SIMPLE PROOF AND REFINEMENT OF HERMITE–HADAMARD INEQUALITY

ABDALLAH EL FARISI

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Abstract. In this note we give a simple proof and a new generalization of the Hermite-Hadamard inequality.

1. Introduction and main results

Throughout this note, we denote by $I$ the closed interval $[a, b]$. A real-valued function $f$ is said to be convex on $I$ if $\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y)$ for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Conversely, if the opposite inequality holds, the function is said to be concave on $I$. A function $f$ that is continuous on $I$ and twice differentiable on $(a, b)$ is convex on $I$ if and only if $f''(x) \geq 0$ for all $x \in (a, b)$. ($f$ is concave if and only if $f''(x) \leq 0$ for all $x \in (a, b)$).

The classical Hermite-Hadamard inequality which was first published in [6] gives us an estimate of the mean value of a convex function $f : I \to \mathbb{R}$:

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

An account on the history of this inequality can be found in [7]. Surveys on various generalizations and developments can be found in [8] and [3]. The description of best possible inequalities of Hadamard-Hermite type are due to Fink [5]. A generalization to higher-order convex functions can be found in [1], while [2] offers a generalization for functions that are Beckenbach-convex with respect to the two dimensional linear space of continuous functions.

Recently in [4], the authors established this inequality for twice differentiable functions. In the case where $f$ is convex then there exists an estimation better than (1.1) and for this, they posed the following question:

If $f$ is a convex function on $I$, do there exist real numbers $l, L$ such that

$$f \left( \frac{a+b}{2} \right) \leq l \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq L \leq \frac{f(a) + f(b)}{2}?$$

The aim of this paper is to give an affirmative answer to this question. Firstly we give a simple proof of inequality (1.1).


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THEOREM A. Assume that $f : I \rightarrow \mathbb{R}$ is a convex function on $I$. Then the inequality (1.1) holds.

See [9, pp. 50–51], for details. This result can be easily improved by applying (1.1) on each of the subintervals $[a, \frac{(a+b)}{2}]$ and $[\frac{(a+b)}{2}, b]$; summing up side by side we get

$$f\left(\frac{a+b}{2}\right) \leq l \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq L \leq \frac{f(a) + f(b)}{2},$$

(1.2)

where

$$l = \frac{1}{2} \left( f\left(\frac{3b+a}{4}\right) + f\left(\frac{b+3a}{2}\right) \right)$$

and

$$L = \frac{1}{2} \left( f\left(\frac{b+a}{2}\right) + \frac{f(a) + f(b)}{2} \right).$$

Secondly, we prove the following result:

THEOREM 1.1. Assume that $f : I \rightarrow \mathbb{R}$ is a convex function on $I$. Then for all $\lambda \in [0,1]$, we have

$$f\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq L(\lambda) \leq \frac{f(a) + f(b)}{2},$$

(1.3)

where

$$l(\lambda) := \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) := \frac{1}{2} \left( f\left(\lambda b + (1-\lambda)a\right) + \lambda f(a) + (1-\lambda) f(b) \right).$$

COROLLARY 1.1. Assume that $f : I \rightarrow \mathbb{R}$ is a convex function on $I$. Then we have the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \sup_{\lambda \in [0,1]} l(\lambda) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \inf_{\lambda \in [0,1]} L(\lambda) \leq \frac{f(a) + f(b)}{2},$$

(1.4)

where $l(\lambda), L(\lambda)$ are defined in Theorem 1.1.

REMARK 1.1. Applying Theorem 1.1 for $\lambda = \frac{1}{2}$ we get inequality (1.2).
2. Lemma

In order to prove Theorem A, we shall need the following Lemma:

**Lemma 2.1.** Let \( f \) be an integrable function on \( I \). Then we have

\[
\frac{1}{b-a} \int_a^b f(x) \, dx = \int_0^1 f(\lambda b + (1 - \lambda) a) \, d\lambda
\]

(2.1)

\[
= \int_0^1 f(\lambda a + (1 - \lambda) b) \, d\lambda.
\]

(2.2)

**Proof.** We use the change of variables \( x = \lambda b + (1 - \lambda) a \) to prove (2.1) and we use \( x = \lambda a + (1 - \lambda) b \) to prove (2.2). \( \Box \)

3. Proof of the theorems

Firstly we give a simple proof of inequality (1.1).

