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BOUNDS IMPROVEMENT FOR ALTERNATING MATHIEU TYPE SERIES

TIBOR K. POGÁNY AND ŽIVORAD TOMOVSKI

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Abstract. The aim of this research paper is to establish some precise, new bounding inequalities for the generalized alternating Mathieu series using their integral representations and the classical CBS inequality. The obtained inequalities improve certain bounds derived recently by Tomovski and Hilfer in [17].

1. Motivation with introduction

The special function defined by series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \qquad r > 0 \tag{1.1}$$

is called Mathieu–series, after his 'ancestor' É.L. Mathieu, who introduced it in his classical book [7] treating problematics in mathematical physics. Bounds on this series have been used in discussing boundary value problems for the biharmonic equations in $2D$ rectangular domains [11, p. 258, Eq. (54)]. The alternating variant of $S(r)$, *viz.*

$$\widetilde{S}(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2} \qquad r > 0 \tag{1.2}$$

has been introduced recently by Pogány *et al.* in [9]. After that few article have been devoted to alternating Mathieu series and its generalizations, such as alternating generalized Mathieu series, alternating Mathieu $\boldsymbol{a}$–series and alternating Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$–series. Good sources for bounding ineqaulities for alternating Mathieu's and alternating Mathieu type series are the recent articles [9, 15, 17, 18]. However, further research clearly showes that in [17] the authors didn't take into account the oscillatory nature of the Bessel function of the first kind in integrands. Clearly the mistake appears in the proofs of related bounding inequalities. These cases are pointed out here and the erroneous places are corrected in a set of theorems remarking in the same time that this article is essentially *not a corrigendum* to [17].

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Several interesting problems and solutions dealing with integral representations and bounding inequalities for the slight generalization of $S(r), \widetilde{S}(r)$ to Mathieu series with a fractional power

$$S_\mu(r) = \sum_{n=1}^\infty \frac{2n}{(n^2+r^2)^\mu} \qquad r, \mu - 1 > 0 \tag{1.3}$$

can be found in an older article by Diananda [4] and in recent works by Tomovski and Trenčevski [19] and Cerone and Lenard [3]. Motivated essentially by [3] and by [10] a family of so–called *generalized Mathieu series*, reads

$$S_\mu^{(\alpha,\beta)}\big(r;(a_n)_{\mathbb{N}}\big) =: S_\mu^{(\alpha,\beta)}(r;\boldsymbol{a}) = \sum_{n=1}^\infty \frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu} \qquad r,\alpha,\beta,\mu > 0 \tag{1.4}$$

has been defined in [14], tacitly assumed that the positive sequence $\boldsymbol{a} = (a_n)_{\mathbb{N}}$ monotonously diverges to the infinity, i.e. $\lim_{n\to\infty} a_n = \infty$ and the series (1.4) converges, that is the auxiliary series

$$\sum_{n=1}^\infty \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent. Specifiying parameters in (1.4) one recognizes $S_2(r) = S(r)$, $S_\mu(r) = S_\mu^{(2,1)}(r,\mathbb{N})$ etc. compare some parts of [3, 4, 10, 15].

Introducing

$$\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;(a_n)_{\mathbb{N}}\big) =: \widetilde{S}_\mu^{(\alpha,\beta)}(r;\boldsymbol{a}) = \sum_{n=1}^\infty (-1)^{n-1} \frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu} \qquad r,\alpha,\beta,\mu > 0 \tag{1.5}$$

as the alternating variant of (1.4), in [9, 14, 16] several integral representations of $S_\mu^{(\alpha,\beta)}\big(r;\boldsymbol{a}\big)$ and $\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\boldsymbol{a}\big)$ in terms of different variants of generalized hypergeometric functions and Bessel functions of the first kind were established.

