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VERY ACCURATE APPROXIMATIONS FOR THE FACTORIAL FUNCTION

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Abstract. We establish the following new Stirling-type approximation formulas for the factorial function

$$n! \approx \sqrt{2\pi}n^n e^{-n}\sqrt{n+\frac{1}{6}+\frac{1}{72n}-\frac{31}{6480n^2}-\frac{139}{155520n^3}+\frac{9871}{6531840n^4}}$$

and

$$n! \approx \sqrt{2\pi}n^n e^{-n}\sqrt[4]{n^2+\frac{n}{3}+\frac{1}{18}-\frac{2}{405n}-\frac{31}{9720n^2}}.$$

Our estimations give much more accurate values for the factorial function than some previously published strong formulas. We also derive new sequences converging to Euler-Mascheroni constant γ very quickly.

1. Introduction

The classical Stirling's approximation formula

$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$

is used to approximate large factorials and has many applications in science and mathematics. For example it has applications in statistics, statistical physics and number theory. In consequence, it has been deeply studied by many mathematicians because of its practical importance. A slightly better result than Stirling's formula was offered by Burnside [5] as

$$n! \approx \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+\frac{1}{2}}.$$

Bauer [4] defined the sequence (δ_n) by the relation

$$n! = n^n e^{-n}\sqrt{2\pi(n+\delta_n)}$$

and numerical computations led him to infer that

$$\lim_{n\to\infty}\delta_n = 0.166666... = 1/6. \tag{1.1}$$

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Consequently, he conjectured the approximation formula

$$n! \approx \sqrt{2\pi} n^n e^{-n} \sqrt{n+1/6}. \tag{1.2}$$

The author [2] proved (1.1) and established the following inequalities, for $n \in \mathbb{N}$

$$n^n e^{-n} \sqrt{2\pi \left(n+\frac{1}{6}\right)} < n! \leqslant n^n e^{-n} \sqrt{2\pi \left(n+\frac{e^2}{2\pi}-1\right)}.$$

The author [2] also shows numerically that the formula (1.2) has great superiority over the Stirling and Burnside's formulas. For other improvement of Stirling formula, please refer to [3, 6, 9, 11]. In this work we continue investigations of approximation of the factorial function. Our first aim is to provide the largest number α^* and the smallest number β^* such that the following inequalities hold:

$$\alpha^* n^n e^{-n} \sqrt[4]{n^2+\frac{n}{3}+\frac{1}{18}-\frac{2}{405n}-\frac{31}{9720n^2}} \leqslant n!$$
$$< \beta^* n^n e^{-n} \sqrt[4]{n^2+\frac{n}{3}+\frac{1}{18}-\frac{2}{405n}-\frac{31}{9720n^2}}.$$

The second aim of this paper is to improve (1.2). In order to fulfill it we introduce a new approximations family of the form

$$n! \approx \sqrt{2\pi} n^n e^{-n} \sqrt{n+\frac{1}{6}+\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4}} := u_n(a,b,c,d), \tag{1.3}$$

depending on four parameters $a, b, c,$ and$d$. We note that $u_n(0,0,0,0)$ is the formula (1.2). Throughout, α, β, γ, and ν denote the following real numbers.

$$\alpha = \frac{1}{72},\ \beta = -\frac{31}{6480},\ \delta = -\frac{139}{155520},\ \text{and } \nu = \frac{9871}{6531840}. \tag{1.4}$$

We prove in our Theorem 2.4 that the best approximation of the form (1.3) is $n! \approx u_n(\alpha,\beta,\delta,\nu)$, where u_n is as given in (1.3). The gamma function

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du \ (z>0)$$

and the factorials are related with $\Gamma(n+1) = n!$. The logarithmic derivative of the gamma function is called the digamma (or psi) function and denoted by ψ. The digamma function and the n^{th} harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are related with $\psi(n+1) = H_n - \gamma$, where $\gamma = lim_{n\to\infty}(H_n - \log n) = 0.57721566...$ is Euler-Mascheroni constant.

