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THE EQUIVALENCE AMONG THE MODIFIED MANN–ISHIKAWA AND NOOR ITERATIONS FOR UNIFORMLY L–LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract. In this paper, the equivalence of the convergence among Mann-Ishikawa and Noor iterations is obtained for uniformly L-Lipschitzian mappings in real Banach spaces. Our results extend and improve the corresponding results in Chang [3] and Ofoedu [4].

1. Introduction

Let E be an arbitrary real Banach space and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequal we shall denote single-valued normalized duality mapping by j.

Let D be a nonempty closed convex subset of E and $T : D \to D$ be a map. The mapping T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leqslant L\|x - y\|$$

for any $x, y \in D$ and $\forall n \geqslant 1$. The mapping T is said to be asymptotically pseudo-contractive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ and for any $x, y \in D$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leqslant k_n\|x - y\|^2, \quad \forall n \geqslant 1.$$

The concepts of asymptotically pseudo-contractive mapping and asymptotically nonexpansive mapping were introduced by Goebel and Kirk [1], and Schu [2], respectively. Recently, Ofoedu [4] has obtained that the strong convergence theorem for uniformly Lipschitzian asymptotically pseudo-contractive mapping in real Banach space and the result is as follows.

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THEOREM 1. (Ofoedu [4,Theorem 3.2]) *Let E be a real Banach space. Let K be a nonempty closed and convex subset of E, $T : K \to K$ a uniformly L-Lipschitz asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geqslant 0} \subset [1,+\infty)$, $\lim_{n\to\infty} k_n = 1$. Let $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geqslant 0} \subset [0,1]$ be such that $\sum_{n\geqslant 0} \alpha_n = \infty$, $\sum_{n\geqslant 0} \alpha_n^2 < \infty$ and $\sum_{n\geqslant 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geqslant 0}$ be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, n \geqslant 0.$$

Suppose there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$, $\Phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x-x^*)\rangle \leqslant k_n\|x-x^*\|^2 - \Phi(\|x-x^*\|), \quad \forall x \in K.$$

Then, $\{x_n\}_{n\geqslant 0}$ converges strongly to $x^* \in F(T)$.

Currently, Chang [3] has proved the following theorem.

THEOREM 2. (Chang [3,Theorem 2.1]) *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$, $i = 1,2$ be two uniformly L_i-Lipschitzian mappings with $F(T_1)\bigcap F(T_2) \neq \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in K and x^* be a point in $F(T_1)\bigcap F(T_2)$. Let $\{k_n\} \subset [1,+\infty)$ be a sequence with $\lim_{n\to\infty} k_n = 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n\geqslant 0} \alpha_n = \infty$;
(ii) $\sum_{n\geqslant 0} \alpha_n^2 < \infty$;
(iii) $\sum_{n\geqslant 0} \beta_n < \infty$;
(iv) $\sum_{n\geqslant 0} \alpha_n(k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + a_n T_1^n y_n, y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n.$$

If there exists a strict increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle T_i^n x - x^*, j(x-x^*)\rangle \leqslant k_n\|x-x^*\|^2 - \Phi(\|x-x^*\|)$$

for all $j(x-x^*) \in J(x-x^*)$ and $x \in K, i = 1,2$, then $\{x_n\}$ converges strongly to x^*.

In the paper, the author discusses the equivalence of convergence results of the modified Mann-Ishikawa, Noor iterative sequences for uniformly L-Lipschitzian mappings in real Banach spaces. In order to obtain the main results, the following iterations and Lemmas are given.

