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MEAN THEORETIC APPROACH TO A FURTHER EXTENSION OF GRAND FURUTA INEQUALITY

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Dedicated to Professor Kichi-Suke Saito on his 60th birthday

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Abstract. Very recently, Furuta has shown a further extension of grand Furuta inequality. In this paper, we obtain a more precise and clear expression of Furuta's extension by considering a mean theoretic proof of grand Furuta inequality. Moreover, we get a variant of Furuta's extension by scrutinizing the former argument.

1. Introduction

Throughout this paper, A and B are positive operators on a complex Hilbert space, and we denote $A \geqslant 0$ (resp. $A > 0$) if A is a positive (resp. strictly positive) operator.

As an extension of Löwner-Heinz theorem "*$A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$*," Furuta inequality was established in [8] (see also [2, 9, 12, 18, 20]).

THEOREM A. (Furuta inequality [8]) *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

$$\text{(i) } (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant B^{\frac{p+r}{q}} \quad \textit{and} \quad \text{(ii) } A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

THEOREM B. ([3]) *Let $A \geqslant B \geqslant 0$ with $A > 0$. Then*

$$f(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}} \tag{1.1}$$

is decreasing for $p \geqslant 1$ and $r \geqslant 0$.

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For $A > 0$ and $B \geqslant 0$, α-power mean $\sharp_\alpha$ for $\alpha \in [0,1]$ is defined by $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$. In this paper, we use this operator mean as our main tool. We remark that the operator mean theory was established by Kubo-Ando [19].

It is known that α-power mean is very usful for investigating Furuta inequality. As stated in [18], when $A > 0$ and $B \geqslant 0$, Furuta inequality can be arranged in terms of α-power mean as follows: If $A \geqslant B \geqslant 0$ with $A > 0$, then

$$A \geqslant B \geqslant A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \quad \text{for } p \geqslant 1 \text{ and } r \geqslant 0.$$

Similarly, (1.1) can be rewritten by

$$f(p,r) = A^{-r} \sharp_{\frac{1+r}{p+r}} B^p. \tag{1.1'}$$

In [10], Furuta has shown an extension of Furuta inequality, which is called grand Furuta inequality (see also [5, 7, 11, 12, 13, 16, 21, 22, 23]). We remark that grand Furuta inequality is also an extension of Ando-Hiai inequality [1] which is equivalent to the main result of log majorization, and we are also discussing Furuta inequality and Ando-Hiai inequality in [4, 6, 17].

THEOREM C. (Grand Furuta inequality [10]) *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,*

$$F(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ and $s \geqslant 1$, and

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geqslant t$ and $s \geqslant 1$.

By putting $\beta = (p-t)s+t$ and $\gamma = r-t$, we can arrange Theorem C in terms of α-power mean as follows [5]: If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$ with $p \neq t$,

$$\hat{F}(\beta,\gamma) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{is decreasing for } \beta \geqslant p \text{ and } \gamma \geqslant 0,$$

and

$$A \geqslant B \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{for } \beta \geqslant p \text{ and } \gamma \geqslant 0, \tag{1.2}$$

where $A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^sA^{\frac{1}{2}}$ for a real number s. (If $s \in [0,1]$, then $\natural_s = \sharp_s$.)

Very recently, Furuta [14, 15] has dug for a further extension of grand Furuta inequality, which is the following Theorem D. We call this “FGF inequality” here.

THEOREM D. (FGF inequality [14, 15]) *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$, $t \in [0,1]$ *and* $p_1, p_2, \dots, p_{2n-1} \geqslant 1$ *for natural number* n. *Then*

$$G(r,p_{2n}) = A^{\frac{-r}{2}}\Big[A^{\frac{r}{2}}\Big(A^{\frac{-t}{2}}\Big\{A^{\frac{t}{2}}\cdots(A^{\frac{-t}{2}}\{A^{\frac{t}{2}} \times (A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4}\cdots A^{\frac{t}{2}}\Big\}^{p_{2n-1}}A^{\frac{-t}{2}}\Big)^{p_{2n}}A^{\frac{r}{2}}\Big]^{\frac{1-t+r}{q[2n]-t+r}}A^{\frac{-r}{2}} \tag{1.3}$$

is decreasing for $r \geqslant t$ *and* $p_{2n} \geqslant 1$, *and*

$$A^{1-t+r} \geqslant \Big[A^{\frac{r}{2}}\Big(A^{\frac{-t}{2}}\Big\{A^{\frac{t}{2}}\cdots(A^{\frac{-t}{2}}\{A^{\frac{t}{2}} \times (A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4}\cdots A^{\frac{t}{2}}\Big\}^{p_{2n-1}}A^{\frac{-t}{2}}\Big)^{p_{2n}}A^{\frac{r}{2}}\Big]^{\frac{1-t+r}{q[2n]-t+r}} \tag{1.4}$$

holds for $r \geqslant t$ *and* $p_{2n} \geqslant 1$, *where*

$$q[2n] = \big(\{\cdots(\{(p_1-t)p_2+t\}p_3-t)p_4+\cdots+t\big\}p_{2n-1}-t\big)p_{2n}+t.$$

