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SOME BETTER BOUNDS ON THE VARIANCE WITH APPLICATIONS

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Abstract. We derive bounds on the variance of a finite universe. Some related inequalities for the roots of the polynomial equations and bounds for the largest and smallest eigenvalues of a square matrix with real spectrum are obtained.

1. Introduction

Let $x_1, x_2, \ldots, x_n$ denote n real numbers with arithmetic mean

$$A = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{1.1}$$

variance

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - A)^2 \tag{1.2}$$

and range

$$r = M - m, \tag{1.3}$$

where $m \leqslant x_i \leqslant M, i = 1, 2, \ldots, n$.

The well-known Popoviciu inequality says that [1]

$$S^2 \leqslant \frac{r^2}{4}. \tag{1.4}$$

The complementary Von Szokefalvi Nagy inequality asserts that [2]

$$S^2 \geqslant \frac{r^2}{2n}. \tag{1.5}$$

Several authors have worked on such inequalities, their further refinements and extensions along with a variety of alternative proofs. In particular, Bhatia and Davis [3] have proved that

$$S^2 \leqslant (M - A)(A - m). \tag{1.6}$$

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Let H be the harmonic mean of n positive real numbers x_i $(i = 1, 2, ..., n)$, defined by

$$H = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}. \tag{1.7}$$

Then, Sharma [4] shows that

$$S^2 \leqslant \frac{M(A-H)(M-A)}{M-H}, \quad H < M \tag{1.8}$$

and

$$S^2 \geqslant \frac{m(A-H)(A-m)}{H-m}, \quad H > m. \tag{1.9}$$

One of the interests in (1.8) and (1.9) is that they provide refinements of the inequality (1.6) which itself is a refinement of the Popoviciu inequality (1.4). We prove (Theorem 2.1 and Corollary 2.1, below) that the inequalities (1.8) and (1.9) yield some further refinements of the Popoviciu inequality, and also give an upper bound (Corollary 2.2, below) for the variance in terms of M and H. We obtain one more refinement of the Popoviciu inequality (Theorem 2.2, below) which involves third central moment,

$$m_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - A)^3. \tag{1.10}$$

This also gives an upper bound (Corollary 2.3, below) for the measure of skewness, defined by

$$\gamma_1 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - A}{S}\right)^3. \tag{1.11}$$

An upper bound for the Karl Pearson coefficient of dispersion $\left(V = \frac{S}{A}\right)$ is also given (Corollary 2.3, below), see also [5]. We remark that these inequalities also hold good for both discrete and continuous probability distributions.

As an application we consider nth degree polynomial equation with all its roots positive and obtain (Theorem 3.1, below) bounds for the largest and smallest roots which also gives a lower bound for the span of roots. In addition, the bounds for the largest and smallest eigenvalues of a Hermitian matrix C are obtained (Theorem 3.2, below) in terms of the traces of C, C^2 *and* C^3. Our results compare favorably with those obtained by Wolkowicz and Styan [6].

2. Main Results

THEOREM 2.1. *For* $0 < m < H < M$ *and under the above notations*

$$S^2 + \frac{1}{4}\left(A - \frac{S^2}{A-H}\right)^2 \leqslant \frac{r^2}{4}, \quad A \neq H. \tag{2.1}$$

The inequality (2.1) *gives a refinement of the Popoviciu inequality* (1.4).

Proof. From the inequality (1.8), we have

$$(A-H)M^2-\left(A^2-AH+S^2\right)M+HS^2 \geqslant 0. \tag{2.2}$$

Therefore, either

$$M \leqslant \frac{A^2-AH+S^2-\sqrt{\left(A^2-AH+S^2\right)^2-4HS^2(A-H)}}{2(A-H)}, \tag{2.3}$$

or

$$M \geqslant \frac{A^2-AH+S^2+\sqrt{\left(A^2-AH+S^2\right)^2-4HS^2(A-H)}}{2(A-H)}. \tag{2.4}$$

But, if (2.3) holds then

$$M-A \leqslant \frac{S^2-A(A-H)-\sqrt{\left(S^2-A(A-H)\right)^2+4S^2(A-H)}}{2(A-H)}. \tag{2.5}$$