In order to prove our main results we need the following elementary but very useful lemma, which was proved in [7]. The algebraic and numerical computations have been carried out with the computer software *Mathematica 5*.

LEMMA 1.1. *If* $(\omega_n)_{n\geqslant 1}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = c \in \mathbb{R},$$

with $k > 1$, *then there exists the limit*

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{c}{k-1}.$$

It is clear from this lemma that the speed of convergence of the sequence (ω_n) is as higher as the value of k is greater.

2. Main results

We are in a position to prove our main results.

THEOREM 2.1. *Let* n *be a natural number. Then, we have*

$$\alpha^* n^n e^{-n}\sqrt[4]{n^2+\frac{n}{3}+\frac{1}{18}-\frac{2}{405n}-\frac{31}{9720n^2}} \leqslant n! < \beta^* n^n e^{-n}\sqrt[4]{n^2+\frac{n}{3}+\frac{1}{18}-\frac{2}{405n}-\frac{31}{9720n^2}}, \tag{2.1}$$

where $\alpha^* = \sqrt{2\pi} = 2.5066282...$, *and* $\beta^* = e\left(\frac{9720}{13421}\right)^{\frac{1}{4}} = 2.5155926...$ *are the best possible constants.*

Proof. We define, for $x \geqslant 1$,

$$G(x) = \log(\Gamma(x+1)) - x\log x + x - \frac{1}{2}\log(2\pi) - \frac{1}{4}\log\left(x^2+\frac{x}{3}+\frac{1}{18}-\frac{2}{405x}-\frac{31}{9720x^2}\right). \tag{2.2}$$

Differentiation yields

$$G'(x) = \psi(x+1) - \log x - \frac{31+24x+1620x^3+9720x^4}{-62x-96x^2+1080x^3+6480x^4+19440x^5} \tag{2.3}$$

and

$$G''(x+1) - G''(x) = \frac{P(x)}{Q(x)}, \tag{2.4}$$

where

$$P(x)=\frac{173098434601}{69735688020000}+\frac{158605714067x}{3874204890000}+\frac{148365177961x^2}{860934420000}$$
$$-\frac{1172334638183x^3}{697356880200}-\frac{1703781670013x^4}{58113073350}-\frac{16583056275098x^5}{80712601875}$$
$$-\frac{6188823795061x^6}{7174453500}-\frac{7130716114694x^7}{2989355625}-\frac{60012295714x^8}{13286025}$$
$$-\frac{43928339036x^9}{7381125}-\frac{4470988726x^{10}}{820125}-\frac{111699032x^{11}}{32805}$$
$$-\frac{226826144x^{12}}{164025}-\frac{17997056x^{13}}{54675}-\frac{8464x^{14}}{243}, \tag{2.5}$$

and

$$Q(x)=4x^4(x+1)^2\left(-\frac{31}{9720}-\frac{2x}{405}+\frac{x^2}{18}+\frac{x^3}{3}+x^4\right)^2$$
$$\times\left(-\frac{31}{9720}-\frac{2(x+1)}{405}+\frac{(x+1)^2}{18}+\frac{(x+1)^3}{3}+(x+1)^4\right)^2,$$

where we have used the functional relation $\psi'(x+1)-\psi'(x)=-1/x^2$. Using the asymptotic formula [1, p.259]

$$\psi(x)\sim\log x-\frac{1}{2x}-\frac{1}{12x^2}+....,$$

we get

$$\lim_{x\to\infty}(\psi(x)-\log x)=0. \tag{2.6}$$

Thus, using Stirling's formula and (2.6), respectively. we obtain the limit relations

$$\lim_{x\to\infty}G(x)=\lim_{x\to\infty}G'(x)=0. \tag{2.7}$$

From (2.5), it follows that all the coefficients of $P(x+1)$ are negative for all $x\geqslant 1$, so $P(x)<0$ for every $x\in[1,\infty)$. Therefore it results from (2.4) that $G''(x+1)-G''(x)<0$ for all $x\in[1,\infty)$. By mathematical induction, we find for every $n\in\mathbb{N}$

$$G''(x)>G''(x+1)>G''(x+2)>...>G''(x+n)>\lim_{n\to\infty}G''(x+n)=0,$$

that is, G' is strictly increasing on $[1,\infty)$. Taking into account (2.7), we conclude that G is strictly decreasing on $[1,\infty)$. Consequently, we have for any $n\in\mathbb{N}$

$$0=\lim_{n\to\infty}G(n)<G(n)\leqslant G(1),$$

which is equivalent with (2.1).