In this paper, we obtain a more precise and clear expression of FGF inequality by considering a mean theoretic proof of grand Furuta inequality. Moreover, we get a variant of FGF inequality by scrutinizing the former argument.

2. FGF inequality

It is known that the following Lemma 2.1 plays an important role in the proof of grand Furuta inequality (1.2).

LEMMA 2.1. ([5]) *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$. *Then*

$$A \geqslant B \geqslant (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$$

holds for $t \in [0,1]$, $\beta \geqslant p \geqslant 1$ *and* $p \neq t$.

For convenience, we prove this lemma given in [5].

Proof. We may also assume that B is invertible. Let $[p,\beta]$ divide into $p = \beta_0 \leqslant \beta_1 \leqslant \ldots \leqslant \beta_n = \beta$ such that $1 \leqslant \frac{\beta_i - t}{\beta_{i-1}-t} \leqslant 2$. Let $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i-t}{p-t}} B^p)^{\frac{1}{\beta_i}}$ for $i = 1,2,\dots,n$. Then we show

$$B_{i-1}^{\beta_i} \geqslant B_i^{\beta_i} \text{ for } i = 1,2,\dots,n \quad \text{and} \quad A \geqslant B \geqslant B_1 \geqslant \cdots \geqslant B_n. \tag{2.1}$$

First, we can show $B^{\beta_1} \geqslant B_1^{\beta_1}$ as follows:

$$\begin{aligned} B_1^{\beta_1} &= A^t \natural_{\frac{\beta_1-t}{p-t}} B^p = B^p \natural_{\frac{p-\beta_1}{p-t}} A^t = B^p(B^{-p} \sharp_{\frac{\beta_1-p}{p-t}} A^{-t})B^p \\ &\leqslant B^p(B^{-p} \sharp_{\frac{\beta_1-p}{p-t}} B^{-t})B^p = B^{\beta_1} \end{aligned}$$

since $1 \leqslant \frac{\beta_1 - t}{p-t} \leqslant 2$ and $t \in [0,1]$.

For some natural number k such that $k \leqslant n-1$, assume that $B_{i-1}^{\beta_i} \geqslant B_i^{\beta_i}$ for $i = 1,2,\dots,k$. We note that $A \geqslant B \geqslant B_1 \geqslant \cdots \geqslant B_k$ is easily obtained by Löwner-Heinz theorem. Then we obtain

$$\begin{aligned} B_{k+1}^{\beta_{k+1}} &= A^t \natural_{\frac{\beta_{k+1}-t}{p-t}} B^p = A^t \natural_{\frac{\beta_{k+1}-t}{\beta_k - t}} (A^t \natural_{\frac{\beta_k - t}{p-t}} B^p) \\ &= A^t \natural_{\frac{\beta_{k+1}-t}{\beta_k - t}} B_k^{\beta_k} = B_k^{\beta_k} \natural_{\frac{\beta_k - \beta_{k+1}}{\beta_k - t}} A^t = B_k^{\beta_k} (B_k^{-\beta_k} \sharp_{\frac{\beta_{k+1}-\beta_k}{\beta_k - t}} A^{-t}) B_k^{\beta_k} \\ &\leqslant B_k^{\beta_k} (B_k^{-\beta_k} \sharp_{\frac{\beta_{k+1}-\beta_k}{\beta_k - t}} B_k^{-t}) B_k^{\beta_k} = B_k^{\beta_{k+1}} \end{aligned}$$

since $1 \leqslant \frac{\beta_{k+1}-t}{\beta_k - t} \leqslant 2$ and $t \in [0,1]$, so that the proof is complete. $\square$

Next we show that a sequence $\{B_i\}$ such that $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$ is decreasing. Theorem 2.2 is a key result in the proof of FGF inequality.