The modified Mann-Ishikawa, Noor iterations are defined as follows:

$$\begin{cases} \forall u_1 \in D \\ u_{n+1} = (1-a_n)u_n + a_n T^n u_n, n \geqslant 1, \end{cases} \tag{1.1}$$

is called the modified Mann iteration; and

$$\begin{cases} \forall v_1 \in D \\ v_{n+1} = (1-a_n)v_n + a_n T^n w_n, \\ w_n = (1-b_n)v_n + b_n T^n v_n, n \geqslant 1, \end{cases} \tag{1.2}$$

is called the modified Ishikawa iteration; and

$$\begin{cases} \forall x_1 \in D \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n, \\ y_n = (1-b_n)x_n + b_n T^n z_n, \\ z_n = (1-c_n)x_n + c_n T^n x_n, n \geqslant 1, \end{cases} \tag{1.3}$$

is called the modified Noor iteration [6], where the sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are three real sequences in $[0,1]$ satisfying some certain conditions.

LEMMA 1. ([3]) *Let E be a real Banach space and $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle,$$

for all $x,y \in E$ and each $j(x+y) \in J(x+y)$.

LEMMA 2. ([5]) *Let $\{\rho_n\}_{n=1}^{\infty}$ be a nonnegative real numbers sequence satisfying the inequality*

$$\rho_{n+1} \leqslant (1-\theta_n)\rho_n + o(\theta_n),$$

where $\theta_n \in (0,1)$ with $\sum_{n=1}^{\infty} \theta_n = \infty$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Main Results

THEOREM 2.1. *Let E be a real Banach space, D be a nonempty closed and convex subset of E and $T : D \to D$ be a uniformly L-Lipschitzian mapping. Let $\{k_n\}_{n\geqslant 1} \subset [1,+\infty)$ be a sequence with $\lim_{n\to\infty} k_n = 1$. Let $q \in F(T) = \{x \in K : Tx = x\}$. The sequence $\{u_n\}_{n=1}^{\infty}$ is defined by (1.1), with sequences $\{a_n\}_{n=1}^{\infty}$ satisfying: $a_n \to 0$ as $n \to \infty$; $\sum_{n=1}^{\infty} a_n = \infty$. Suppose there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that*

$$\langle T^n x - q, j(x-q)\rangle \leqslant k_n\|x-q\|^2 - \Phi(\|x-q\|), \quad \forall x \in D, \forall n \geqslant 1,$$

then $\{x_n\}_{n\geqslant 1}$ converges strongly to q.

Proof. Applying (1.1) and Lemma 1, we have

$$\begin{aligned} \|u_{n+1}-q\|^2 &\leqslant (1-a_n)^2\|u_n-q\|^2 + 2a_n\langle T^n u_n - T^n q, j(u_{n+1}-q)\rangle \\ &= (1-a_n)^2\|u_n-q\|^2 + 2a_n\langle T^n u_{n+1} - T^n q, j(u_{n+1}-q)\rangle \\ &\quad + 2a_n\langle T^n u_n - T^n u_{n+1}, j(u_{n+1}-q)\rangle \\ &\leqslant (1-a_n)^2\|u_n-q\|^2 + 2a_n(k_n\|u_{n+1}-q\|^2 - \Phi(\|u_{n+1}-q\|)) \\ &\quad + 2a_n L\|u_n-u_{n+1}\| \cdot \|u_{n+1}-q\|, \end{aligned} \tag{2.1}$$

and

$$\|u_n - u_{n+1}\| = \|a_n(u_n - T^n u_n)\| \leqslant a_n(1+L)\|u_n-q\|. \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain

$$\begin{aligned}\|u_{n+1}-q\|^2 &\leqslant (1-a_n)^2\|u_n-q\|^2+2a_nk_n\|u_{n+1}-q\|^2-2a_n\Phi(\|u_{n+1}-q\|)\\ &\quad +2a_nLa_n(1+L)\|u_n-q\|\cdot\|u_{n+1}-q\|\\ &\leqslant (1-a_n)^2\|u_n-q\|^2+2a_nk_n\|u_{n+1}-q\|^2-2a_n\Phi(\|u_{n+1}-q\|)\\ &\quad +a_n^2L(1+L)(\|u_n-q\|^2+\|u_{n+1}-q\|^2). \end{aligned} \tag{2.3}$$

