Journal of
Mathematical
Inequalities
Volume 4, Number 3 (2010), 371–389

A GOOD λ ESTIMATE FOR MULTILINEAR COMMUTATOR OF FRACTIONAL INTEGRAL ON SPACES OF HOMOGENEOUS TYPE

ZHANG QIAN AND LIU LANZHE

(*Communicated by P. Gurka*)

Abstract. In this paper, a good λ estimate for the multilinear commutator associated to the fractional integral operator on the spaces of homogeneous type is obtained. Under this result, we get the $(L^p(X),L^q(X))$-boundedness of the multilinear commutator.

1. Introduction

Let T be the Calderón-Zygmund operator, Coifman, Rochberg and Weiss (see [9]) proves that the commutator $[b,T](f) = bT(f) - T(bf)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [5]) proves a similiar result when T is replaced by the fractional operators. In [13], [16], Janson and Paluszynski study these results for the Triebel-Lizorkin spaces and the case $b \in Lip_\beta(R^n)$, where $Lip_\beta(R^n)$ is the homogeneous Lipschitz space. The main purpose of this paper is to find the good λ estimate for the multilinear commutator associated to the fractional integral operator on the spaces of homogeneous type, where $b \in Lip_\beta(X)$ and $b \in BMO(X)$. Under this result, we get $(L^p(X),L^q(X))$-boundedness of the multilinear commutator.

2. Preliminaries and Theorems

Give a set X, a function $d : X \times X \to R^+$ is called a quasi-distance on X if the following conditions are satisfied:

(i) for every x and y in X, $d(x,y) \geqslant 0$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) for every x and y in X, $d(x,y) = d(y,x)$,

(iii) there exists a constant $k \geqslant 1$ such that

$$d(x,y) \leqslant k(d(x,z) + d(z,y)) \tag{1}$$

for every x,y and z in X.

Mathematics subject classification (2010): 42B20, 42B25.

Keywords and phrases: Fractional integral, multilinear commutator, Lipschitz space, BMO, space of homogeneous type, good λ inequality.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the r-balls $B(x,r) = \{y : d(x,y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x,2r)) \leqslant A\mu(B(x,r)) < \infty \tag{2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X,d,μ), with d and μ as above, is called a space of homogeneous type. The constants k and A in (1) and (2) will be called the constants of the space.

Then let us introduce some notations (see [3], [11], [16], [21]). Throughout this paper, B will denote a ball of X , and for a ball B let $f_B = \mu(B)^{-1}\int_B f(x)d\mu(x)$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f_B| d\mu(y).$$

It is well-known that (see [11])

$$f^{\#}(x) \approx \sup_{B \ni x} \inf_{c \in C} \frac{1}{\mu(B)} \int_B |f(y) - C| d\mu(y).$$

We say that b belongs to $BMO(X)$ if $b^{\#}$ belongs to $L^\infty(X)$ and define $||b||_{BMO} = ||b^{\#}||_{L^\infty}$. It has been known that(see [11])

$$||b - b_{2^k B}||_{BMO} \leqslant Ck||b||_{BMO}.$$

For $1 \leqslant p < \infty$ and $0 \leqslant \gamma < 1$, let

$$M_{\gamma,p}(f)(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)^{1-p\gamma}} \int_B |f(y)|^p d\mu(y) \right)^{1/p}.$$

If $\gamma = 0$, $M_{p,\gamma}(f) = M_p(f)$ which is the Hardy-Littlewood maximal function when $p = 1$. For $0 < \beta < 1$, the Lipschitz space $\dot{\wedge}_\beta$ is the space of functions f such that

$$||f||_{\dot{\wedge}_\beta} = \sup_{\substack{x,h \in X \\ h \neq 0}} \rho\left(\Delta_h^{[\beta]+1} f(x)\right) / \rho(x+h,x)^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [16]) and the existence of ρ is guaranteed by the following Lemma 1.

In this paper, we will study some multilinear commutators as follows.