2. Integral representations of $\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;(a_n)_{\mathbb{N}}\big)$

The generalized hypergeometric function is defined by

$${}_pF_q\left[\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q\end{matrix}\,\middle|\, x\right] = {}_pF_q\left[\begin{matrix} a_p \\ b_q\end{matrix}\,\middle|\, x\right] := \sum_{m=0}^\infty \frac{\prod_{\ell=1}^p (a_\ell)_m}{\prod_{\ell=1}^q (b_\ell)_m}\frac{x^m}{m!} \tag{2.1}$$

where

$$(\tau)_0 := 1,\, (\tau)_m := \tau(\tau+1)\cdots(\tau+m-1) = \frac{\Gamma(\tau+m)}{\Gamma(\tau)} \qquad m \in \mathbb{N}$$

denotes the *shifted factorial* or Pochhammer symbol. Here, and in what follows, ${}_p\Psi_q$ denotes the Fox–Wright generalization of the hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters (e.g. [14, p. 4, Eq. (2.4)]) defined by

$${}_p\Psi_q\left[\begin{matrix} (a_1,\alpha_p),\cdots,(a_p,\alpha_p) \\ (b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\,\middle|\, x\right] = {}_p\Psi_q\left[\begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q)\end{matrix}\,\middle|\, x\right] := \sum_{m=0}^\infty \frac{\prod_{\ell=1}^p \Gamma\big(a_\ell+\alpha_\ell m\big)}{\prod_{\ell=1}^q \Gamma\big(b_\ell+\beta_\ell m\big)}\frac{x^m}{m!} \tag{2.2}$$

for suitably bounded values of $|x|$, when the parameters involved satisfy

$$\alpha_\ell \in \mathbb{R}_+,\, \ell = \overline{1,p}; \quad \beta_j \in \mathbb{R}_+,\, j = \overline{1,q}; \quad 1 + \sum_{\ell=1}^{q} \beta_\ell - \sum_{j=1}^{p} \alpha_j > 0.$$

So that, obviously

$${}_p\Psi_q\Big[\begin{matrix}(a_p,1)\\(b_q,1)\end{matrix}\Big|x\Big] = \frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)} \cdot {}_pF_q\Big[\begin{matrix}a_p\\b_q\end{matrix}\Big|x\Big] \qquad a_j > 0, b_k \notin \mathbb{Z}_0^-. \tag{2.3}$$

The following integral representations are closely connected to one reported in [9, Eqs. (8.12), (9.7)] and to certain results (for Mathieu type series) reported in [14]. Denote in the sequel $\mathbb{N}^\delta := \big(n^\delta\big)_{n\in\mathbb{N}}$, $\delta > 0$. Hence

$$\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^\gamma\big) = \frac{2}{\Gamma(\mu)} \int_0^\infty \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x+1}\, {}_1\Psi_1\Big[\begin{matrix}(\mu,1)\\(\gamma(\mu\alpha-\beta),\gamma\alpha)\end{matrix}\Big| -r^2x^{\gamma\alpha}\Big]\,\mathrm{d}x, \tag{2.4}$$

valid for $r,\alpha,\beta,\gamma,\gamma(\mu\alpha-\beta)-1>0$ and

$$\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^{q/\alpha}\big) = \frac{2}{\Gamma\big(q[\mu-\beta/\alpha]\big)} \int_0^\infty \frac{x^{q(\mu-\beta/\alpha)-1}}{e^x+1}\, {}_1F_q\Big[\begin{matrix}\mu\\\Delta\big(q;q(\mu-\beta/\alpha)\big)\end{matrix}\Big| -\frac{r^2x^q}{q^q}\Big]\,\mathrm{d}x, \tag{2.5}$$

where

$$r,\, \alpha,\, \beta,\, q(\mu-\beta/\alpha)-1>0; \quad \Delta(q;\lambda) := \Big(\frac{\lambda}{q},\cdots,\frac{\lambda+q-1}{q}\Big),\, q\in\mathbb{N}.$$

Finally, for all $r,\mu>0$ there holds [9, Eq. (5.12)]

$$\widetilde{S}_{\mu+1}^{(\alpha,\alpha/2)}\big(r;\mathbb{N}^{2/\alpha}\big) = \widetilde{S}_{\mu+1}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)} \int_0^\infty \frac{x^{\mu+1/2}}{e^x+1} J_{\mu-1/2}(rx)\,\mathrm{d}x. \tag{2.6}$$

