\,|s_2 - s_1|$ for all s_1 and s_2. Hence K-*curves* are generalization of C^2 curves with curvature bounded by K. Dubins [10] showed that among K-*curves* with prescribed initial and terminal points and prescribed initial and terminal vectors there exists a K-*curve* of minimal length. Chakerian, Johnson, and Vogt [6] prove that the convex hull of a closed K-*curve* in R^n is a closed K-*curve*. The same proof is extended to the Minkowski plane with unit circle E in lemma 2. □

Parametrize a given curve C with a yielding $C(s)$, s is the Euclidean arc length along C.

Let $E(s)$ be the point on E such that the unit tangent to E makes the same angle $\theta(s)$ with the horizontal as the tangent to C at $C(s)$. Hence $E(s)$ is the relative normal to C at $C(s)$.

$$\left\| \frac{dE}{ds} \right\|_e = \frac{ds_E}{ds} \tag{2}$$

where ds_E is the Euclidean arc length along E. But,

$$\lim_{S_2 \to S} \frac{\|E(s_2) - E(s)\|_e}{s_2 - s} = \left\| \frac{dE}{ds} \right\|_e = \left\| \frac{dE}{ds} \right\|_e \frac{ds_E}{ds} = \frac{ds_E}{ds} = \frac{\frac{ds_E}{d\theta}}{\frac{ds}{d\theta}} = \frac{\kappa_e(C,s)}{\kappa_e(E,s)} = \kappa_m(C,s) \tag{3}$$

Thus we define a continuously differentiable curve to be a Minkowskian K-*curve* if and only if

$$\|E(s_1) - E(s_2)\| \leqslant K\,|s_1 - s_2| \tag{4}$$

For all s_1 and s_2. As a consequence of (3) and (4) we see that a C^2 curve C with Minkowskian curvature bounded by K is Minkowskian K-*curve*.

Lemma 2 below shows that the convex hull of closed Minkowski K-*curve* is a Minkowskian K-*curve*.

LEMMA 2. *Let C be a closed Minkowskian K-curve in a Minkowski plane unit circle E. Let $\bar{C}$ be the convex hull of C, then $\bar{C}$ is a Minkowskian K-curve.*

Proof. Every point on $\bar{C}$ is either a point C or else an interior point of a line segment in $\bar{C}$ whose endpoints lie on C. At the endpoints of the line segment, $\bar{C}$ has its tangent line parallel to the line segment. At points of $\bar{C}\cap C$ there is a unique supporting line of C. If the supporting lines formed a cone at such a point, C could not have a derivative there.

Let τ be an arc length parameter for C and let s be an arc length parameter for $\bar{C}$. Let $E(\tau)$ and $E(s)$ be the points on the unit circle corresponding to $C(\tau)$ and $\bar{C}(s)$ respectively. Let ε be a positive number and s_0 a particular value of s. We show that for s sufficiently close to s_0

$$|E(s_1)-E(s_0)|_e \leqslant (K+\varepsilon)\,|s_1-s_0|\ .$$

Suppose not. There exists a sequence $\{s_n\}$ convergent to s_0 such that $\|E(s_n)-E(s_0)\|_e > (K+\varepsilon)\,|s_n-s_0|$ for all n. If $\bar{C}(s_0)$ is not on C, then for n large $\bar{C}(s_n)$ is on the line segment of $\bar{C}$ through $\bar{C}(s_0)$. Then $E(s_n)=E(s_0)$ for a contradiction. Hence $\bar{C}(s_0)$ must belong to $\bar{C}\cap C$. We can also suppose that for each n, $\bar{C}(s_n)$ belongs to $\bar{C}\cap C$. Otherwise, the point $\bar{C}(s_n)$ would be interior points of the line segments of $\bar{C}$. Without affecting $E(s_n)$ or increasing $|s_n-s_0|$, we can shift points along the segments until they meet C.