Since $\lim_{n\to\infty} a_nk_n = \lim_{n\to\infty} a_n^2L(1+L) = 0$, there exists a natural number N_0 such that

$$\frac{1}{2} < 1-2a_nk_n-a_n^2L(1+L) < 1$$

for all $n > N_0$. Then (2.3) implies that

$$\begin{aligned}\|u_{n+1}-q\|^2 &\leqslant \frac{(1-a_n)^2+a_n^2L(1+L)}{1-2a_nk_n-a_n^2L(1+L)}\|u_n-q\|^2\\ &\qquad -\frac{2a_n}{1-2a_nk_n-a_n^2L(1+L)}\Phi(\|u_{n+1}-q\|)\\ &\leqslant \|u_n-q\|^2+2a_n\frac{(k_n-1)+a_n+a_nL(1+L)}{1-2a_nk_n-a_n^2L(1+L)}\|u_n-q\|^2\\ &\qquad -\frac{2a_n}{1-2a_nk_n-a_n^2L(1+L)}\Phi(\|u_{n+1}-q\|)\\ &\leqslant \|u_n-q\|^2+4a_n\delta_n\|u_n-q\|^2-2a_n\Phi(\|u_{n+1}-q\|), \end{aligned} \tag{2.4}$$

where $\delta_n = (k_n-1)+a_n+a_nL(1+L) \to 0$ as $n\to\infty$.

Set $\inf_{n\geqslant N}\frac{\Phi(\|u_{n+1}-q\|)}{1+\|u_{n+1}-q\|^2} = \lambda_0$. Then $\lambda_0 = 0$. If it is not the case, we assume that $\lambda_0 > 0$. Let $0<\gamma_0<\min\{1,\lambda_0\}$, then $\frac{\Phi(\|u_{n+1}-q\|)}{1+\|u_{n+1}-q\|^2}\geqslant\gamma_0$, i.e., $\Phi(\|u_{n+1}-q\|)\geqslant\gamma_0+\gamma_0\|u_{n+1}-q\|^2\geqslant\gamma_0\|u_{n+1}-q\|^2$. Thus, we obtain that from (2.4)

$$\|u_{n+1}-q\|^2 \leqslant \frac{1+4a_n\delta_n}{1+2a_n\gamma_0}\|u_n-q\|^2 = (1-a_n\frac{2\gamma_0-4\delta_n}{1+2a_n\gamma_0})\|x_n-u_n\|^2. \tag{2.5}$$

By $a_n,\delta_n\to 0$ as $n\to\infty$, we choose $N_1>N_0$ such that $\frac{2\gamma_0-4\delta_n}{1+2\alpha_n\gamma_0}>\gamma_0$ for all $n\geqslant N_1$. It follows from (2.5) that

$$\begin{aligned}\|u_{n+1}-q\|^2 &\leqslant (1-a_n\gamma_0)\|u_n-q\|^2\\ &\leqslant (1-a_n\gamma_0)(1-a_{n-1}\gamma_0)\|u_{n-1}-q\|^2\\ &\leqslant (1-a_n\gamma_0)(1-a_{n-1}\gamma_0)\cdots(1-a_{N_1}\gamma_0)\|u_{N_1}-q\|^2\\ &\leqslant \|u_{N_1}-q\|^2\exp(-\sum_{i=N_1}^{n}a_i\gamma_0) \end{aligned} \tag{2.6}$$

for all $n>N_1$. Hence, $\|u_{n+1}-q\|\to 0$ as $n\to\infty$ is a contradiction and so $\lambda_0=0$. Therefore there exists an infinite subsequence such that $\|u_{n_j+1}-q\|\to 0$ as $j\to\infty$.