Since G' is strictly increasing on $[1,\infty)$, we conclude the following from (2.3) and the relation $1-\gamma-\frac{11395}{26842}=G'(1)\leqslant G'(n)<\lim_{n\to\infty}G'(n)=0$:

COROLLARY 2.2. *For all natural number n, we have*

$$a^*+\log n+\frac{31+24n+1620n^3+9720n^4}{-62n-96n^2+1080n^3+6480n^4+10440n^5}\leqslant H_n$$
$$b^*+\log n+\frac{31+24n+1620n^3+9720n^4}{-62n-96n^2+1080n^3+6480n^4+10440n^5},$$

where the constants $a^*=\frac{15447}{26842}=0.57549...$, and $b^*=\gamma=0.57721566...$ are the best possible.

THEOREM 2.3. *For $n\in\mathbb{N}$, we define*

$$\omega_n=G(n)=\log n!-\frac{1}{2}\log 2\pi-n\log n+n$$
$$-\frac{1}{4}\log\left(n^2+\frac{n}{3}+\frac{1}{18}-\frac{2}{405n}-\frac{31}{9720n^2}\right).$$

Then we have

$$\lim_{n\to\infty}n^5\omega_n=\frac{1058}{25515}. \tag{2.8}$$

Proof. Applications of l'Hospital rule yield

$$\begin{aligned}\lim_{n\to\infty}n^6(\omega_n-\omega_{n+1})&=\lim_{x\to\infty}x^6(G(x)-G(x+1))\\&=\lim_{x\to\infty}\frac{G(x)-G(x+1)}{1/x^6}\\&=\lim_{x\to\infty}\frac{G'(x)-G'(x+1)}{-6/x^7}\\&=-\frac{1}{42}\lim_{x\to\infty}[x^8(G''(x+1)-G''(x))]=\frac{1058}{5103},\end{aligned}$$

where G is as given in (2.2). So by Lemma 1.1 we get

$$\lim_{n\to\infty}n^5\omega_n=\frac{1058}{5\cdot 5103}=\frac{1058}{25515}.$$

This completes the proof.

THEOREM 2.4. *For $n\in\mathbb{N}$, let us define θ_n by*

$$\theta_n(a,b,c,d)=\log(\Gamma(n+1))-\frac{1}{2}\log 2\pi-n\log n+n$$
$$-\frac{1}{2}\log\left(n+\frac{1}{6}+\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4}\right), \tag{2.9}$$

where a,b,c and d are real parameters. Then,

(i) If $a\neq 1/72$, then the speed of convergence of the sequence θ_n is n^{-2}.

(ii) If $a = 1/72$ and $b \neq -31/6480$, then the speed of convergence of the sequence (θ_n) is n^{-3}.

(iii) If $a = 1/72$ and $b = -31/6480$, $c \neq -\frac{139}{155520}$ then the speed of convergence of the sequence (θ_n) is n^{-4}.

(iv) If $a = 1/72$, $b = -31/6480$, $c = -\frac{139}{155520}$, and $d \neq \frac{9871}{6531840}$ then the speed of convergence of the sequence (θ_n) is n^{-5}.

(v) If $a = 1/72$, $b = -31/6480$, $c = -\frac{139}{155520}$, and $d = \frac{9871}{6531840}$ then the speed of convergence of the sequence (θ_n) is n^{-6}.