This is not possible as right hand side expression in (2.5) is negative while left hand side expression is positive. Thus, M satisfies (2.4). On using similar arguments it follows from the inequality (1.9) that

$$m \leqslant \frac{A^2-AH+S^2-\sqrt{\left(A^2-AH+S^2\right)^2-4HS^2(A-H)}}{2(A-H)}. \tag{2.6}$$

From the inequalities (2.4) and (2.6) we find that

$$M-m \geqslant \frac{\sqrt{\left(A^2-AH+S^2\right)^2-4HS^2(A-H)}}{A-H}. \tag{2.7}$$

On combining (1.3) and (2.7); the inequality (2.1) follows immediately. □

COROLLARY 2.1. *For* $0<m \leqslant x_i \leqslant M, (i=1,2,\ldots,n)$,

$$S^2 \leqslant \frac{r^2}{4}-\left(A-H-\sqrt{\frac{r^2}{4}-H(A-H)}\right)^2. \tag{2.8}$$

The inequality (2.8) *gives one more refinement of the Popoviciu inequality* (1.4).

Proof. From the inequality (2.1) we find that the variance S^2 satisfies the following quadratic inequality:

$$S^4-2(A-H)(2H-A)S^2+(A-H)^2\left(A^2-r^2\right) \leqslant 0. \tag{2.9}$$

The inequality (2.8) now follows at once from (2.9). □

COROLLARY 2.2. *For* $0 < x_i \leqslant M, (i = 1, 2, \ldots, n)$,

$$S^2 \leqslant \frac{M(M-H)}{4}. \tag{2.10}$$

Proof. It is easily seen from the inequality (1.8) that the point (A, S) in AS-plane lies on or inside the elliptical region,

$$(M-H)S^2 + M\left(A - \frac{M+H}{2}\right)^2 \leqslant \frac{M(M-H)^2}{4}. \tag{2.11}$$

The inequality (2.10) is a particular case of (2.11). □

THEOREM 2.2. *For* $S > 0$,

$$S^2 + \left(\frac{m_3}{2S^2}\right)^2 \leqslant \frac{r^2}{4}. \tag{2.12}$$

The inequality (2.12) *provides a refinement of the Popoviciu inequality* (1.4).

Proof. For $x_i \leqslant M, i = 1, 2, ..., n,$ we have

$$(x_i - \alpha)^2 (M - x_i) \geqslant 0, \tag{2.13}$$

where α is any real number. On adding these inequalities we get on simplification

$$M \geqslant A + \frac{m_3 + 2(A-\alpha)S^2}{S^2 + (A-\alpha)^2}. \tag{2.14}$$

Let

$$f(\alpha) = \frac{m_3 + 2(A-\alpha)S^2}{S^2 + (A-\alpha)^2}. \tag{2.15}$$

The derivative

$$f'(\alpha) = 2S^2 \frac{(\alpha - \alpha_1)(\alpha - \alpha_2)}{\left(S^2 + (A-\alpha)^2\right)^2} \tag{2.16}$$

vanishes at $\alpha = \alpha_1$ and $\alpha = \alpha_2$, where

$$\alpha_1 = \frac{2AS^2 + m_3 - \sqrt{m_3^2 + 4S^6}}{2S^2} \tag{2.17}$$

and

$$\alpha_2 = \frac{2AS^2 + m_3 + \sqrt{m_3^2 + 4S^6}}{2S^2}. \tag{2.18}$$

The sign of $f'(\alpha)$ changes from positive to negative while α passes through the value α_1 and therefore $\alpha = \alpha_1$ is the point of maximum of the function $f(\alpha)$. We have

$$f(\alpha_1) = \frac{\sqrt{m_3^2 + 4S^6}}{2S^2 + \frac{m_3\left(m_3 - \sqrt{m_3^2+4S^6}\right)}{2S^4}} = \frac{m_3 + \sqrt{m_3^2 + 4S^6}}{2S^2}. \tag{2.19}$$