Next we want to prove $\|u_{n_j+m}-q\| \to 0$ as $j \to \infty$ for any natural number m. Let $\forall \varepsilon \in (0,1)$, choose $n_j > N$ such that $\|u_{n_j+1}-q\| < \varepsilon$, $\delta_{n_j+1} < \frac{\Phi(\varepsilon)}{4(1+\varepsilon^2)}$. First we want to prove $\|u_{n_j+2}-q\| < \varepsilon$. Suppose it is not this case, then $\|u_{n_j+2}-q\| \geqslant \varepsilon$. It implies $\Phi(\|u_{n_j+2}-q\|) \geqslant \Phi(\varepsilon)$. Using the formula (2.4), we now obtain the following estimates:

$$\begin{aligned} \|u_{n_j+2}-q\|^2 &\leqslant \|u_{n_j+1}-q\|^2 + 4a_{n_j+1}\delta_{n_j+1}\|u_{n_j+1}-q\|^2 \\ &\qquad - 2a_{n_j+1}\Phi(\|u_{n_j+2}-q\|) \\ &< \varepsilon^2 - a_{n_j+1}\Phi(\varepsilon) \leqslant \varepsilon^2 \end{aligned} \tag{2.7}$$

is a contradiction. Hence $\|u_{n_j+2}-q\| < \varepsilon$. Repeat the above course, we can easily prove that it holds for $m = k+1$. Therefore, we obtain $\|u_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

REMARK 1. Our Theorem 2.1 in the proof methods is different from Theorems of Ofoedu [4]; On the other hand, the assumptions that $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\sum_{n=1}^{\infty} a_n(k_n-1) < \infty$ in [4] are replaced by the more weaker condition $\lim_{n\to\infty} a_n = 0$.

THEOREM 2.2. *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitzian mapping. Let $\{k_n\}_{n\geqslant 1} \subset [1,+\infty)$ be a sequence with $\lim_{n\to\infty} k_n = 1$. Let $q \in F(T) = \{x \in K : Tx = x\}$. The sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ are defined by (1.1), (1.2) and (1.3), respectively, with sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ satisfying: (i) $a_n, b_n, c_n \to 0$ as $n \to \infty$; (ii) $\sum_{n=1}^{\infty} a_n = \infty$. If there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty), \Phi(0) = 0$ such that*

$$\langle T^n x - T^n y, j(x-y)\rangle \leqslant k_n\|x-y\|^2 - \Phi(\|x-y\|), \forall x,y \in D, \forall n \geqslant 1,$$

then the following assertions are equivalent:

(i) The modified Mann iteration (1.1) converges strongly to the fixed point q of T;

(ii) The modified Ishikawa iteration (1.2) converges strongly to the fixed point q of T;

(iii) The modified Noor iteration (1.3) converges strongly to the fixed point q of T.

Proof. We can only prove the conclusion (i)$\Leftrightarrow$ (iii). If the modified Noor iteration (1.3) converges to the fixed point q, then by putting $b_n = c_n = 0$, we can get the convergence of the modified Mann iteration (1.1). Conversely, we only need to show (i) $\Rightarrow$ (iii), i.e., $\|u_n - q\| \to 0$ as $n \to \infty \Rightarrow \|x_n - q\| \to 0$ as $n \to \infty$.