DEFINITION. Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on X. Let I_γ be the fractional integral operator as

$$I_\gamma(f)(x) = \int_X K_\gamma(x,y) f(y) d\mu(y), \quad 0 < \gamma < 1,$$

where

$$K_\gamma(x,y)=\begin{cases}[\mu(B(x,d(x,y)))]^{\gamma-1} & \text{if } x\neq y\\ \mu(x)^{\gamma-1} & \text{if } x=y \text{ and } \mu(x)>0.\end{cases}$$

The multilinear commutator of fractional integral operator is defined by

$$I_\gamma^{\vec{b}}(f)(x)=\int_X\prod_{j=1}^m(b_j(x)-b_j(y))K_\gamma(x,y)f(y)d\mu(y). \tag{3}$$

and $I_\star^{\gamma\vec{b}}(f)(x)=\sup_{\varepsilon>0}|I_\varepsilon^{\gamma\vec{b}}(f)(x)|$, where

$$I_\varepsilon^{\gamma\vec{b}}(f)(x)=\int_{\rho(x,y)>\varepsilon}\prod_{j=1}^m(b_j(x)-b_j(y))K_\gamma(x,y)f(y)d\mu(y)$$

Note that when $b_1=\cdots=b_m$, $I_\gamma^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors(see [1–6], [10–12], [17–19]). Our main purpose is to find the good λ estimate for the multilinear commutator $I_\star^{\gamma\vec{b}}$, and with this result to find $(L^P(X),L^q(X))$-boundedness for the multilinear commutator $I_\gamma^{\vec{b}}$.

Given some functions b_j $(j=1,\cdots,m)$ and a positive integer m and $1\leqslant j\leqslant m$, we set $||\vec{b}||_{Lip_\beta}=\prod_{j=1}^m||b_j||_{Lip_\beta}$, $||\vec{b}||_{BMO}=\prod_{j=1}^m||b_j||_{BMO}$ and denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements, $|\sigma|=j$ is the element number of σ. For $\sigma\in C_j^m$, set $\sigma^c=\{1,\cdots,m\}\setminus\sigma$. For $\vec{b}=(b_1,\cdots,b_m)$ and $\sigma=\{\sigma(1),\cdots,\sigma(j)\}\in C_j^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\cdots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(j)}$, $||\vec{b}_\sigma||_{\wedge_\beta}=||b_{\sigma(1)}||_{\wedge_\beta}\cdots||b_{\sigma(j)}||_{\wedge_\beta}$ and $||\vec{b}_\sigma||_{BMO}=||b_{\sigma(1)}||_{BMO}\cdots||b_{\sigma(j)}||_{BMO}$.

In what follows, $C>0$ always denotes a constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p\in[1,\infty]$, we denote by p' its conjugate index, namely, $1/p+1/p'=1$.

Now we state our results as following.

THEOREM 1. *Let* $0<\gamma<1$, $\theta=1-\gamma$, $0<\beta<1$ *and* $b_j\in\wedge_\beta$ *for* $j=1,\cdots,m$.

(a) *Suppose* $1<r<p<\infty$. *Then there exists* $\xi_0>0$ *such that, for any* $0<\xi<\xi_0$ *and* $\lambda>0$,

$$\mu\left(\left\{x\in X:I_\star^{\gamma\vec{b}}(f)(x)>3\lambda,||\vec{b}||_{\wedge_\beta}M_{m\beta+\gamma,p}(f)(x)\leqslant\xi\lambda\right\}\right)\leqslant C\xi^r\mu(\{x\in X:I_\star^{\gamma\vec{b}}(f)(x)>\lambda\})$$

(b) $I_\gamma^{\vec{b}}$ *is bounded from* $L^p(X)$ *to* $L^q(X)$ *for* $1<p<1/(m\beta+\gamma)$ *and* $1/q=1/p-(m\beta+\gamma)$.

THEOREM 2. *Let* $0<\gamma<1,\ \theta=1-\gamma$, *and* $b_j \in BMO(X)$ *for* $j=1,\cdots,m$.