The inequality (2.14) is valid for all real values of α and therefore also holds good when $\alpha = \alpha_1$. It then follows from (2.14), (2.15) and (2.19) that

$$M \geqslant A + \frac{m_3 + \sqrt{m_3^2 + 4S^6}}{2S^2}. \tag{2.20}$$

On using similar arguments we see that the inequality $(x_i - \beta)^2 (m - x_i) \leqslant 0,$ where $i = 1, 2, ..., n$ and β is any real number, gives us the following inequality:

$$m \leqslant A + \frac{m_3 - \sqrt{m_3^2 + 4S^6}}{2S^2}. \tag{2.21}$$

On subtracting (2.21) from (2.20) we get that

$$M - m \geqslant \frac{\sqrt{m_3^2 + 4S^6}}{S^2}. \tag{2.22}$$

The inequality (2.12) now follows immediately on simplifying (2.22). □

COROLLARY 2.3. *For* $m \leqslant x_i \leqslant M,\ (i = 1, 2, \ldots, n)$

$$\gamma_1 \leqslant \sqrt{\left(\frac{r}{S}\right)^2 - 4}. \tag{2.23}$$

Also, for $0 < x_i \leqslant M,\ (i = 1, 2, \ldots, n)$

$$V \leqslant \sqrt{\frac{M - H}{4H}}. \tag{2.24}$$

Proof. Combining (1.10), (1.11) and (2.12) the inequality (2.23) follows easily. The inequality (2.24) follows from (1.8) on dividing both sides by A^2 and then maximizing the right hand side expression. □

3. Applications

We show that some immediate consequences of the above inequalities have interesting applications in the field of theory of polynomial equations and matrix analysis.

THEOREM 3.1. *Let the roots of the nth degree monic polynomial equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x^{n-1} + a_n = 0 \tag{3.1}$$

be all real and positive. Let x_1 and x_n respectively denote the smallest and largest root of (3.1)*. Then*

$$x_n \geqslant \frac{\alpha + \sqrt{\alpha^2+\beta}}{2\gamma}, \tag{3.2}$$

$$x_1 \leqslant \frac{\alpha - \sqrt{\alpha^2+\beta}}{2\gamma} \tag{3.3}$$

and

$$x_n - x_1 \geqslant \frac{\sqrt{\alpha^2+\beta}}{\gamma}, \tag{3.4}$$

where

$$\alpha = \frac{a_1^2 - 2a_2}{n} - \frac{a_1 a_n}{a_{n-1}}, \tag{3.5}$$

$$\beta = \frac{4a_n\left(a_1a_{n-1} - n^2a_n\right)\left(2na_2 - (n-1)\,a_1^2\right)}{\left(na_{n-1}\right)^2} \tag{3.6}$$

and

$$\gamma = n\frac{a_n}{a_{n-1}} - \frac{a_1}{n}. \tag{3.7}$$

The inequality (3.4) *gives a lower bound for the span of roots of the polynomial equation* (3.1)*.*

Proof. Let A, H and S^2 respectively denote the arithmetic mean, harmonic mean and variance of the n positive real roots $x_1 \leqslant x_i \leqslant x_n$ $(i = 1, 2, \dots, n)$ of the polynomial equation (3.1). On using the relations between the roots and coefficients of a polynomial equation we find that

$$A = -\frac{a_1}{n}, \tag{3.8}$$

$$S^2 = \frac{(n-1)\,a_1^2 - 2na_2}{n^2} \tag{3.9}$$

and

$$H = -n\frac{a_n}{a_{n-1}}. \tag{3.10}$$

It follows from the inequalities (1.8) and (1.9) that

$$x_n \geqslant \frac{A\,(A-H) + S^2 + \sqrt{\left(A\,(A-H)+S^2\right)^2 - 4HS^2\,(A-H)}}{2\,(A-H)} \tag{3.11}$$

and

$$x_1 \leqslant \frac{A\,(A-H) + S^2 - \sqrt{\left(A\,(A-H)+S^2\right)^2 - 4HS^2\,(A-H)}}{2\,(A-H)}. \tag{3.12}$$

On substituting values of A, S^2 and H respectively from (3.8), (3.9) and (3.10) in (3.11) and (3.12) we respectively get the inequalities (3.2) and (3.3). The remaining assertions of the theorem are now immediate. □