Proof. We define for $x \geqslant 1$

$$\begin{aligned} F(x) &= \log(\Gamma(x+1)) - x\log x + x - \frac{1}{2}\log(2\pi) \\ &\quad - \frac{1}{2}\log\left(x + \frac{1}{6} + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{x^4}\right). \end{aligned} \tag{2.10}$$

Differentiation gives

$$F'(x) = \psi(x+1) - \log x + \frac{12d + 9cx + 6bx^2 + 3ax^3 - 3x^5}{6dx + 6cx^2 + 6bx^3 + 6ax^4 + x^5 + 6x^6} \tag{2.11}$$

and

$$F''(x) = \psi'(x+1) + \frac{p(x)}{q(x)}, \tag{2.12}$$

where

$$\begin{aligned} p(x) &= -72d^2\mathbf{x} + (-144cd - 36d^2)\mathbf{x}^2 + (-54c^2 - 180bd - 72cd)\mathbf{x}^3 \\ &\quad + (-180bc - 36c^2 - 252ad - 72bd)\mathbf{x}^4 \\ &\quad + (-36b^2 - 144ac - 72bc - 60d - 72ad)\mathbf{x}^5 \\ &\quad + (-72ab - 36b^2 - 36c - 72ac - 516d)\mathbf{x}^6 \\ &\quad + (-18a^2 - 18b - 72ab - 336c - 72d)\mathbf{x}^7 \\ &\quad + (-6a - 36a^2 - 192b - 72c)\mathbf{x}^8 + (-84a - 72b)\mathbf{x}^9 \\ &\quad + (-1 - 72a)\mathbf{x}^{10} + 6\mathbf{x}^{11} - 36\mathbf{x}^{12}, \end{aligned} \tag{2.13}$$

and

$$q(x) = 36.x^3(d + cx + bx^2 + ax^3 + x^4/6 + x^5)^2. \tag{2.14}$$

Hence, we obtain

$$F''(x+1) - F''(x) = \frac{f_1(x)}{f_2(x)}, \tag{2.15}$$

where

$$\begin{aligned} f_1(x) &= \mu_1(a)\mathbf{x}^{23} + \mu_2(a,b)\mathbf{x}^{22} + \mu_3(a,b,c)\mathbf{x}^{21} + \mu_4(a,b,c,d)\mathbf{x}^{20} \\ &\quad + \mu_5(a,b,c,d)\mathbf{x}^{19} + \ldots, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\mu_1(a) &= -216 + 15552a, \\
\mu_2(a,b) &= -2568 + 198288a + 38880b, \\
\mu_3(a,b,c) &= -13943 + 1173744a + 23328a^2 + 479520b + 77760c, \\
\mu_4(a,b,c,d) &= -45661 + 4277706a + 270216a^2 + 2754756b \\
&\quad + 81648ab + 923400c + 136080d, \\
\mu_5(a,b,c,d) &= -100265 + 10730322a + 1442232a^2 + 10368a^3 + 9783060b \\
&\quad + 911088ab + 46656b^2 + 5131944c + 155520ac + 1551312d, \quad (2.17)
\end{aligned}$$

and

$$f_2(x) = (x+1)^2 q(x) q(x+1),$$

with q is as given in (2.14). Now let $a \neq 1/72$. Using Stirling's formula and (2.6) we find $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} F'(x) = 0$. Applying l'Hospital rule and (2.15) we get

$$\begin{aligned}
\lim_{n\to\infty} n^3(\theta_n - \theta_{n+1}) &= \lim_{x\to\infty} x^3(F(x) - F(x+1)) \\
&= \lim_{x\to\infty} \frac{F(x) - F(x+1)}{1/x^3} \\
&= \lim_{x\to\infty} \frac{F'(x) - F'(x+1)}{-3/x^4} \\
&= -\frac{1}{12} \lim_{x\to\infty} [x^5(F''(x+1) - F''(x))] \\
&= -\frac{\mu_1(\alpha)}{12.1296} = -\frac{1-72a}{12} \neq 0.
\end{aligned}$$