THEOREM 2.2. *Let $A \geqslant B \geqslant 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_i \geqslant \alpha_i \geqslant 1$ and $\alpha_i \neq t$ for $i = 1,2,\dots,n$,*

$$A \geqslant B \geqslant B_1 \geqslant \cdots \geqslant B_{n-1} \geqslant B_n,$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. By applying Lemma 2.1 to that $A \geqslant B \geqslant 0$ with $A > 0$, we have

$$A \geqslant B \geqslant (A^t \natural_{\frac{\beta_1 - t}{\alpha_1 - t}} B^{\alpha_1})^{\frac{1}{\beta_1}} = B_1$$

for $t \in [0,1]$, $\beta_1 \geqslant \alpha_1 \geqslant 1$ and $\alpha_1 \neq t$, and also by applying Lemma 2.1 repeatedly to that $A \geqslant B_{i-1} \geqslant 0$ with $A > 0$ for $i = 1,2,\dots,n$, we have

$$B_{i-1} \geqslant (A^{t_i} \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}} = B_i$$

for $t \in [0,1]$, $\beta_i \geqslant \alpha_i \geqslant 1$ and $\alpha_i \neq t$, so that

$$A \geqslant B \geqslant B_1 \geqslant \cdots \geqslant B_{n-1} \geqslant B_n.$$

Hence the proof is complete. $\square$

Furuta [15] has given an extension of Lemma 2.1 as an application of Theorem D.

THEOREM E. ([15]) *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \dots, p_{2n-1}, p_{2n} \geqslant 1$ for natural number n. Then*

$$\begin{aligned} &A \geqslant B \geqslant \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{\frac{1}{q[2]}} \geqslant \cdots \geqslant \\ &\left[A^{\frac{t}{2}} \left(A^{\frac{-t}{2}} \left\{A^{\frac{t}{2}} \cdots (A^{\frac{-t}{2}} \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}})^{p_4} \cdots A^{\frac{t}{2}}\right\}^{p_{2n-1}} A^{\frac{-t}{2}}\right)^{p_{2n}} A^{\frac{t}{2}}\right]^{\frac{1}{q[2n]}}, \end{aligned}$$

where

$$q[2n] = \big(\{\cdots(\{(p_1-t)p_2+t\}p_3-t)p_4+\cdots+t\}p_{2n-1}-t\big)p_{2n}+t.$$

We can rewrite Theorem E by putting

$$\beta_0 = 1\,,\ \alpha_i = \beta_{i-1}p_{2i-1}\,,\ \beta_i = (\alpha_i - t)p_{2i}+t \text{ and } \gamma = r-t \tag{2.2}$$

as follows:

THEOREM E'. *Let $A \geqslant B \geqslant 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_n \geqslant \alpha_n \geqslant \beta_{n-1} \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \beta_1 \geqslant \alpha_1 \geqslant 1$ and $\alpha_i \neq t$ for $i = 1,2,\ldots,n$,*

$$A \geqslant B \geqslant B_1 \geqslant \cdots \geqslant B_{n-1} \geqslant B_n,$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Therefore we recognize that Theorem 2.2 is a fine extension of Theorem E. More precisely, $\beta_i \geqslant \alpha_i \geqslant 1$ in Theorem 2.2 is looser than $\beta_n \geqslant \alpha_n \geqslant \beta_{n-1} \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \beta_1 \geqslant \alpha_1 \geqslant 1$ in Theorem E.

By using Theorem 2.2, we obtain an improvement of (1.4) in Theorem D and Theorem E. Theorem 2.3 is a satellite form of Theorem D in our sense. Theorem 2.3 leads (1.4) in Theorem D by the same replacement to (2.2).

THEOREM 2.3. *Let $A \geqslant B \geqslant 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_n \geqslant \alpha_n \geqslant \beta_{n-1} \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \beta_1 \geqslant \alpha_1 \geqslant 1$, $\gamma \geqslant 0$ and $\alpha_1 \neq t$,*

$$\begin{aligned} A &\geqslant B \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_1+\gamma}} B^{\alpha_1} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B_1^{\beta_1} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_2+\gamma}} B_1^{\alpha_2} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_2^{\beta_2} \\ &\geqslant \cdots \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{n-1}+\gamma}} B_{n-1}^{\beta_{n-1}} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_n+\gamma}} B_{n-1}^{\alpha_n} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_n^{\beta_n}, \end{aligned}$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. Let $\beta_0 = 1$. By Theorem 2.2, $A \geqslant B_{i-1}$ holds for $i = 1,2,\ldots,n$, so that we have

$$\begin{aligned} & A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{i-1}+\gamma}} B_{i-1}^{\beta_{i-1}} \\ &\geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_i+\gamma}} B_{i-1}^{\alpha_i} && \text{by Theorem B} \\ &\geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i}) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_i^{\beta_i} && \text{by Theorem C} \end{aligned}$$

since $\beta_i \geqslant \alpha_i \geqslant \beta_{i-1} \geqslant 1$. Hence the proof is complete. $\square$

3. Variant of FGF inequality

In this section, we obtain a variant of FGF inequality by scrutinizing the argument in Section 2.