The curve C has only a finite number of branches which go through $\bar{C}(s_0)$. By passing to a subsequence, we can suppose that the points $\bar{C}(s_n)$ all lie on the same branch of C. Hence there is a sequence $\{\tau_n\}$ convergent to a parameter value of τ_0 with $C(\tau_0)=\bar{C}(s_0)$ and $C(\tau_n)=\bar{C}(s_n)$ for all n. By passing to a subsequence once more, we can assume that for all n, $E(s_n)=uE(\tau_n)$ where $u=\pm 1$ is fixed. But $\{E(\tau_n)\}$ converges to $E(\tau_0)$. Hence $\{E(s_n)\}$ converges to a limit which, up to sign, equals $E(s_0)$. Since $\bar{C}$ is a closed convex curve, it cannot reverse direction abruptly. So $\{E(s_n)\}$ converges to $E(s_0)$ and $E(s_0)=uE(\tau_0)$. Then

$$\begin{aligned}(K+\varepsilon)|s_n-s_0| < \|E(s_n)-E_{s_0}\|_e &= \|uE(\tau_n)-E(\tau_0)\|_e\\ &= \|E(\tau_n)-E(\tau_0)\|_e \leqslant \mathrm{K}\,|\tau_n-\tau_0|\end{aligned}$$

But for any positive number $\alpha<1$ and for n sufficiently large,

$$\left\|C(\tau_n)-C(\tau_0)/_{(\tau_n-\tau_0)}\right\|_e > 1-\alpha$$

Hence

$$\begin{aligned}(K+\varepsilon)\,|s_n-s_0| < \mathrm{K}\,|\tau_n-\tau_0| &< \mathrm{K}\,\|C(\tau_n)-C(\tau_0)\|_e/_{1-\alpha}\\ &= \mathrm{K}\left\|\bar{C}(s_n)-\bar{C}(s_0)\right\|_e/_{1-\alpha} \leqslant K\,|s_n-s_0|/_{1-\alpha}.\end{aligned}$$

Hence $K+\varepsilon < \frac{K}{1-\alpha}$. Letting α vary, we conclude $K+\varepsilon \leqslant K$ which gives a contradiction. Thus for each s_0 and for each s sufficiently close to s_0, $\|E(s)-E(s_0)\|_e \leqslant (K+\varepsilon)\,|s-s_0|$.

If s_1 and s_2 are any two values, by a compactness argument and partitioning the interval $[s_1, s_2]$ into small enough subintervals and repeated application of the triangle inequality, it follows that $\bar{C}$ is Minkowskian *K-curve*. $\square$

Blaschke's Rolling Theorem states that if C is a C^2 simple convex curve with curvature κ satisfying $\kappa \leqslant$ K then a circle of radius $\frac{1}{\kappa}$ rolls freely inside C in the sense that, if it touches C from inside at any point, it lies entirely within the closed convex set bounded by C.

The following Theorem 4 is a generalization of Blaschke's Rolling Theorem which will be used for the proof of Theorem 2.

THEOREM 4. (Koutroufiotis [18]) *Let the plane convex set D_1 have as boundary a C^2 curve C_1 with curvature κ_1. Let C_2 be a regular convex curve with curvature κ_2. Assume that C_1 and C_2 are tangent to each other at p_0 with the same unit normal and that $\kappa_1(p_1) < \kappa_2(p_2)$ if the unit tangents at p_1 and p_2 are equal. Then C_2, except for p_0, lies in D_1.*

Proof of Theorem 2. Let $\bar{C}$ be the convex hull of C. Lemma 2 implies $\left|\kappa_e(\bar{C}, .)\right| \leqslant \kappa_e(E, .) = \kappa_e(\frac{1}{\kappa}E, .)$. Let $\bar{D}$ be the region bounded by $\bar{C}$. Using Theorem 4, we conclude $\bar{D} = \frac{1}{\kappa}E + D$ for some D. Hence $l(D) = l(\bar{C}) - \frac{1}{\kappa}l(E)$. By Theorem 3, D can be covered by a copy of the unit circle magnified by a factor of $\frac{1}{4}l\ (\bar{C}) - \frac{1}{4\kappa}l\ (E) + \frac{1}{\kappa}$. Using Lemma 1, $l\ (\bar{C}) \leqslant l\ (C)$, and the fact that any region containing $\bar{C}$ also contains C, we conclude that C can be covered by a copy of the unit circle E magnified by a factor of $\frac{1}{4}l\ (\bar{C}) - \frac{1}{4\kappa}l\ (E) + \frac{1}{\kappa} = \frac{l\ (C)}{4} - \frac{1}{4\kappa}(l\ (E) - 4)$. $\square$

REFERENCES

[1] R. V. BENSON, *Euclidean Geometry and Convexity*, McGraw-Hill, 1966.

[2] O. BIBERSTEIN, *Eléments de géométrie différentielle Minkowskienne, Thése*, Universite de Montreal, 1957.

[3] W. BLASCHKE, *Kreis and Kugel*, Viet, Leipzig, 1916; reprint, Chelsa, New York, 1949.