(a) *Suppose* $1<r<p<\infty$. *Then there exists* $\xi_0>0$ *such that, for any* $0<\xi<\xi_0$ *and* $\lambda>0$,

$$\mu\left(\left\{x\in X: I_\star^{\gamma\vec{b}}(f)(x)>3\lambda, ||\vec{b}||_{BMO}M_{\gamma,p}(f)(x)\leqslant\xi\lambda\right\}\right)$$
$$\leqslant C\xi^r\mu(\{x\in X: I_\star^{\gamma\vec{b}}(f)(x)>\lambda\})$$

(b) $I_\gamma^{\vec{b}}$ *is bounded from* $L^p(X)$ *to* $L^q(X)$ *for* $1<p<1/\gamma$ *and* $1/q=1/p-\gamma$.

3. Proofs of Theorems

To prove the theorem, we need the following lemmas.

LEMMA 1. (see [15]). *Let* d *be a quasi-distance on a set* X. *Then there exists a quasi-distance* ρ *on* X, *a finite constant* C *and a number* $0<\alpha<1$, *such that* ρ *is equivalent to* d *and, for every* x, y *and* z *in* X

$$|\rho(x,y)-\rho(z,y)|\leqslant C\rho(x,z)^\alpha(\rho(x,y)+\rho(z,y))^{1-\alpha}.$$

LEMMA 2. (see [16]). *Let* $0<\beta<1, 1\leqslant p\leqslant\infty$, *then*

$$\begin{aligned}||b||_{\dot{\Lambda}_\beta} &\approx \sup_B \frac{1}{\mu(B)^{1+\beta}}\int_B |b(x)-b_B|d\mu(x)\\ &\approx \sup_B \frac{1}{\mu(B)^{\beta}}\left(\frac{1}{\mu(B)}\int_B |b(x)-b_B|^p d\mu(x)\right)^{1/p}\\ &\approx \sup_B\inf_c \frac{1}{\mu(B)^{1+\beta}}\int_B |b(x)-c|d\mu(x)\\ &\approx \sup_B\inf_c \frac{1}{\mu(B)^{\beta}}\left(\frac{1}{\mu(B)}\int_B |b(x)-c|^p d\mu(x)\right)^{1/p}\end{aligned}$$

LEMMA 3. (see [5]). *Let* $0\leqslant\nu<1,\ 1\leqslant r<p<1/\nu$ *and* $1/q=1/p-\nu$, *then*

$$||M_{\nu,r}(f)||_{L^q}\leqslant C||f||_{L^p}.$$

REMARK. In order to prove Theorems 1, it is clear that we may replace the kernel K_γ by any kernel Q_γ equivalent to K_γ, in the sense that $1/cK_\gamma\leqslant Q_\gamma\leqslant cK_\gamma$ for some positive constant. We shall use an equivalent kernel having some smoothness properties that fractional kernel K_γ does not have.

In fact, notice that the function $K(x,y)$ defined as $\mu(B(x,d(x,y)))$ if $x\neq y$ and $K(x,y)=0$ is not a quasi-distance because it might not be symmetric. However, it is easy to prove that the function

$$\delta(x,y)=\begin{cases}\frac{1}{2}[\mu(B(x,d(x,y)))+\mu(B(y,d(x,y)))] & \text{if } x\neq y,\\ 0 & \text{if } x=y.\end{cases}$$

is a quasi-distance equivalent to $K(x,y)$. Now let ρ be a continuous quasi-distance equivalent to δ (the existence of ρ is guaranteed by lemma 1). Associated to ρ we define the kernel

$$Q_\gamma(x,y) = \begin{cases} \rho(x,y)^{\gamma-1} & \text{if } x \neq y, \\ \mu(x)^{\gamma-1} & \text{if } x = y \text{ and } \mu(x) > 0. \end{cases}$$

and the operator

$$\mathscr{I}_\gamma^{\vec{b}}(f)(x) = \int_X \prod_{j=1}^m (b_j(x) - b_j(y)) Q_\gamma(x,y) f(y) d\mu(y) \tag{4}$$

It is clear that the above operator is equivalent to the one defined in (3). Consequently, we shall work in the proof Theorem 1 and the following Theorems 2 with the operator defined in (4).