Using (1.1), (1.3) and Lemma1, we have

$$
\begin{aligned}
\|x_{n+1}-u_{n+1}\|^2 &= \|(1-a_n)(x_n-u_n)+a_n(T^ny_n-T^nu_n)\|^2 \\
&\leqslant (1-a_n)^2\|x_n-u_n\|^2+2a_n\langle T^ny_n-T^nu_n, j(x_{n+1}-u_{n+1})\rangle \\
&\leqslant (1-a_n)^2\|x_n-u_n\|^2+2a_n\langle T^ny_n-T^nx_{n+1}, j(x_{n+1}-u_{n+1})\rangle \\
&\quad +2a_n\langle T^nx_{n+1}-T^nu_{n+1}, j(x_{n+1}-u_{n+1})\rangle \\
&\quad +2a_n\langle T^nu_{n+1}-T^nu_n, j(x_{n+1}-u_{n+1})\rangle \\
&\leqslant (1-a_n)^2\|x_n-u_n\|^2+2a_nL\|y_n-x_{n+1}\|\cdot\|x_{n+1}-u_{n+1}\| \\
&\quad +2a_n(k_n\|x_{n+1}-u_{n+1}\|^2-\Phi(\|x_{n+1}-u_{n+1}\|)) \\
&\quad +2a_nL\|u_{n+1}-u_n\|\cdot\|x_{n+1}-u_{n+1}\|. \qquad (2.8)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|y_n-x_{n+1}\| &= \|a_n(x_n-T^ny_n)-b_n(x_n-T^nz_n)\| \\
&\leqslant a_n\|x_n-q\|+a_n\|T^ny_n-T^nq\|+b_n\|x_n-q\|+b_n\|T^nz_n-T^nq\| \\
&\leqslant a_n\|x_n-q\|+a_nL\|y_n-q\|+b_n\|x_n-q\|+b_nL\|z_n-q\| \\
&\leqslant a_n\|x_n-q\|+a_nL((1-b_n)\|x_n-q\|+b_n\|T^nz_n-T^nq\|) \\
&\quad +b_n\|x_n-q\|+b_nL((1-c_n)\|x_n-q\|+c_n\|T^nx_n-T^nq\|) \\
&\leqslant a_n\|x_n-q\|+a_nL(\|x_n-q\|+b_nL(\|x_n-q\|+c_nL\|x_n-q\|)) \\
&\quad +b_n\|x_n-q\|+b_nL(\|x_n-q\|+c_nL\|x_n-q\|) \\
&= A_n\|x_n-q\|, \qquad (2.9)
\end{aligned}
$$

where $A_n = a_n+b_n+a_nL+b_nL(1+c_nL)(1+a_nL)$.

Substituting (2.9) into (2.8), we obtain

$$
\begin{aligned}
\|x_{n+1}-u_{n+1}\|^2 &\leqslant (1-a_n)^2\|x_n-u_n\|^2+2a_nLA_n(\|x_n-u_n\|+\|u_n-q\|)\cdot\|x_{n+1}-u_{n+1}\| \\
&\quad +2a_n(k_n\|x_{n+1}-u_{n+1}\|^2-\Phi(\|x_{n+1}-u_{n+1}\|)) \\
&\quad +2a_nL\|u_{n+1}-u_n\|\cdot\|x_{n+1}-u_{n+1}\| \\
&\leqslant (1-a_n)^2\|x_n-u_n\|^2+a_nLA_n((\|x_n-u_n\|+\|u_n-q\|)^2+\|x_{n+1}-u_{n+1}\|^2) \\
&\quad +2a_n(k_n\|x_{n+1}-u_{n+1}\|^2-\Phi(\|x_{n+1}-u_{n+1}\|)) \\
&\quad +a_nL\|u_{n+1}-u_n\|(1+\|x_{n+1}-u_{n+1}\|^2) \\
&\leqslant (1-a_n)^2\|x_n-u_n\|^2+a_nLA_n(2\|x_n-u_n\|^2+2\|u_n-q\|^2+\|x_{n+1}-u_{n+1}\|^2) \\
&\quad +2a_n(k_n\|x_{n+1}-u_{n+1}\|^2-\Phi(\|x_{n+1}-u_{n+1}\|)) \\
&\quad +a_nL\|u_{n+1}-u_n\|(1+\|x_{n+1}-u_{n+1}\|^2). \qquad (2.10)
\end{aligned}
$$