3. Laplace transforms of generalized hypergeometric functions

In the Bessel functions theory it is fairly well–known [6, p. 688, Eq. 6.612.3] that

$$\int_0^\infty e^{-\alpha x} J_b(cx) J_b(hx)\mathrm{d}x = \frac{1}{\pi\sqrt{hc}}\, Q_{b-1/2}\Big(\frac{\alpha^2+c^2+h^2}{2hc}\Big), \tag{3.1}$$

when $\alpha,c,h,b+1/2>0$. Here

$$Q_\beta(z) = \frac{\mathrm{B}(\beta+1,1/2)}{(2z)^{\beta+1}}\, {}_2F_1\Big[\begin{matrix}\beta/2+1,\ (\beta+1)/2\\\beta+3/2\end{matrix}\Big| z^{-2}\Big]$$

stands for the Legendre function of the second kind in which B is the familiar Euler Beta–function. Substituting $\alpha=1, b=\mu-1/2, c=h=r$ in (3.1), we get

$$\begin{aligned}\int_0^\infty e^{-x} J_{\mu-1/2}^2(rx)\,\mathrm{d}x &= \frac{1}{\pi r}\, Q_{\mu-1}\Big(1+\frac{1}{2r^2}\Big)\\ &= \frac{\mathrm{B}(\mu,1/2)r^{2\mu}}{(1+2r^2)^\mu}\, {}_2F_1\Big[\begin{matrix}(\mu+1)/2,\ \mu/2\\\mu+1/2\end{matrix}\Big| \frac{4r^4}{(1+2r^2)^2}\Big].\end{aligned} \tag{3.2}$$

The next generation of hypergeometric type functions we will need in the sequel is the Srivastava–Daoust generalization of Kampé de Fériet hypergeometric function in two variables defined by double hypergeometric series [12, p. 199]

$$\mathscr{S}^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\Phi]:[(b):\psi];[(b'):\psi'] \\ [(c):\delta,\varepsilon]:[(d):\eta];[(d'):\eta']\end{matrix}\,\Big|\, x,y\right) = \mathscr{S}^{A:B;B'}_{C:D;D'}(x,y)$$
$$:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{A}\Gamma(a_j+m\theta_j+n\Phi_j)\prod_{j=1}^{B}\Gamma(b_j+m\psi_j)\prod_{j=1}^{B'}\Gamma(b'_j+n\psi'_j)}{\prod_{j=1}^{C}\Gamma(c_j+m\delta_j+n\varepsilon_j)\prod_{j=1}^{D}\Gamma(d_j+m\eta_j)\prod_{j=1}^{D'}\Gamma(d'_j+n\eta'_j)}\,\frac{x^m}{m!}\,\frac{y^n}{n!}, \tag{3.3}$$

where the coefficients

$$\theta_1,\cdots,\theta_A,\cdots,\eta'_1,\cdots,\eta'_{D'}>0;$$

for the sake of brevity (a) is taken to denote the sequence of A parametars $a_1,\cdots,a_A$, with the similar interpretations for $(b),\cdots,(d')$. Srivastava and Daoust find [13, p. 155] that the series (3.3) converges absolutely for all $x,y\in\mathbb{C}$ when

$$\Delta = 1+\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j>0,$$
$$\Delta' = 1+\sum_{j=1}^{C}\varepsilon_j+\sum_{j=1}^{D'}\eta'_j-\sum_{j=1}^{A}\Phi_j-\sum_{j=1}^{B'}\psi'_j>0.$$

It could be mention that the case $\Delta=\Delta'=0$ has been discussed also in [13, pp. 154–155], while the remaining case, when at least one of Δ,Δ' is negative results in formal power series, that is (3.3) converges only in trivial situation $x=y\equiv 0$.