Proof of Theorem 1(a). When $\mathbf{m = 1}$, by the Whitney decomposition, $\{x \in X : \mathscr{I}_\star^{\gamma b_1}(f)(x) > \lambda\}$ may be written as a union of balls $\{B_k\}$ with mutually disjoint interiors and with distance from each to $X \setminus \bigcup_k B_k$ comparable to the diameter of B_k. It suffices to prove the good λ estimate for each B_k. There exists a constant $C = C(n)$ such that for each k, the cube $\tilde{B}_k$ intersects $X \setminus \bigcup_k B_k$, where $\tilde{B}_k$ denotes the ball with the same center as B_k and with the diam $\tilde{B}_k = C$ diam B_k. Then, for each k, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathscr{I}_\star^{\gamma b_1}(f)(x_0) \leqslant \lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$||b_1||_{\wedge_\beta} M_{\beta+\gamma,p}(f)(z) \leqslant \xi\lambda.$$

Set $\overline{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{B}_k}$ and $f_2 = f\chi_{X\setminus\overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1. For $x \in B_k$, $1/r = 1/p + 1/q < 1$

$$\begin{aligned} ||\mathscr{I}_\star^{\gamma b_1}(f_1)||_{L^r} &\leqslant C\left(\int_{\rho(x,y)>\varepsilon} \left|(b_1(x) - b_1(y))Q_\gamma(x,y)f_1(y)\right|^r d\mu(y)\right)^{1/r} \\ &\leqslant C\mu(\overline{B}_k)^{\gamma-1}\left(\int_{\overline{B}_k} |b_1(x) - b_1(y)|^q d\mu(y)\right)^{1/q}\left(\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p} \\ &\leqslant C\mu(\overline{B}_k)^{\gamma-1}\mu(\overline{B}_k)^{\beta+1/q}||b_1||_{\wedge_\beta}\mu(\overline{B}_k)^{1/p-\beta-\gamma} \\ &\quad \times\left(\frac{1}{\mu(\overline{B}_k)^{1-(\beta+\gamma)p}}\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p} \\ &\leqslant C\mu(\overline{B}_k)^{1/r-1}||b_1||_{\wedge_\beta} M_{\beta+\gamma,p}(f)(z). \end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned}\mu(\{x\in X:\mathscr{I}_\star^{\gamma b_1}(f_1)(x)>\eta\lambda\}) &\leqslant C(\eta\lambda)^{-r}\|\mathscr{I}_\star^{\gamma b_1}(f_1)\|_{L^r}^r\\ &\leqslant C(\eta\lambda)^{-r}[\|b_1\|_{\dot\wedge_\beta}M_{\beta+\gamma,p}(f)(z)]^r\mu(\overline{B}_k)^{1-1/r}\\ &\leqslant C(\eta\lambda)^{-r}(\xi\lambda)^r\mu(\overline{B}_k)\\ &\leqslant C(\xi/\eta)^r\mu(B_k).\end{aligned}$$

The estimates on f_2. Let $H = H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k)\leqslant\varepsilon\leqslant H\mathrm{diam}(\tilde{B}_k)$. Set $V(x,y)=(b_1(x)-b_1(y))Q_\gamma(x,y)f(y)$. Choose $x_0\in\tilde{B}_k$ such that $x_0\in X\setminus\bigcup_k B_k$. For $x\in B_k$, following [8], we have

$$\begin{aligned}|\mathscr{I}_\varepsilon^{\gamma b_1}(f_2)(x)| &\leqslant |\int_{\rho(x,y)>\varepsilon}[V(x,y)-V(x_0,y)]f_2(y)d\mu(y)|+\int_{R(x)}|V(x_0,y)f(y)|d\mu(y)\\ &\quad+\int_{R(x_0)}|V(x_0,y)f(y)|d\mu(y)+|\mathscr{I}_\varepsilon^{\gamma b_1}(f)(x_0)|\\ &= I+II+III+IV,\end{aligned}$$

where $R(u)=\{y\in X:\mathrm{diam}(\tilde{B}_k)<\rho(u,y)\leqslant H\mathrm{diam}(\tilde{B}_k)\}$. Now let us treat I,II and III respectively. For I, we write