Without loss of generality, we assume that

$$
0 < 1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\| < 1.
$$

Then (2.10) implies that

$$\begin{aligned}\|x_{n+1}-u_{n+1}\|^2 &\leqslant \frac{(1-a_n)^2+2a_nA_nL}{1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\|}\|x_n-u_n\|^2\\ &\quad+\frac{2a_nLA_n\|u_n-q\|^2+a_nL\|u_{n+1}-u_n\|}{1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\|}\\ &\quad-\frac{2a_n}{1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\|}\Phi(\|x_{n+1}-u_{n+1}\|) \end{aligned}\tag{2.11}$$

Since $2a_nk_n+a_nA_nL+a_nL\|u_{n+1}-u_n\|\to 0$ as $n\to\infty$, then there exists N such that $2a_nk_n+a_nA_nL+a_nL\|u_{n+1}-u_n\|\leqslant\frac{1}{2}$ for any $n>N$, i.e.,

$$1>1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\|\geqslant\frac{1}{2}\quad(n>N).$$

Thus, we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\|^2 &\leqslant \|x_n-u_n\|^2+2a_n\frac{(k_n-1)+a_n+3a_nA_nL+L\|u_{n+1}-u_n\|}{1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\|}\|x_n-u_n\|^2\\ &\quad+a_n\frac{2LA_n\|u_n-q\|^2+L\|u_{n+1}-u_n\|}{1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\|}\\ &\quad-\frac{2a_n}{1-2a_nk_n-a_nA_nL-a_nL\|u_{n+1}-u_n\|}\Phi(\|x_{n+1}-u_{n+1}\|)\\ &\leqslant \|x_n-u_n\|^2+4a_nB_n\|x_n-u_n\|^2+2a_nC_n-2a_n\Phi(\|x_{n+1}-u_{n+1}\|) \end{aligned}\tag{2.12}$$

where

$$\begin{aligned}B_n&=(k_n-1)+a_n+3a_nA_nL+L\|u_{n+1}-u_n\|\to 0\quad\text{as}\quad n\to\infty,\\ C_n&=2LA_n\|u_n-q\|^2+L\|u_{n+1}-u_n\|\to 0\quad\text{as}\quad n\to\infty.\end{aligned}$$

Set $\inf_{n\geqslant N}\frac{\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\|x_{n+1}-u_{n+1}\|^2}=\lambda$. Then $\lambda=0$. If it is not the case, we assume that $\lambda>0$. Let $0<\gamma<\min\{1,\lambda\}$, then $\frac{\Phi(\|x_{n+1}-u_{n+1}\|)}{1+\|x_{n+1}-u_{n+1}\|^2}\geqslant\gamma$, i.e.,

$$\Phi(\|x_{n+1}-u_{n+1}\|)\geqslant\gamma+\gamma\|x_{n+1}-u_{n+1}\|^2\geqslant\gamma\|x_{n+1}-u_{n+1}\|^2.$$

Thus

$$\begin{aligned}\|x_{n+1}-u_{n+1}\|^2&\leqslant\frac{1+4a_nB_n}{1+2a_n\gamma}\|x_n-u_n\|^2+\frac{2a_nC_n}{1+2a_n\gamma}\\ &=(1-a_n\frac{2\gamma-4B_n}{1+2a_n\gamma})\|x_n-u_n\|^2+\frac{4a_nC_n}{1+2a_n\gamma}.\end{aligned}\tag{2.13}$$

Since $a_n,B_n\to 0$ as $n\to\infty$, we choose $N_1>N$ such that $\frac{2\gamma-4B_n}{1+2\alpha_n\gamma}>\gamma$ for all $n>N_1$. It follows from (2.6) that

$$\|x_{n+1}-u_{n+1}\|^2\leqslant(1-a_n\gamma)\|x_n-u_n\|^2+\frac{4a_nC_n}{1+2a_n\gamma}$$