[4] V. G. BOLTYANSKII, *Mathematical Methods of Optimal Control*, Nauka, Moscow, 1966; English transl., Holt, Rinehart, and Winston, New York, 1948.

[5] H. BUSEMANN, *The foundations of Minkowskian geometry*, *Comm. Math Helv.*, 24 (1950), pp. 156–187.

[6] G. D. CHAKERIAN, H. H. JOHNSON, AND A. VOGT, *A geometric inequality for plane curves with restricted curvature*, Proc. Amer. Math. Soc., 57 (1076), pp. 122–126.

[7] G. D. CHAKERIAN, *A characterization of curves of constant width*, Amer. Math. Monthly, 81 (1974), pp. 146–149.

[8] G. D. CHAKERIAN, AND M. S. KLAMKIN, *Minimal covers for closed curves*, Math. Mag., 46 (1973), pp. 55–61.

[9] G. D. CHAKERIAN, *Integral geometry in the Minkowski plane*, Duke Math. J., 29 (1962), pp. 375–382.

[10] L. E. DUBINS, *On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents*, *Amer. J. Math.*, 79(1957), pp. 497–516.

[11] W. H. FLEMING, AND R. W. RISHEL, *Deterministic and stochastic Optimal Control*, Springer-Verlag, 1975.

[12] I. M. GELFAND, AND S. V. FOMIN, *Calculus of Variations*, Prentice Hall, 1963.

[13] M. GHANDEHARI, *Controlling Curvature in the Minkowski Plane*, (English summary), J. Math. Anal. Appl. 252 (2000), no. 2, 951–958.

[14] P. A. GRIFFITHS, *Exterior Differential Systems and the Calculus of Variations*, Progress in Mathematics, Birkhäuser, 1983.

[15] R. HERMANN, *Differential Geometry and the Clculus of Variations*, Academic Press, 1968.

[16] R. ISAACS, *Differential Games*, Wiley, 1965, pp. 8–13.

[17] H. H. JOHNSON, *An application of the maximum principle to the geometry of plane curves*, *Proc. Amer. Math. Soc.*, 44(1974), pp. 432–435.

[18] D. KOUTROUFIOTIS, *On Blaschke's rolling theorems*, Arch. Math., 23 (1972), pp. 655–660.

[19] D. LAUGWITZ, *Konvexe mittelpunktsbereiche und normierte Räume*, Math. Z., 61 (1954), pp. 235–244.

[20] G. LEITMANN, *The Calculus of Variations and optimal Control*, Plenum press, 1981.

[21] Z. A. MELZAK, *Plane motion with curvature limitation*, J. Soc. Indust. Appl. Math., 9 (1961), pp. 422–432.

[22] H. MINKOWSKI, *Theorie der Konvexen Körper, Insbesondere Begründung Ihres Oberflächen Bgiffs*, Ges. Abhandl., Leipzig-Berlin, Vol. 2 (1911), pp. 131–229.

[23] J. C. C. NITSCHE, *The smallest sphere containg a rectifiable curve*, Amer. Math. Monthly, 78 (1971), pp. 881–888.

[24] C. M. PETTY, *On the geometry of the Minkowski plane*, Riv. Mat. Univ. Parma, Vol. 6 (1955), pp. 269–292.

[25] L. S. PONTRYAGIN, V. G. BOLTYANSKII, R. V. GAMKERLIDZE, AND E. F. MISHCHENKO, *The Mathematical Theory of Optimal Processes*, translated from the Russian by K. N. Trirogoff, edited by L. W. Neustadt, Interscience Publishers, New York, 1962.

[26] H. RUTISHAUSER, AND H. SAMELSON, *Sur le rayon d'une sphere dont la surface contient une courbe fermée*, C. R. Acad. Sci. Paris, 227 (1948), pp. 755–757.

[27] S. L. A. SANTALÓ, *Introduction to Integral Geometry*, Paris, Hermann, 1953.

[28] B. SEGRE, *Sui circoli heodetici di una superfilcie a curvature, che contengono nell'interno una linea assegnata*, Boll. Un. Mat. Ital., 13 (1934), pp. 279–283.

[29] J. SCHAER, *An "obvious" but useful theorem about closed curves*, Math. Mag., (1972), pp. 154–155.

[30] F. A. VALENTINE, *Convex Sets*, Krieger, 1976.

(Received February 10, 2009)

Mostafa Ghandehari
Department of Mathematics
University of Texas at Arlington
Arlington, Texas 76019
U.S.A.