$$\begin{aligned}|V(x,y)-V(x_0,y)| &= |(b_1(x)-b_1(y))Q_\gamma(x,y)f_2(y)-(b_1(x_0)-b_1(y))Q_\gamma(x_0,y)f_2(y)|\\ &= |b_1(x)Q_\gamma(x,y)f_2(y)-b_1(y)Q_\gamma(x,y)f_2(y)-b_1(x_0)Q_\gamma(x_0,y)f_2(y)\\ &\quad+b_1(y)Q_\gamma(x_0,y)f_2(y)-b_1(x_0)Q_\gamma(x,y)f_2(y)+b_1(x_0)Q_\gamma(x,y)f_2(y)|\\ &\leqslant |(b_1(x)-b_1(x_0))Q_\gamma(x,y)f_2(y)|\\ &\quad+|(b_1(x_0)-b_1(y))(Q_\gamma(x,y)-Q_\gamma(x_0,y))f_2(y)|\\ &= I_1+I_2.\end{aligned}$$

For I_1, by Lemma 2, Hölder's inequality and the following inequality, for $b\in\dot\wedge_\beta$

$$|b(x)-b_B|\leqslant\frac{1}{\mu(B)}\int_B\|b\|_{\dot\wedge_\beta}|x-y|^\beta d\mu(y)\leqslant\|b\|_{\dot\wedge_\beta}\mu(B)^\beta,$$

we have

$$\begin{aligned}\int_{\rho(x,y)>\varepsilon}I_1d\mu(y) &\leqslant C\sum_{\nu=1}^\infty\int_{B_{2^{\nu+1}\varepsilon}(x_0)\setminus B_{2^\nu\varepsilon}(x_0)}|b_1(x)-b_1(x_0)||Q_\gamma(x,y)||f(y)|d\mu(y)\\ &\leqslant C\|b_1\|_{\dot\wedge_\beta}\mu(\tilde{B}_k)^\beta\sum_{\nu=1}^\infty\mu(B_{2^\nu\varepsilon}(x_0))^{\gamma-1}\\ &\quad\times\left(\int_{B_{2^{\nu+1}\varepsilon}(x_0)}|f(y)|^pd\mu(y)\right)^{1/p}\mu(B_{2^{\nu+1}\varepsilon}(x_0))^{1-1/p}\end{aligned}$$

$$\begin{aligned}
&\leqslant C||b_1||_{\dot\wedge_\beta}\mu(\tilde{B}_k)^\beta\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-\beta-\gamma}\\
&\qquad\times\left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-(\beta+\gamma)p}}\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C||b_1||_{\dot\wedge_\beta}\mu(\tilde{B}_k)^\beta\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-\beta}M_{\beta+\gamma,p}(f)(z)\\
&\leqslant C||b_1||_{\dot\wedge_\beta}\sum_{v=1}^{\infty}2^{-v\beta}M_{\beta+\gamma,p}(f)(z)\\
&\leqslant C||b_1||_{\dot\wedge_\beta}M_{\beta+\gamma,p}(f)(z)\\
&\leqslant C\xi\lambda.
\end{aligned}$$

For I_2, notice $\theta=1-\gamma$, by μ's doubling condition, Hölder's inequality, Lemma 2 and the inequality $||a|^\alpha-|c|^\alpha|\leqslant|a-c|^\alpha$ for $a,\ c\in\mathbf{R}$ and $0<\alpha<1$, we obtain

$$\begin{aligned}
&\int_{\rho(x,y)>\varepsilon}I_2 d\mu(y)\\
&\quad\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)\setminus B_{2^v\varepsilon}(x_0)}|(b_1(x_0)-b_1(y))(Q_\gamma(x,y)-Q_\gamma(x_0,y))f(y)|d\mu(y)\\
&\quad\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|\frac{|\eta(x_0,y)^{1-\gamma}-\eta(x,y)^{1-\gamma}|}{|\eta(x,y)^{1-\gamma}\eta(x_0,y)^{1-\gamma}|}|f(y)|d\mu(y)\\
&\quad\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|\frac{|\eta(x_0,x)^{1-\gamma}|}{|\eta(x_0,y)^{2(1-\gamma)}|}|f(y)|d\mu(y)\\
&\quad\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|\frac{\eta(x_0,x)^{\theta}}{\eta(x_0,y)^{1-\gamma+\theta}}|f(y)|d\mu(y)\\
&\quad\leqslant C\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^\theta}{\mu(B_{2^{v+1}\varepsilon}(x_0)^{1-\gamma+\theta}}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'}\\
&\qquad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\quad\leqslant C\sum_{v=1}^{\infty}\mu(\tilde{B}_k)^\theta\mu(B_{2^{v+1}\varepsilon})(x_0)^{-1+\gamma-\theta+\beta+1/p'+1/p-\beta-\gamma}||b_1||_{\dot\wedge_\beta}M_{\beta+\gamma,p}(f)(z)\\
&\quad\leqslant C\sum_{v=1}^{\infty}2^{-v\theta}||b_1||_{\dot\wedge_\beta}M_{\beta+\gamma,p}(f)(z)\\
&\quad\leqslant C||b_1||_{\dot\wedge_\beta}M_{\beta+\gamma,p}(f)(z)\\
&\quad\leqslant C\xi\lambda.
\end{aligned}$$