for all $n > N_1$. It follows from Lemma 2 that $\|x_{n+1} - u_{n+1}\| \to 0$ as $n \to \infty$, this is a contradiction and so $\lambda = 0$. Thus, there exists an infinite subsequence such that $\|x_{n_j+1} - u_{n_j+1}\| \to 0$ as $j \to \infty$. Next we want to prove that $\|x_{n_j+m} - u_{n_j+m}\| \to 0$ as $j \to \infty$ by induction. Let $\forall \varepsilon \in (0,1)$, choose $n_j > N$ such that $\|x_{n_j+1} - u_{n_j+1}\| < \varepsilon$, $B_{n_j+1} < \frac{\Phi(\varepsilon)}{8(1+\varepsilon^2)}$, $C_{n_j+1} < \frac{\Phi(\varepsilon)}{8}$. First we want to prove $\|x_{n_j+2} - u_{n_j+2}\| < \varepsilon$. Suppose it is not this case. Then $\|x_{n_j+2} - u_{n_j+2}\| \geqslant \varepsilon$, this implies $\Phi(\|x_{n_j+2} - u_{n_j+2}\|) \geqslant \Phi(\varepsilon)$. Using the formula (2.12), we now obtain the following estimates:

$$\begin{aligned} \|x_{n_j+2} - u_{n_j+2}\|^2 &\leqslant \|x_{n_j+1} - u_{n_j+1}\|^2 + 4a_{n_j+1}B_{n_j+1}\|x_{n_j+1} - u_{n_j+1}\|^2 \\ &\quad + 2a_{n_j+1}C_{n_j+1} - 2a_{n_j+1}\Phi(\|x_{n_j+2} - u_{n_j+2}\|) \\ &< \varepsilon^2 - a_{n_j+1}\Phi(\varepsilon) \leqslant \varepsilon^2 \end{aligned} \tag{2.14}$$

is a contradiction. Hence $\|x_{n_j+2} - u_{n_j+2}\| < \varepsilon$. Assume that it holds for $m = k$. Then by the argument above, we easily prove that it holds for $m = k+1$. Hence, we obtain $\|x_n - u_n\| \to 0$ as $n \to \infty$. Owing to $\|u_n - q\| \to 0$ as $n \to \infty$ and the inequality $0 \leqslant \|x_n - q\| \leqslant \|x_n - u_n\| + \|u_n - q\|$, we can get $\|x_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

THEOREM 2.3. *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitzian mapping. Let $\{k_n\}_{n\geqslant 1} \subset [1,+\infty)$ be a sequence with $\lim_{n\to\infty} k_n = 1$. Let $q \in F(T) = \{x \in K : Tx = x\}$. The sequences $\{v_n\}_{n=1}^{\infty}, \{x_n\}_{n=1}^{\infty}$ are defined by (1.2) and (1.3), respectively, with sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ satisfying: (i) $a_n, b_n, c_n \to 0$ as $n \to \infty$; (ii) $\sum_{n=1}^{\infty} a_n = \infty$. If there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty), \Phi(0) = 0$ such that*

$$\langle T^n x - q, j(x-q)\rangle \leqslant k_n\|x-q\|^2 - \Phi(\|x-q\|), \forall x \in D, \forall n \geqslant 1,$$

then the modified Ishikawa iteration (1.2) and the Noor iteration (1.3) converge strongly to the fixed point q of T.

Proof. From Theorem 2.1 and Theorem 2.2, we obtain directly the conclusion of Theorem 2.3. □