THEOREM 1. *Assume* $\mu,X,Y>0, Z,W\in\mathbb{R}, q\in\mathbb{N}$ *and* $\Re\{s\}>0$. *Then we have:*

$$\int_0^\infty \mathrm{e}^{-sx}\left\{{}_1\Psi_1\left[\begin{matrix}(\mu,1)\\(X,Y)\end{matrix}\,\Big|\,-Zx^Y\right]\right\}^2\mathrm{d}x$$
$$= s^{-1}\,\mathscr{S}^{1:1;1}_{0:1;1}\left(\begin{matrix}[1:Y,Y]:[\mu:1];[\mu:1]\\ -:[X:Y];[X:Y]\end{matrix}\,\Big|\,-\frac{Z}{s^Y},-\frac{Z}{s^Y}\right), \tag{3.4}$$

$$\int_0^\infty \mathrm{e}^{-sx}\left\{{}_1F_q\left[\begin{matrix}a_1\\b_q\end{matrix}\,\Big|\,-Wx^q\right]\right\}^2\mathrm{d}x$$
$$= \frac{\prod_{j=1}^{q}\Gamma^2(b_j)}{s\,\Gamma^2(a_1)}\,\mathscr{S}^{1:1;1}_{0:q;q}\left(\begin{matrix}[1:q,q]:[a_1:1];[a_1:1]\\ -:[(b):1];[(b):1]\end{matrix}\,\Big|\,-\frac{W}{s^q},-\frac{W}{s^q}\right). \tag{3.5}$$

Proof. By definition it is

$$\begin{aligned}
\int_0^\infty & \mathrm{e}^{-sx} \left\{ {}_1\Psi_1 \left[\begin{matrix} (\mu,1) \\ (X,Y) \end{matrix} \,\Big|\, -Zx^Y \right] \right\}^2 \mathrm{d}x \\
&= \int_0^\infty \mathrm{e}^{-sx} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\Gamma(\mu+m)\Gamma(\mu+n)(-Z)^{m+n}}{\Gamma(X+Ym)\Gamma(X+Yn)} \frac{x^{Y(m+n)}}{m!\,n!} \mathrm{d}x \\
&= s^{-1} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\Gamma(1+Y(m+n))\Gamma(\mu+m)\Gamma(\mu+n)}{\Gamma(X+Ym)\Gamma(X+Yn)} \frac{(-Z/s^Y)^{m+n}}{m!\,n!}, \qquad (3.6)
\end{aligned}$$

such that coincides with the right–hand expression in the display (3.4).

The formula (3.5) we prove analogously, firstly expressing the Γ–function terms of the ${}_1F_q$ function in the integrand in Pochhammer symbol terms.

4. Bounding inequalities

Now, we expose our new results concerning upper bounds for alternating Mathieu series $\widetilde{S}(r), \widetilde{S}_{\mu+1}(r), \widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^\gamma\big), \widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^{q/\alpha}\big)$ derived by means of the celebrated Cauchy–Bunyakovsky–Schwarz (CBS) inequality.

THEOREM 2. *For all $r \in \mathbb{R}$ there holds*

$$\widetilde{S}(r) \leqslant \sqrt{\frac{3\zeta(3)}{1+4r^2}} \qquad \Big(=: B_1(r) \Big). \tag{4.1}$$

Here is $\zeta(3) = 1.2020569\ldots$ the famous Apéry's constant.

Proof. Consider the integral representation formula [9, Eq. (2.8)]

$$\widetilde{S}(r) = \frac{1}{r} \int_0^\infty \frac{x\sin(rx)}{e^x+1} \mathrm{d}x \qquad r \in \mathbb{R}.$$

By the CBS inequality one concludes

$$\widetilde{S}^2(r) \leqslant \frac{1}{r^2} \int_0^\infty \frac{x^2}{e^x+1} \mathrm{d}x \cdot \int_0^\infty \frac{\sin^2(rx)}{e^x+1} \mathrm{d}x.$$