Therefore $I\leqslant C\xi\lambda$.

For II and III, note that, for $y \in R(x)$,

$$\rho(x,y) \leqslant H\mathrm{diam}(\tilde{B}_k),$$

we get, by Lemma 2 and Hölder's inequality

$$\begin{aligned}
II &\leqslant C\int_{H\tilde{B}_k} |b_1(x_0)-b_1(y)||Q_\gamma(x,y)||f(y)|d\mu(y)\\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1}\left(\int_{H\tilde{B}_k}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{H\tilde{B}_k}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1+\beta+1/p'+1/p-\beta-\gamma}||b_1||_{\wedge_\beta}\left(\frac{1}{\mu(H\tilde{B}_k)^{1-(\beta+\gamma)p}}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C||b_1||_{\wedge_\beta}M_{\beta+\gamma,p}(f)(z)\\
&\leqslant C\xi\lambda.
\end{aligned}$$

Similar $III \leqslant C\xi\lambda$.

Thus $I+II+III \leqslant C\xi\lambda$.

For IV, since $x \notin \bigcup_k Q_k$, then $|\mathscr{I}_\varepsilon^{\gamma b_1}(f)(x_0)| \leqslant \lambda$. For $x \in B_j$,

$$\sup_{\varepsilon\approx \mathrm{diam}(\tilde{B}_k)} |\mathscr{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leqslant C\xi\lambda+\lambda.$$

Case 2. $\varepsilon > H\mathrm{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of *Case 1*, we get

$$\sup_{\varepsilon> \mathrm{diam}(\tilde{B}_k)} |\mathscr{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leqslant C\xi\lambda+\lambda.$$

Thus, we have shown that for $x \in B_k$,

$$\mathscr{I}_\star^{\gamma b_1}(f_2)(x) \leqslant C\xi\lambda+\lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned}
&\mu\left(\left\{x\in B_k : I_\star^{\gamma b_1}(f)(x) > 3\lambda, ||b_1||_{\wedge_\beta}M_{\beta+\gamma,p}(f)(x)\leqslant \xi\lambda\right\}\right)\\
&\quad\leqslant \mu(\{x\in B_k : I_\star^{\gamma b_1}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x\in X : I_\star^{\gamma b_1}(f_2)(x) > \lambda + C\xi\lambda\})\\
&\quad\leqslant \mu(\{x\in B_k : I_\star^{\gamma b_1}(f_1)(x) > \lambda\}) \leqslant \xi^r\mu(B_k).
\end{aligned}$$

When $\mathbf{m} > \mathbf{1}$, similar to the case $\mathbf{m} = \mathbf{1}$, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathscr{I}_\star^{\gamma\vec{b}}(f)(x_0) \leqslant \lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$||\vec{b}||_{\wedge_\beta} M_{m\beta+\gamma,p}(f)(z) \leqslant \xi\lambda.$$