THEOREM 3.1. *Let $A \geqslant B \geqslant 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\alpha_i \geqslant 1$, $1 \leqslant \frac{\beta_i - t}{\alpha_i - t} \leqslant 2$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,*

$$B_{i-1}^{\beta_i} \geqslant B_i^{\beta_i},$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. We may also assume that B is invertible. By Theorem 2.2, $A \geqslant B_{i-1}$ holds for $i = 1, 2, \ldots, n$, so that we have

$$\begin{aligned} B_i^{\beta_i} &= A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i} = B_{i-1}^{\alpha_i} \natural_{\frac{\alpha_i - \beta_i}{\alpha_i - t}} A^t = B_{i-1}^{\alpha_i} (B_{i-1}^{-\alpha_i} \sharp_{\frac{\beta_i - \alpha_i}{\alpha_i - t}} A^{-t}) B_{i-1}^{\alpha_i} \\ &\leqslant B_{i-1}^{\alpha_i} (B_{i-1}^{-\alpha_i} \sharp_{\frac{\beta_i - \alpha_i}{\alpha_i - t}} B_{i-1}^{-t}) B_{i-1}^{\alpha_i} = B_{i-1}^{\beta_i} \end{aligned}$$

since $1 \leqslant \frac{\beta_i - t}{\alpha_i - t} \leqslant 2$ and $t \in [0,1]$. Hence the proof is complete. $\square$

THEOREM 3.2. *Let $A \geqslant B \geqslant 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\alpha_i \geqslant 1$, $\beta_n \geqslant \cdots \geqslant \beta_2 \geqslant \beta_1 \geqslant 1$, $1 \leqslant \frac{\beta_i - t}{\alpha_i - t} \leqslant 2$, $\gamma \geqslant 0$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,*

$$\begin{aligned} A \geqslant B &\geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B^{\beta_1} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B_1^{\beta_1} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_1^{\beta_2} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_2^{\beta_2} \\ &\geqslant \cdots \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{n-1}+\gamma}} B_{n-1}^{\beta_{n-1}} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_{n-1}^{\beta_n} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_n^{\beta_n}, \end{aligned}$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. Let $\beta_0 = 1$. By Theorems 2.2 and 3.1, $A \geqslant B_{i-1}$ and $B_{i-1}^{\beta_i} \geqslant B_i^{\beta_i}$ for $i = 1, \ldots, n$. Then we have

$$\begin{aligned} A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{i-1}+\gamma}} B_{i-1}^{\beta_{i-1}} &\geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_{i-1}^{\beta_i} \quad \text{by Theorem B since } \beta_i \geqslant \beta_{i-1} \\ &\geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_i^{\beta_i}, \end{aligned}$$

so that the proof is complete. $\square$

REMARK. By Theorem 3.1, $B_{i-1}^{\beta} \geqslant B_i^{\beta}$ by putting $\beta_i = \beta$ and $\alpha_i = p$ for $i = 1, 2, \ldots, n$, so we can obviously get a basis of this argument as follows: Let $A \geqslant B \geqslant 0$

with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $p \geqslant 1$, $1 \leqslant \frac{\beta-t}{p-t} \leqslant 2$, $\gamma \geqslant 0$ and $p \neq t$.

$$\begin{aligned} A \geqslant B \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B_1^p) \\ \geqslant \cdots \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B_n^p). \end{aligned}$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta-t}{p-t}} B_{i-1}^p)^{\frac{1}{\beta}}$ for $i = 1,2,\ldots,n$.

4. FGF-type operator function

Here, we attempt to trace a proof of grand Furuta inequality from the viewpoint of a sequence $\{B_i\}$ such that $B_i = (A^t \natural_{\frac{\beta_i-t}{p-t}} B^p)^{\frac{1}{\beta_i}}$.

THEOREM 4.1. *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$, $p \geqslant 1$ with $p \neq t$ and $\gamma \geqslant 0$. Then*

$$\hat{F}(\beta) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \tag{4.1}$$

is decreasing for $\beta \geqslant p$.