Substituting $a=2, b=1$ into

$$\int_0^\infty \frac{x^{a-1}}{e^{bx}+1} \mathrm{d}x = \frac{1}{b^a} \left(1-2^{1-a}\right) \Gamma(a)\, \zeta(a) \qquad \min\big(\Re\{a\}, \Re\{b\}\big) > 0, \tag{4.2}$$

we get

$$\int_0^\infty \frac{x^2}{e^x+1} \mathrm{d}x = \frac{3}{2}\,\zeta(3).$$

On the other hand

$$\int_0^\infty \frac{\sin^2(rx)}{e^x+1} \mathrm{d}x < \int_0^\infty e^{-x} \sin^2(rx)\, \mathrm{d}x = \frac{2r^2}{1+4r^2}.$$

Collecting all derived bounds we obtain

$$\widetilde{S}(r) \leqslant \big|\widetilde{S}(r)\big| \leqslant \sqrt{\frac{3\zeta(3)}{1+4r^2}}\,.$$

The Theorem is proved.

REMARK 1. The upper bound (4.1) corrects the bound [17, Eq. (2.7)] when $\mu = 2$. On the other hand the authors report on [18, §4, B]

$$\widetilde{S}(r) \leqslant \frac{\pi^2}{12r} =: B_2(r)\,.$$

Comparing the bounds $B_1(r)$ in (4.1) with $B_2(r)$ we conclude

$$\begin{array}{ll} B_1(r) \leqslant B_2(r) & \quad r \in \big(0, r_0\big] \\ B_1(r) > B_2(r) & \quad r > r_0 \end{array}$$

although both bounds have the same asymptotical magnitude when $r \to \infty$. Here

$$r_0 = \frac{\pi^2}{\sqrt{432\zeta(3) - 4\pi^4}} = 0.8667817\ldots\,.$$

THEOREM 3. *For all* $r, \mu > 0$ *we have*

$$\widetilde{S}_{\mu+1}(r) \leqslant \widetilde{C}_\mu(r) \left({}_2F_1\left[\begin{matrix} (\mu+1)/2,\ \mu/2 \\ \mu+1/2 \end{matrix} \,\Big|\, \frac{4r^4}{(1+2r^2)^2} \right] \right)^{1/2}, \tag{4.3}$$

where

$$\widetilde{C}_\mu(r) = \frac{\sqrt{\pi r(2+1/\mu)}}{2^{\mu-1/2}(1+2r^2)^{\mu/2}} \sqrt{\zeta(2\mu+2)\,\frac{\mathrm{B}(2\mu+1,1/2)}{\mathrm{B}(\mu+1,\mu+1/2)}}\,.$$

Proof. Applying the CBS inequality to the integral in (2.6) we deduce

$$\left(\int_0^\infty \frac{x^{\mu+1/2}}{e^x+1} J_{\mu-1/2}(rx)\,\mathrm{d}x \right)^2 \leqslant \int_0^\infty \frac{x^{2\mu+1}}{e^x+1}\,\mathrm{d}x \cdot \int_0^\infty \frac{J^2_{\mu-1/2}(rx)}{e^x+1}\,\mathrm{d}x\,.$$

Putting $a = 2\mu+2, b = 1$ in (4.2) we readily get

$$\int_0^\infty \frac{x^{2\mu+1}}{e^x+1}\,\mathrm{d}x = \big(1 - 2^{-1-2\mu}\big)\,\Gamma(2\mu+2)\,\zeta(2\mu+2) \leqslant \Gamma(2\mu+2)\,\zeta(2\mu+2)\,.$$

Making use of Laplace–transform formula (3.2) we obtain

$$\begin{aligned} \int_0^\infty \frac{J^2_{\mu-1/2}(rx)}{e^x+1}\,\mathrm{d}x &< \int_0^\infty e^{-x} J^2_{\mu-1/2}(rx)\,\mathrm{d}x \\ &= \frac{\mathrm{B}(\mu,1/2)\,r^{2\mu}}{(1+2r^2)^\mu}\, {}_2F_1\left[\begin{matrix} (\mu+1)/2,\ \mu/2 \\ \mu+1/2 \end{matrix} \,\Big|\, \frac{4r^4}{(1+2r^2)^2} \right]. \end{aligned}$$

Majorizing (2.6) by the achieved bounds we arrive at (4.3).

