Journal of
Mathematical
Inequalities
Volume 4, Number 3 (2010), 365–369

SIMPLE PROOF AND REFINEMENT OF HERMITE–HADAMARD INEQUALITY

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(*Communicated by J. Pečarić*)

Abstract. In this note we give a simple proof and a new generalization of the Hermite-Hadamard inequality.

1. Introduction and main results

Throughout this note, we denote by I the closed interval $[a,b]$. A real-valued function f is said to be convex on I if $\lambda f(x)+(1-\lambda)f(y) \geqslant f(\lambda x+(1-\lambda)y)$ for all $x,y \in I$ and $0 \leqslant \lambda \leqslant 1$. Conversely, if the opposite inequality holds, the function is said to be concave on I. A function f that is continuous on I and twice differentiable on (a,b) is convex on I if and only if $f''(x) \geqslant 0$ for all $x \in (a,b)$. (f is concave if and only if $f''(x) \leqslant 0$ for all $x \in (a,b)$).

The classical Hermite-Hadamard inequality which was first published in [6] gives us an estimate of the mean value of a convex function $f : I \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.1}$$

An account on the history of this inequality can be found in [7]. Surveys on various generalizations and developments can be found in [8] and [3]. The description of best possible inequalities of Hadamard-Hermite type are due to Fink [5]. A generalization to higher-order convex functions can be found in [1], while [2] offers a generalization for functions that are Beckenbach-convex with respect to the two dimensional linear space of continuous functions.

Recently in [4], the authors established this inequality for twice differentiable functions. In the case where f is convex then there exists an estimation better than (1.1) and for this, they posed the following question:

If f is a convex function on I, do there exist real numbers l, L such that

$$f\left(\frac{a+b}{2}\right) \leqslant l \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L \leqslant \frac{f(a)+f(b)}{2}?$$

The aim of this paper is to give an affirmative answer to this question. Firstly we give a simple proof of inequality (1.1).

Mathematics subject classification (2010): 52A40, 52A41.
Keywords and phrases: Convex functions, Hermite-Hadamard integral inequality.

Paper JMI-04-33

THEOREM A. *Assume that $f: I \to \mathbb{R}$ is a convex function on I. Then the inequality* (1.1) *holds.*

See [9, pp. 50–51], for details. This result can be easily improved by applying (1.1) on each of the subintervals $[a, \frac{(a+b)}{2}]$ and $[\frac{(a+b)}{2}, b]$; summing up side by side we get

$$f\left(\frac{a+b}{2}\right) \leqslant l \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L \leqslant \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where

$$l = \frac{1}{2}\left(f\left(\frac{3b+a}{4}\right) + f\left(\frac{b+3a}{2}\right)\right)$$

and

$$L = \frac{1}{2}\left(f\left(\frac{b+a}{2}\right) + \frac{f(a)+f(b)}{2}\right).$$

Secondly, we prove the following result:

THEOREM 1.1. *Assume that $f: I \to \mathbb{R}$ is a convex function on I. Then for all $\lambda \in [0.1]$, we have*

$$f\left(\frac{a+b}{2}\right) \leqslant l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L(\lambda) \leqslant \frac{f(a)+f(b)}{2}, \tag{1.3}$$

where

$$l(\lambda) := \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) := \frac{1}{2}\left(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)\right).$$

COROLLARY 1.1. *Assume that $f: I \to \mathbb{R}$ is a convex function on I. Then we have the following inequality*

$$f\left(\frac{a+b}{2}\right) \leqslant \sup_{\lambda\in[0.1]} l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \inf_{\lambda\in[0.1]} L(\lambda) \leqslant \frac{f(a)+f(b)}{2}, \tag{1.4}$$

where $l(\lambda), L(\lambda)$ are defined in Theorem 1.1.

REMARK 1.1. Applying Theorem 1.1 for $\lambda = \frac{1}{2}$ we get inequality (1.2).

2. Lemma

In order to prove Theorem A, we shall need the following Lemma:

LEMMA 2.1. *Let f be an integrable function on I. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \int_0^1 f(\lambda b+(1-\lambda)a)\,d\lambda \tag{2.1}$$

$$= \int_0^1 f(\lambda a+(1-\lambda)b)\,d\lambda. \tag{2.2}$$

Proof. We use the change of variables $x=\lambda b+(1-\lambda)a$ to prove (2.1) and we use $x=\lambda a+(1-\lambda)b$ to prove (2.2). □

3. Proof of the theorems

Firstly we give a simple proof of inequality (1.1).