THEOREM 2.4. *Let E be a real Banach space, D be a nonempty closed and convex subset of E and $T_i : D \to D$ ($i = 1,2,3$) be three uniformly L-Lipschitzian mappings with $q \in F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{k_n\}_{n\geqslant 1} \subset [1,+\infty)$ be a sequence with $\lim_{n\to\infty} k_n = 1$. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be three sequences in $[0,1]$ satisfying the following conditions: (i) $a_n, b_n, c_n \to 0$ as $n \to \infty$; (ii) $\sum_{n=1}^{\infty} a_n = \infty$. For any given $u_1 \in D, v_1 \in D, x_1 \in D$, define the sequence $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}, \{x_n\}_{n=1}^{\infty} \subset D$ by the iterative schemes [7], individually.*

$$u_{n+1} = (1-a_n)u_n + a_n T_1^n u_n, n \geqslant 1; \tag{2.15}$$

$$\begin{cases} w_n = (1-b_n)v_n + b_nT_2^nv_n, n \geqslant 1, \\ v_{n+1} = (1-a_n)v_n + a_nT_1^nw_n, n \geqslant 1; \end{cases} \tag{2.16}$$

$$\begin{cases} z_n = (1-c_n)x_n + c_nT_3^nx_n, n \geqslant 1, \\ y_n = (1-b_n)x_n + b_nT_2^nz_n, n \geqslant 1, \\ x_{n+1} = (1-a_n)x_n + a_nT_1^ny_n, n \geqslant 1. \end{cases} \tag{2.17}$$

If there exists a strict increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ *with* $\Phi(0) = 0$*, and sequence* $\{k_n\}_{n\geqslant 1} \subset [1,+\infty)$*,* $\lim\limits_{n\to\infty} k_n = 1$ *such that*

$$\langle T_i^nx - T_i^ny, j(x-y)\rangle \leqslant k_n\|x-y\|^2 - \Phi(\|x-y\|), \forall x,y \in D, \forall n \geqslant 1, i = 1,2,3.$$

Then the following three assertions are equivalent:

(i) (2.15) converges strongly to the fixed point q *of* T_1*;*

(ii)(2.16) converges strongly to the fixed point q *of* $T_1 \cap T_2$*;*

(iii)(2.17) converges strongly to the fixed point q *of* $T_1 \cap T_2 \cap T_3$*.*

COROLLARY 2.5. *Let* E *be a real Banach space,* D *be a nonempty closed and convex subset of* E *and* $T_i : D \to D$ *(*$i = 1,2$*) be three uniformly L-Lipschitzian mappings with* $q \in F(T_1) \cap F(T_2) \neq \emptyset$*. Let* $\{k_n\}_{n\geqslant 1} \subset [1,+\infty)$ *be a sequence with* $\lim\limits_{n\to\infty} k_n = 1$*. Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences in* $[0,1]$ *satisfying the following conditions: (i)* $a_n, b_n \to 0$ *as* $n \to \infty$*; (ii)* $\sum\limits_{n=1}^{\infty} a_n = \infty$*. For any given* $v_1 \in D$*, define the sequence* $\{v_n\}_{n=1}^{\infty} \subset D$ *by the iterative schemes [7],*

$$\begin{cases} w_n = (1-b_n)v_n + b_nT_2^nv_n, n \geqslant 1, \\ v_{n+1} = (1-a_n)v_n + a_nT_1^nw_n, n \geqslant 1. \end{cases} \tag{2.18}$$

If there exists a strict increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ *with* $\Phi(0) = 0$*, and sequence* $\{k_n\}_{n\geqslant 1} \subset [1,+\infty)$*,* $\lim\limits_{n\to\infty} k_n = 1$ *such that*

$$\langle T_i^nx - q, j(x-q)\rangle \leqslant k_n\|x-q\|^2 - \Phi(\|x-q\|), x \in D, \forall n \geqslant 1, i = 1,2.$$

Then the (2.18) converges strongly to the fixed point q *of* $T_1 \cap T_2$*.*

REMARK 2. Corollary 2.5 reduces the conditions of Theorem 2.1 in [3], i.e., it is that from conditions $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum\limits_{n=1}^{\infty} \beta_n < \infty$, $\sum\limits_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$ to $a_n, b_n \to 0$ as $n \to \infty$. Therefore, Our results extend and improve the results of Chang [3], and also cover all results of Ofoedu [4].

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