REMARK 2. To correct the erroneous bound [17, Eq. (2.23)] it is enough to replace it by the upper bound (4.3).

THEOREM 4. *We have*

$$\widetilde{S}_{\mu+1}(r) = \mathscr{O}\Big(r^{\mu-1/2}\ln\frac{1+4r^2}{(1+2r^2)^2}\Big) \qquad r\to\infty. \tag{4.4}$$

Proof. Ramanujan's asymptotic relation for zero–balanced ${}_2F_1$ (cf. e.g. [2]), reads

$$\begin{aligned}-\mathrm{B}(a,b)\cdot {}_2F_1\left[\begin{matrix}a,b\\ a+b\end{matrix}\,\middle|\,x\right] &= \psi(a)+\psi(b)+2\gamma\\ &\quad +\ln(1-x)+\mathscr{O}\big((1-x)\ln(1-x)\big) \qquad x\to 1-\end{aligned}$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the familiar digamma–function, while γ denotes the famous Euler–Mascheroni constant. Applying this asymptotic formula with

$$x=\frac{4r^4}{(1+2r^2)^2}$$

to the Gaussian term in (4.3), we conclude the assertion.

In [1, Eq. (1.2)] Alzer has been proved the sharpness of the inequality

$$\mathrm{B}(x,y)-\frac{1}{xy}\geqslant 0 \qquad \min(x,y)\geqslant 1 \tag{4.5}$$

given already by Dragomir *et al.* [5]. It is obvious that (4.5) is equivalent to

$$\Gamma(x+y)\leqslant \Gamma(x+1)\cdot\Gamma(y+1) \tag{4.6}$$

on the same range of x,y.

THEOREM 5. *For all* $r,\alpha,\beta,\gamma>0,\ \gamma(\mu\alpha-\beta)>1$ *we have*

$$\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^\gamma\big)\leqslant \widetilde{C}_\mu^{(\alpha,\beta)}(\gamma)\,{}_2\Psi_1\left[\begin{matrix}(3/2,\gamma\alpha),(\mu,1)\\ \big(\gamma(\mu\alpha-\beta),\gamma\alpha\big)\end{matrix}\,\middle|\,-r^2\right] \tag{4.7}$$

where

$$\widetilde{C}_\mu^{(\alpha,\beta)}(\gamma)=\frac{2}{\Gamma(\mu)}\sqrt{\Gamma\big(2\gamma(\mu\alpha-\beta)-1\big)\,\zeta\big(2\gamma(\mu\alpha-\beta)-1\big)}\,.$$

Proof. The Srivastava–Daoust $\mathscr{S}$–function in (3.4) has been majorized by a square of a ${}_2\Psi_1$–function expression *via* (4.6) specifying $x=Ym+1/2,\ y=Yn+1/2$ to evaluate the double indexed Γ–term in (3.6), that is:

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-sx}\Big({}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (X,Y)\end{matrix}\,\middle|\,-Zx^Y\right]\Big)^2\mathrm{d}x\\ &\qquad\leqslant \frac{1}{s}\left(\sum_{m=0}^\infty\frac{\Gamma(3/2+Ym)\Gamma(\mu+m)}{\Gamma(X+Ym)}\,\frac{(-Z/s^Y)^m}{m!}\right)^2\\ &\qquad= \frac{1}{s}\Big({}_2\Psi_1\left[\begin{matrix}\big(3/2,Y\big),(\mu,1)\\ (X,Y)\end{matrix}\,\middle|\,-Zs^{-Y}\right]\Big)^2. \end{aligned}\tag{4.8}$$

Specifying here $X=\gamma(\mu\alpha-\beta), Y=\gamma\alpha, Z=r^2$ the upper bound in (4.8) becomes

$$\frac{1}{s}\left({}_2\Psi_1\left[\begin{matrix}(3/2,\gamma\alpha),(\mu,1)\\ (\gamma(\mu\alpha-\beta),\gamma\alpha)\end{matrix}\,\Big|\,-r^2 s^{-\gamma\alpha}\right]\right)^2. \tag{4.9}$$