Let $a = 1/72$ and $b \neq -31/6480$. Then, $\mu_1(a) = 0$, so, by the same way we get

$$\begin{aligned}
\lim_{n\to\infty} n^4(\theta_n - \theta_{n+1}) &= \lim_{x\to\infty} x^4(F(x) - F(x+1)) \\
&= \lim_{x\to\infty} \frac{F(x) - F(x+1)}{1/x^4} \\
&= \lim_{x\to\infty} \frac{F'(x) - F'(x+1)}{-4/x^5} \\
&= -\frac{1}{20} \lim_{x\to\infty} [x^6(F''(x+1) - F''(x))] \\
&= -\frac{1}{20} \frac{\mu_2(\alpha,\beta)}{20.1296} = -\frac{31+6480b}{4320} \neq 0.
\end{aligned}$$

Let $a=1/72$, $b=-31/6480$, and $c\neq -139/155520$, $\mu_1(a)=\mu_2(a,b)=0$, thus we find that

$$\begin{aligned}\lim_{n\to\infty} n^5(\theta_n-\theta_{n+1}) &= \lim_{x\to\infty} x^5(F(x)-F(x+1))\\ &= \lim_{x\to\infty}\frac{F(x)-F(x+1)}{1/x^5}\\ &= \lim_{x\to\infty}\frac{F'(x)-F'(x+1)}{-5/x^6}\\ &= -\frac{1}{20}\lim_{x\to\infty}[x^7(F''(x+1)-F''(x))]\\ &= -\frac{1}{30}\frac{\mu_3(\alpha,\beta,\delta)}{1296} = -\frac{139+155520c}{77760}\neq 0,\end{aligned}$$

for $a=\frac{1}{72}$, $b=-\frac{31}{6480}$, $c=-\frac{139}{155520}$, and $d\neq -\frac{9871}{6531840}$ we find that

$$\begin{aligned}\lim_{n\to\infty} n^6(\theta_n-\theta_{n+1}) &= \lim_{x\to\infty} x^6(F(x)-F(x+1))\\ &= \lim_{x\to\infty}\frac{F(x)-F(x+1)}{1/x^6}\\ &= \lim_{x\to\infty}\frac{F'(x)-F'(x+1)}{-6/x^7}\\ &= -\frac{1}{20}\lim_{x\to\infty}[x^8(F''(x+1)-F''(x))]\\ &= -\frac{1}{42}\frac{\mu_4(\alpha,\beta,\delta,\nu)}{1296} = -\frac{9871+6531840d}{2612736}\neq 0.\end{aligned}$$

And, finally for $a=1/72$, $b=-31/6480$, $c=-139/155520$, and $d=-\frac{9871}{6531840}$ we obtain

$$\begin{aligned}\lim_{n\to\infty} n^7(\theta_n-\theta_{n+1}) &= \lim_{x\to\infty} x^7(F(x)-F(x+1))\\ &= \lim_{x\to\infty}\frac{F(x)-F(x+1)}{1/x^7}\\ &= \lim_{x\to\infty}\frac{F'(x)-F'(x+1)}{-7/x^8}\\ &= -\frac{1}{56}\lim_{x\to\infty}[x^9(F''(x+1)-F''(x))]\\ &= -\frac{1}{56}\frac{\mu_5(\alpha,\beta,\delta,\nu)}{1296} = -\frac{217798933}{263363788800}. \qquad (2.18)\end{aligned}$$

Here $\alpha, \beta, \delta, \nu$, and μ_i are as given in (1.4) and (2.17), respectively. Now the assertions of (i)-(v) follow from Lemma1.1.

3. Concluding remarks

REMARK 3.1. As mentioned in the first section of this paper, the formula (1.2) is better than Stirling and Burnside formulas. Other known formulas much more stronger than (1.2) are:

$$n! \approx \sqrt{\pi} n^n e^{-n} \left(8n^3+4n^2+n+\frac{1}{30}\right)^{\frac{1}{6}} = \alpha_n \quad \text{(S. Ramanujan [10])},$$

$$n \approx \sqrt{2\pi} e^{-n-a}(n+a)^{n+b}(n+c)^{\frac{1}{2}-b} = \beta_n \qquad \text{(C. Mortici [8])},$$

where $a = 1.570854265...$, $b = 1.383200696...$, and $c = 0.968841875...$, and

$$n! \approx \sqrt{2\pi} n^n e^{-n}\sqrt{n+1/2}\exp(-\frac{1}{6(n+3/8)}) = \gamma_n, \qquad \text{(N. Batir [2])}$$