Proof of Theorem A. Because f is a convex function we have for all $\lambda\in[0,1]$

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{\lambda b+(1-\lambda)a+\lambda a+(1-\lambda)b}{2}\right)$$
$$\leqslant \frac{f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)}{2} \leqslant \frac{f(a)+f(b)}{2},$$

then we can write

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)}{2} \leqslant \frac{f(a)+f(b)}{2}. \tag{3.1}$$

Integrating (3.1) over $[0,1]$ and using Lemma 2.1 we get (1.1). □

Proof of Theorem 1.1. Let f be a convex function on I. Applying (1.1) on the subinterval $[a,\lambda b+(1-\lambda)a]$, with $\lambda\neq 0$, we get

$$f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) \leqslant \frac{1}{\lambda(b-a)}\int_a^{\lambda b+(1-\lambda)a} f(x)\,dx \leqslant \frac{f(a)+f(\lambda b+(1-\lambda)a)}{2}. \tag{3.2}$$

Applying (1.1) again on $[\lambda b+(1-\lambda)a,b]$, with $\lambda\neq 1$ we get

$$f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \leqslant \frac{1}{(1-\lambda)(b-a)}\int_{\lambda b+(1-\lambda)a}^{b} f(x)\,dx$$
$$\leqslant \frac{f(b)+f(\lambda b+(1-\lambda)a)}{2}. \tag{3.3}$$

Multiplying (3.2) by λ, (3.3) by $(1-\lambda)$, and adding the resulting inequalities, we get:

$$l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L(\lambda). \tag{3.4}$$

where $l(\lambda)$ and $L(\lambda)$ are defined as in Theorem 1.1.

Using the fact that f is a convex function, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\lambda\frac{(\lambda b+(2-\lambda)a)}{2}+(1-\lambda)\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \\ &\leqslant \lambda f\left(\frac{\lambda b+(1-\lambda)a+a}{2}\right)+(1-\lambda)f\left(\frac{\lambda b+(1-\lambda)a+b}{2}\right) \\ &\leqslant \frac{1}{2}\left(f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)\right) \leqslant \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.5}$$

Then by $(3.4),(3.5)$ we get (1.3). □

EXAMPLES.

(1) Let $b > a \geqslant 0$. For $\lambda = \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ we get

$$l\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) = \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}f\left(\frac{\sqrt{a}\left(\sqrt{a}+\sqrt{b}\right)}{2}\right)+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}f\left(\frac{\sqrt{b}\left(\sqrt{a}+\sqrt{b}\right)}{2}\right)$$

and

$$L\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) = \frac{1}{2}\left(f\left(\sqrt{ab}\right)+\frac{\sqrt{a}f(a)+\sqrt{b}f(b)}{\sqrt{a}+\sqrt{b}}\right).$$

(2) Let $b > a \geqslant 0$ and $\lambda = \frac{a}{a+b}$. We obtain

$$l\left(\frac{a}{a+b}\right) = \frac{a}{a+b}f\left(\frac{\left(3ba+a^2\right)}{2(a+b)}\right)+\frac{b}{a+b}f\left(\frac{\left(b^2+3ab\right)}{2(a+b)}\right)$$

and

$$L\left(\frac{a}{a+b}\right) = \frac{1}{2}\left(f\left(\frac{2ba}{a+b}\right)+\frac{af(a)+bf(b)}{a+b}\right)$$

(3) Let $\lambda = \cos^2\theta,\ \theta \in \mathbb{R}$, then we have

$$l\left(\cos^2\theta\right) = f\left(\frac{b\cos^2\theta+\left(1+\sin^2\theta\right)a}{2}\right)\cos^2\theta+f\left(\frac{\left(1+\cos^2\theta\right)b+a\sin^2\theta}{2}\right)\sin^2\theta$$

and

$$L\left(\cos^2\theta\right) = \frac{1}{2}\left(f\left(b\cos^2\theta+a\sin^2\theta\right)+f(a)\cos^2\theta+f(b)\sin^2\theta\right).$$

REMARK 3.1. The right-hand side of the inequality (1.4), for $L\left(\frac{1}{2}\right)$, has been proved by P. S. Bullen in 1978, [10, p. 140]. For $L\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)$, the right-hand side of the inequality (1.4) has been proved by J. Śandor in 1988 (see [11]) and for $L\left(\frac{a}{a+b}\right)$, the right-hand side of the inequality (1.4) has been proved by S. S. Dragomir and C. E. M. Pearce in 2000, [3, p. 10–11].

Acknowledgement. The author would like to thank the referee for his/her helpful remarks and suggestions to improve the paper.

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(Received April 1, 2009)

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