The CBS inequality one transforms (2.4) into

$$\begin{aligned}\left(\widetilde{S}_\mu^{(\alpha,\beta)}(r;\mathbb{N}^\gamma)\right)^2 &\leqslant \frac{2}{\Gamma(\mu)}\int_0^\infty \frac{x^{2[\gamma(\mu\alpha-\beta)-1]}}{e^x+1}\,\mathrm{d}x\\ &\quad\cdot\int_0^\infty \frac{\left({}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta),\gamma\alpha)\end{matrix}\,\Big|\,-r^2x^{\gamma\alpha}\right]\right)^2}{e^x+1}\,\mathrm{d}x. \end{aligned}\tag{4.10}$$

Since, by usual reasons

$$\int_0^\infty \frac{x^{2[\gamma(\mu\alpha-\beta)-1]}}{e^x+1}\,\mathrm{d}x \leqslant \Gamma\big(2\gamma(\mu\alpha-\beta)-1\big)\,\zeta\big(2\gamma(\mu\alpha-\beta)-1\big),$$

and by (4.8) and (4.9) it is

$$\begin{aligned}\int_0^\infty &\frac{\left({}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta),\gamma\alpha)\end{matrix}\,\Big|\,-r^2x^{\gamma\alpha}\right]\right)^2}{e^x+1}\,\mathrm{d}x\\ &\leqslant \int_0^\infty e^{-x}\left({}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta),\gamma\alpha)\end{matrix}\,\Big|\,-r^2x^{\gamma\alpha}\right]\right)^2\mathrm{d}x\\ &\leqslant \left({}_2\Psi_1\left[\begin{matrix}(3/2,\gamma\alpha),(\mu,1)\\ (\gamma(\mu\alpha-\beta),\gamma\alpha)\end{matrix}\,\Big|\,-r^2\right]\right)^2,\end{aligned}$$

we immediately deduce the asserted bound (4.7).

REMARK 3. By this estimation procedure we just have to replace [17, Eq. (2.27)] by (4.7). However, a question on the existence of the upper bound offered by the authors in [17] in the form

$$\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^\gamma\big) \leqslant M_\mu^{(\alpha,\beta)}(r;\gamma) = \frac{\Phi}{(1+r^2)^\kappa}$$

remains open; here $\Phi=\Phi(\mu,\alpha,\beta,\gamma)$ and $\kappa=\kappa(\mu,\alpha,\beta,\gamma)$ are absolute constants.

THEOREM 6. *For all $q\in\mathbb{N}$; r, α, β, $q(\mu-\beta/\alpha)-1>0$ there holds the bounding inequality*

$$\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^{q/\alpha}\big) \leqslant \widetilde{K}_\mu^{(\alpha,\beta)}(q)\cdot{}_2\Psi_q\left[\begin{matrix}(3/2,q),(\mu,1)\\ \Big(\Delta\big(q;q[\mu-\beta/\alpha]\big),1\Big)\end{matrix}\,\Big|\,-r^2q^{-q}\right], \tag{4.11}$$

where

$$\widetilde{K}_\mu^{(\alpha,\beta)}(q) = \frac{2\,(2\pi)^{\frac{q-1}{2}}}{q^{q[\mu-\beta/\alpha]-1/2}\,\Gamma(\mu)}\sqrt{\Gamma\big(2q(\mu-\beta/\alpha)-1\big)\,\zeta\big(2q(\mu-\beta/\alpha)-1\big)}.$$

Proof. By (3.5), having on mind the Dragomir–Alzer inequality we conclude

$$\int_0^\infty e^{-sx}\Big({}_1F_q\Big[\begin{matrix} a_1 \\ b_q \end{matrix}\,\Big|\, -W x^q\Big]\Big)^2 \mathrm{d}x \leqslant \frac{1}{s}\left(\frac{\prod_{j=1}^q \Gamma(b_j)}{\Gamma(a_1)}\, {}_2\Psi_q\Big[\begin{matrix} (3/2,q),(a_1,1) \\ ((b),1) \end{matrix}\,\Big|\, -\frac{W}{s^q}\Big]\right)^2 .$$