The following table shows that our new approximation formulas

$$n! \approx \sqrt{2\pi} n^n e^{-n}\sqrt{n+\frac{1}{6}+\frac{1}{72n}-\frac{31}{6480n^2}-\frac{139}{155520n^3}+\frac{9871}{6531840n^4}} := a_n$$

and

$$n! \approx \sqrt{2\pi} n^n e^{-n}\sqrt[4]{n^2+\frac{n}{3}+\frac{1}{18}-\frac{2}{405n}-\frac{31}{9720n^2}} := b_n.$$

are stronger than all previous formulas for $n \geqslant 2$. Also, it can be shown numerically that all approximations $n! \approx u_n(\alpha,0,0,0)$, $n! \approx u_n(\alpha,\beta,0,0)$, and $n! \approx u_n(\alpha,\beta,\delta,0)$ are better than all of them. Here u_n, and $\alpha,\beta,\delta,$ and ν are as in (1.3) and (1.4), respectively.

n	$\lvert\alpha_n - n!\rvert$	$\lvert\beta_n - n!\rvert$	$\lvert\gamma_n - n!\rvert$	$\lvert a_n - n!\rvert$	$\lvert b_n - n!\rvert$
1	0.000283346	0.0004	0.0000731	0.00016	0.000402
2	0.0000661	0.000111	0.000034	0.00000298	0.000034
5	0.000147066	0.0003	0.00015	0.00000017	0.000024
10	0.311613	0.9493	0.4417	0.000283224	0.02346
25	3.6384×10^{16}	2.0959×10^{17}	6.380×10^{16}	7.2219×10^{12}	1.03×10^{15}
50	4.5524×10^{54}	4.6899×10^{55}	8.652×10^{54}	2.3968×10^{50}	6.311×10^{52}

REMARK 3.2. First, we recall that $\psi(n+1) = H_n - \gamma$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the n^{th} harmonic number and $\gamma = 0.57721566...$ is Euler-Mascheroni constant. From (2.18) we conclude that the sequence (σ_n) defined by

$$\sigma_n = H_n - \log n - \frac{-19742+8757n+31248n^2-45360n^3+3265920n^5}{n(9871-5838n-31248n^2+90720n^3+1088640n^4+6531840n^5)}, \quad (n \in \mathbb{N})$$

converges to γ as n^{-7} since we have from (2.18)

$$\lim_{n\to\infty} n^8(\sigma_n - \sigma_{n+1}) = \lim_{x\to\infty} x^8(F'(x) - F'(x+1)) = \frac{7.217798933}{263363788800},$$

which implies by Lemma1.1

$$\lim_{n\to\infty} n^7(\sigma_n - \gamma) = \frac{217798933}{263363788800}.$$

Numerically, we have $|\sigma_1 - \gamma| = 0.00102102\ldots$, $|\sigma_2 - \gamma| = 5.54131\times 10^{-6}$, $|\sigma_5 - \gamma| = 4.14287\times 10^{-10}$, $|\sigma_{10} - \gamma| = 4.10054\times 10^{-11}$, and $|\sigma_{15} - \gamma| = 1.33227\times 10^{-15}$.

REMARK 3.3. The coefficients $\frac{1}{3}$, $\frac{1}{18}$, $-\frac{2}{405}$, and $-\frac{31}{9720}$ in the function from (2.1)

$$h(n) = \sqrt[4]{n^2 + \frac{n}{3} + \frac{1}{18} - \frac{2}{405n} - \frac{31}{9720n^2}}$$

have been obtained by the same method used in the proof of Theorem 2.4. So they are the best scalers.

Added in proofs. The author is informed that the part of the result of this paper is obtained in a paper by C. Mortici: *Sharp inequalities related to Gosper's formula*, C. R. Acad. Sci. Paris, Ser. I., 348 (2010), 137–140.

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