Proof. Let n and m be natural numbers such that $n < m$ and $\beta' \geqslant \beta \geqslant p$. For $i = 1,2,\ldots,n,n+1,\ldots,m$, divide $[p,\beta]$ into $p = \beta_0 \leqslant \beta_1 \leqslant \ldots \leqslant \beta_n = \beta$ and $[\beta,\beta']$ into $\beta = \beta_n \leqslant \beta_{n+1} \leqslant \ldots \leqslant \beta_m = \beta'$ such that $1 \leqslant \frac{\beta_i-t}{\beta_{i-1}-t} \leqslant 2$, and let $B_i = (A^t \natural_{\frac{\beta_i-t}{p-t}} B^p)^{\frac{1}{\beta_i}}$. Then, by (2.1) in the proof of Lemma 2.1,

$$\begin{aligned} & B_{i-1}^{\beta_i} \geqslant B_i^{\beta_i} \text{ for } i = 1,2,\ldots,n,n+1,\ldots,m \\ & \text{and} \quad A \geqslant B \geqslant B_1 \geqslant \cdots \geqslant B_n \geqslant B_{n+1} \geqslant \cdots \geqslant B_m. \end{aligned} \tag{4.2}$$

Noting (4.2) and that $\hat{F}(\beta_i) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} (A^t \natural_{\frac{\beta_i-t}{p-t}} B^p) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_i^{\beta_i}$, we have

$$\begin{aligned} \hat{F}(\beta_{i-1}) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{i-1}+\gamma}} B_{i-1}^{\beta_{i-1}} \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_{i-1}^{\beta_i} \quad \text{by Theorem B since } \beta_i \geqslant \beta_{i-1} \\ \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_i^{\beta_i} = \hat{F}(\beta_i) \end{aligned}$$

for $i = n+1,\ldots,m$. Therefore we get $\hat{F}(\beta) = \hat{F}(\beta_n) \geqslant \hat{F}(\beta_{n+1}) \geqslant \cdots \geqslant \hat{F}(\beta_m) = \hat{F}(\beta')$, so that the proof is complete. □

We remark that (4.1) is also decreasing for $\gamma \geqslant 0$ by Theorem B since $A \geqslant B \geqslant 0$ with $A > 0$ ensures $A \geqslant (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$ by Lemma 2.1, so Theorem 4.1 is Theorem C by putting $\beta = (p-t)s+t$ and $\gamma = r-t$.

REMARK. Theorem 3.2 ensures Theorem 4.1 by putting $\alpha_1 = p \geqslant 1$ and $\alpha_i = \beta_{i-1}$ for $i = 2,3,\ldots,n$ since

$$B_i = (A^t \natural_{\frac{\beta_i - t}{\beta_{i-1}-t}} B_{i-1}^{\beta_{i-1}})^{\frac{1}{\beta_i}} = (A^t \natural_{\frac{\beta_i - t}{\beta_{i-1}-t}} (A^t \natural_{\frac{\beta_{i-1} - t}{\beta_{i-2}-t}} B_{i-2}^{\beta_{i-2}}))^{\frac{1}{\beta_i}} = (A^t \natural_{\frac{\beta_i - t}{\beta_{i-2}-t}} B_{i-2}^{\beta_{i-2}})^{\frac{1}{\beta_i}}$$
$$= \cdots = (A^t \natural_{\frac{\beta_i - t}{\beta_1 - t}} B_1^{\beta_1})^{\frac{1}{\beta_i}} = (A^t \natural_{\frac{\beta_i - t}{p-t}} B^p)^{\frac{1}{\beta_i}}.$$

By applying Theorem 4.1 to Theorem 2.2, we can reform Theorem D immediately.

THEOREM 4.2. *Let $A \geqslant B \geqslant 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_i \geqslant \alpha_i \geqslant 1$ for $i = 1,2,\ldots,n-1$, $\alpha_n \geqslant 1$, $\gamma \geqslant 0$ and $\alpha_i \neq t$ for $i = 1,2,\ldots,n$,*

$$\hat{G}(\beta_n) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} (A^t \natural_{\frac{\beta_n - t}{\alpha_n - t}} B_{n-1}^{\alpha_n}) \tag{4.3}$$

is decreasing for $\beta_n \geqslant \alpha_n$, where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. By Theorem 2.2, $A \geqslant B_{i-1}$ holds for $i = 1,2,\ldots,n$, so that we have Theorem 4.2 immediately by Theorem 4.1. $\square$

(4.3) is also decreasing for $\gamma \geqslant 0$ by Theorem B since $A \geqslant B \geqslant 0$ with $A > 0$ ensures $A \geqslant B_n = (A^t \natural_{\frac{\beta_n - t}{\alpha_n - t}} B_{n-1}^{\alpha_n})^{\frac{1}{\beta_n}}$ by Theorem 2.2. Therefore, similarly to Theorem 2.2, we recognize that Theorem 4.2 is a slight extension of (1.3) in Theorem D.

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