Taking here $a_1 = \mu$, $(b) = \Delta\big(q; q[\mu-\beta/\alpha]\big)$, $W = r^2 q^{-q}$ one gets

$$\begin{aligned}\int_0^\infty e^{-sx}&\left({}_1F_q\left[\begin{matrix} \mu \\ \big(\Delta\big(q;q[\mu-\beta/\alpha]\big)\big) \end{matrix}\,\middle|\, -\frac{r^2x^q}{q^q}\right]\right)^2 \mathrm{d}x \\ &\leqslant \frac{1}{s}\left(\frac{\prod_{j=1}^q \Gamma(b_j)}{\Gamma(a_1)}\, {}_2\Psi_q\left[\begin{matrix} (3/2,q),(\mu,1) \\ \big(\Delta\big(q;q[\mu-\beta/\alpha]\big),1\big) \end{matrix}\,\middle|\, -\frac{r^2}{(sq)^q}\right]\right)^2 .\end{aligned}$$

Applying once more the CBS inequality, now for the right hand side of the integral representation (2.5), taking $s = 1$, we conclude

$$\begin{aligned}\Big(\widetilde{S}_\mu^{(\alpha,\beta)}\big(r;\mathbb{N}^{q/\alpha}\big)\Big)^2 &\leqslant \frac{2}{\Gamma\big(q[\mu-\beta/\alpha]\big)} \int_0^\infty \frac{x^{2(q[\mu-\beta/\alpha]-1)}}{e^x+1}\,\mathrm{d}x \\ &\qquad \cdot \int_0^\infty \frac{\left({}_1F_q\left[\begin{matrix} \mu \\ \big(\Delta\big(q;q[\mu-\beta/\alpha]\big)\big) \end{matrix}\,\middle|\, -r^2(x/q)^q\right]\right)^2}{e^x+1}\,\mathrm{d}x \\ &\leqslant \Big(\widetilde{K}_\mu^{(\alpha,\beta)}(q)\Big)^2 \cdot \left({}_2\Psi_q\left[\begin{matrix} (3/2,q),(\mu,1) \\ \big(\Delta\big(q;q[\mu-\beta/\alpha]\big),1\big) \end{matrix}\,\middle|\, -r^2q^{-q}\right]\right)^2 .\end{aligned}$$

Having in mind (4.2) with $a = 2(q[\mu-\beta/\alpha]-1), b = 1$, we arrive at the asserted upper bound, since

$$\begin{aligned}\widetilde{K}_\mu^{(\alpha,\beta)}(q) &= \frac{2\prod_{j=0}^{q-1}\Gamma\big(q[\mu-\beta/\alpha]+j/q\big)}{\Gamma(\mu)\,\Gamma\big(q[\mu-\beta/\alpha]\big)} \sqrt{\Gamma\big(2q(\mu-\beta/\alpha)-1\big)\,\zeta\big(2q(\mu-\beta/\alpha)-1\big)} \\ &= \frac{2\,(2\pi)^{\frac{q-1}{2}}}{q^{q[\mu-\beta/\alpha]-1/2}\,\Gamma(\mu)} \sqrt{\Gamma\big(2q(\mu-\beta/\alpha)-1\big)\,\zeta\big(2q(\mu-\beta/\alpha)-1\big)}\end{aligned}$$

by the well–known Gauss–Legendre multiplication formula for Gamma function.

REMARK 4. The bounding ineqaulity (4.11) replaces [17, Eq. (2.25)].

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Tibor K. Pogány
Faculty of Maritime Studies
University of Rijeka
HR-51000 Rijeka
Studentska 2
Croatia
e-mail: poganj@pfri.hr

Zivorad Tomovski
Institute of Mathematics
St. Cyril and Methodius University
MK-1000 Skopje
Macedonia
e-mail: tomovski@iunona.pmf.ukim.edu.mk