**Proof of Theorem A.** Because \( f \) is a convex function we have for all \( \lambda \in [0, 1] \)

\[
f\left(\frac{a+b}{2}\right) = f\left(\frac{\lambda b + (1 - \lambda) a + \lambda a + (1 - \lambda) b}{2}\right)
\]

\[
\leq \frac{f(\lambda b + (1 - \lambda) a) + f(\lambda a + (1 - \lambda) b)}{2}
\]

then we can write

\[
f\left(\frac{a+b}{2}\right) \leq \frac{f(\lambda b + (1 - \lambda) a) + f(\lambda a + (1 - \lambda) b)}{2} \leq \frac{f(a) + f(b)}{2}.
\]

(3.1)

Integrating (3.1) over \([0, 1]\) and using Lemma 2.1 we get (1.1). \( \Box \)

**Proof of Theorem 1.1.** Let \( f \) be a convex function on \( I \). Applying (1.1) on the subinterval \([a, \lambda b + (1 - \lambda) a]\), with \( \lambda \neq 0 \), we get

\[
f\left(\frac{\lambda b + (2 - \lambda) a}{2}\right) \leq \frac{1}{\lambda (b-a)} \int_a^{\lambda b + (1 - \lambda) a} f(x) \, dx \leq \frac{f(a) + f(\lambda b + (1 - \lambda) a)}{2}.
\]

(3.2)

Applying (1.1) again on \([\lambda b + (1 - \lambda) a, b]\), with \( \lambda \neq 1 \) we get

\[
f\left(\frac{(1+\lambda) b + (1 - \lambda) a}{2}\right) \leq \frac{1}{(1 - \lambda) (b-a)} \int_{\lambda b + (1 - \lambda) a}^{b} f(x) \, dx
\]

\[
\leq \frac{f(b) + f(\lambda b + (1 - \lambda) a)}{2}.
\]

(3.3)

Multiplying (3.2) by \( \lambda \), (3.3) by \( (1 - \lambda) \), and adding the resulting inequalities, we get:

\[
l(\lambda) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq L(\lambda).
\]

(3.4)
where \( l(\lambda) \) and \( L(\lambda) \) are defined as in Theorem 1.1.

Using the fact that \( f \) is a convex function, we obtain

\[
f\left(\frac{a+b}{2}\right) = f\left(\frac{(\lambda b + (2-\lambda)a) + (1-\lambda) (1+\lambda)b + (1-\lambda)a}{2}\right)
\leq \lambda f\left(\frac{\lambda b + (1-\lambda)a + a}{2}\right) + (1-\lambda) f\left(\frac{\lambda b + (1-\lambda)a + b}{2}\right)
\leq \frac{1}{2} \left( f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b) \right) \leq \frac{f(a) + f(b)}{2}.
\]

(3.5)

Then by (3.4), (3.5) we get (1.3). \( \square \)

**Examples.**

(1) Let \( b > a \geq 0 \). For \( \lambda = \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} \), we get

\[
l\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) = \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} f\left(\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{2}\right) + \frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}} f\left(\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{2}\right)
\]

and

\[
L\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) = \frac{1}{2} \left( f\left(\sqrt{ab}\right) + \frac{\sqrt{af(a) + bf(b)}}{\sqrt{a}+\sqrt{b}} \right).
\]

(2) Let \( b > a \geq 0 \) and \( \lambda = \frac{a}{a+b} \). We obtain

\[
l\left(\frac{a}{a+b}\right) = \frac{a}{a+b} f\left(\frac{3ba + a^2}{2(a+b)}\right) + \frac{b}{a+b} f\left(\frac{b^2 + 3ab}{2(a+b)}\right)
\]

and

\[
L\left(\frac{a}{a+b}\right) = \frac{1}{2} \left( f\left(\frac{2ba}{a+b}\right) + \frac{af(a) + bf(b)}{a+b} \right).
\]

(3) Let \( \lambda = \cos^2 \theta, \ \theta \in \mathbb{R} \), then we have

\[
l(\cos^2 \theta) = f\left(\frac{b \cos^2 \theta + (1 + \sin^2 \theta) a}{2}\right) \cos^2 \theta + f\left(\frac{(1 + \cos^2 \theta) b + a \sin^2 \theta}{2}\right) \sin^2 \theta
\]

and

\[
L(\cos^2 \theta) = \frac{1}{2} \left( f(\cos^2 \theta + a \sin^2 \theta) + f(a) \cos^2 \theta + f(b) \sin^2 \theta \right).
\]

**Remark 3.1.** The right-hand side of the inequality (1.4), for \( L\left(\frac{a}{a+b}\right) \), has been proved by P. S. Bullen in 1978, [10, p. 140]. For \( L\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) \), the right-hand side of the inequality (1.4) has been proved by J. Šandor in 1988 (see [11]) and for \( L\left(\frac{a}{a+b}\right) \), the right-hand side of the inequality (1.4) has been proved by S. S. Dragomir and C. E. M. Pearce in 2000, [3, p. 10–11].

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REFERENCES


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