THEOREM 3.2. *Let C be a complex $n\times n$ matrix with real eigenvalues λ_i such that $\lambda_1 \leqslant \lambda_i \leqslant \lambda_n, i=1,2,...,n$. Then*

$$\lambda_n \geqslant \frac{\operatorname{tr} C}{n} + \frac{b+\sqrt{b^2+4a^3}}{2a}, \tag{3.13}$$

$$\lambda_1 \leqslant \frac{\operatorname{tr} C}{n} + \frac{b-\sqrt{b^2+4a^3}}{2a}, \tag{3.14}$$

$$Spread\,(C) = \lambda_n - \lambda_1 \geqslant \frac{\sqrt{b^2+4a^3}}{a}, \tag{3.15}$$

and, for $\lambda_1 > 0$,

$$\frac{\lambda_n}{\lambda_1} \geqslant 1 + \frac{2\sqrt{b^2+4a^3}}{b + 2\frac{\operatorname{tr} C}{n}a - \sqrt{b^2+4a^3}}, \tag{3.16}$$

where

$$a = \frac{\operatorname{tr} C^2}{n} - \left(\frac{\operatorname{tr} C}{n}\right)^2 \tag{3.17}$$

and

$$b = \frac{1}{n^3}\left(n^2 \operatorname{tr} C^3 - 3n \operatorname{tr} C^2 \operatorname{tr} C + 2\left(\operatorname{tr} C\right)^3\right). \tag{3.18}$$

Proof. Let A, S^2 and m_3 be respectively the arithmetic mean, variance and third central moment of the eigenvalues λ_i $(i=1,2,...,n)$ of a complex $n\times n$ matrix C. Then

$$A = \frac{\operatorname{tr} C}{n}, \tag{3.19}$$

$$S^2 = \frac{\operatorname{tr} C^2}{n} - \left(\frac{\operatorname{tr} C}{n}\right)^2 \tag{3.20}$$

and

$$m_3 = \frac{1}{n^3}\left(n^2 \operatorname{tr} C^3 - 3n \operatorname{tr} C^2 \operatorname{tr} C + 2\left(\operatorname{tr} C\right)^3\right). \tag{3.21}$$

On substituting values of A, S^2 and m_3 respectively from (3.19), (3.20) and (3.21) in (2.20) and (2.21) we respectively get the inequalities (3.13) and (3.14). The remaining assertions of the theorem are now immediate. □

It is easy to compute traces of C and C^2. The calculation is costly when we have to determine the value of trace of C^3. We compare the present bounds involving traces of C, C^2 and C^3 with those obtained by Wolkowicz and Styan [6] which involve only traces of C and C^2.

EXAMPLES. Let

$$C_1 = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}.$$

We have $\operatorname{tr} C_1 = 22$, $\operatorname{tr} C_1^2 = 154$ and $\operatorname{tr} C_1^3 = 1201$. From the inequalities (3.13) and (3.14) we respectively have $\lambda_4 \geqslant 8.239$ and $\lambda_1 \leqslant 2.48$, whereas Wolkowicz and Styan [6] have shown that $\lambda_4 \geqslant 7.158$ and $\lambda_1 \leqslant 3.842$.

Let

$$C_2 = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 2 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 1 & 8 \end{bmatrix}.$$

We have $\operatorname{tr} C_2 = 30$, $\operatorname{tr} C_2^2 = 222$ and $\operatorname{tr} C_2^3 = 1929$. From the inequalities (3.13) and (3.14) we respectively have $\lambda_5 \geqslant 10.209$ and $\lambda_1 \leqslant 4.005$, whereas Wolkowicz and Styan [6] have shown that $\lambda_5 \geqslant 7.449$ and $\lambda_1 \leqslant 4.551$.

Let

$$C_3 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 9 \end{bmatrix}.$$

We use the inequality (3.13) to find the lower bound on the spectral radius of the matrix C_3. We have $\operatorname{tr} C_3 = 16$, $\operatorname{tr} C_3^2 = 114$ and $\operatorname{tr} C_3^3 = 946$. The Gerschgorin theory assures that the eigenvalues of C_3 are positive real numbers. From the inequality (3.13) we have $\lambda_3 \geqslant 9.0372$. The largest eigenvalue of C_3 is around 9.4495, and on using the Gerschgorin theorem we have $\lambda_3 \geqslant 7$.

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