Set $\overline{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{B}_k}$ and $f_2 = f\chi_{X\setminus\overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1. For $x \in B_k$, $1/r = 1/p + 1/q < 1$

$$\begin{aligned}
||\mathscr{I}_\star^{\gamma\vec{b}}(f_1)||_{L^r} &\leqslant C\left(\int_{\rho(x,y)>\varepsilon}\left|\prod_{j=1}^m (b_j(x)-b_j(y))Q_\gamma(x,y)f_1(y)\right|^r d\mu(y)\right)^{1/r} \\
&\leqslant C\mu(\overline{B}_k)^{\gamma-1}\left(\int_{\overline{B}_k}|\prod_{j=1}^m b_j(x)-b_j(y)|^q d\mu(y)\right)^{1/q}\left(\int_{\overline{B}_k}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leqslant C\mu(\overline{B}_k)^{\gamma-1}\mu(\overline{B}_k)^{m\beta+1/q}||\vec{b}||_{\wedge_\beta}\mu(\overline{B}_k)^{1/p-m\beta-\gamma} \\
&\quad \times\left(\frac{1}{\mu(\overline{B}_k)^{1-(m\beta+\gamma)p}}\int_{\overline{B}_k}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leqslant C\mu(\overline{B}_k)^{1/r-1}||\vec{b}||_{\wedge_\beta} M_{m\beta+\gamma,p}(f)(z).
\end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned}
\mu(\{x \in X : \mathscr{I}_\star^{\gamma\vec{b}}(f_1)(x) > \eta\lambda\}) &\leqslant C(\eta\lambda)^{-r}||\mathscr{I}_\star^{\gamma\vec{b}}(f_1)||_{L^r}^r \\
&\leqslant C(\eta\lambda)^{-r}[||\vec{b}||_{\wedge_\beta} M_{m\beta+\gamma,p}(f)(z)]^r \mu(\overline{B}_k)^{1-1/r} \\
&\leqslant C(\eta\lambda)^{-r}(\xi\lambda)^r \mu(\overline{B}_k) \\
&\leqslant C(\xi/\eta)^r \mu(B_k).
\end{aligned}$$

The estimates on f_2. Let $H = H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k) \leqslant \varepsilon \leqslant H\mathrm{diam}(\tilde{B}_k)$. Set $U(x,y) = \prod_{j=1}^m (b_j(x)-b_j(y))Q_\gamma(x,y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned}
|\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| &\leqslant |\int_{\rho(x,y)>\varepsilon}[U(x,y)-U(x_0,y)]f_2(y)d\mu(y)| + \int_{R(x)}|U(x_0,y)f(y)|d\mu(y) \\
&\quad + \int_{R(x_0)}|U(x_0,y)f(y)|d\mu(y) + |\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f)(x_0)| \\
&= J + JJ + JJJ + JJJJ,
\end{aligned}$$

where $R(u) = \{y \in X : \mathrm{diam}(\tilde{B}_k) < \rho(u,y) \leqslant H\mathrm{diam}(\tilde{B}_k)\}$. Now let us treat J, JJ and JJJ respectively. For J, we write

$$J \leqslant |(b_1(x)-b_1(x_0))\cdots(b_m(x)-b_m(x_0))\int_{\rho(x,y)>\varepsilon} Q_\gamma(x,y)f_2(y)d\mu(y)|$$

$$+\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}\int_{\rho(x,y)>\varepsilon}(\vec{b}(y)-\vec{b}(x_0))_{\sigma}(Q_\gamma(x,y)-Q_\gamma(x_0,y)f_2(y)d\mu(y)|$$
$$=J_1+J_2.$$

For J_1, we have

$$\begin{aligned}
J_1 &\leqslant \prod_{j=1}^{m}|b_j(x)-b_j(x_0)|\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)\setminus B_{2^v\varepsilon}(x_0)}|Q_\gamma(x,y)||f(y)|d\mu(y)\\
&\leqslant C||\vec{b}||_{\wedge_\beta}\mu(\tilde{B}_k)^{m\beta}\sum_{v=1}^{\infty}\mu(B_{2^v\varepsilon}(x_0))^{\gamma-1}\\
&\quad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p}\\
&\leqslant C||\vec{b}||_{\wedge_\beta}\mu(\tilde{B}_k)^{m\beta}\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-m\beta-\gamma}\\
&\quad\times\left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-(m\beta+\gamma)p}}\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C||\vec{b}||_{\wedge_\beta}\mu(\tilde{B}_k)^{m\beta}\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-m\beta}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C||\vec{b}||_{\wedge_\beta}\sum_{v=1}^{\infty}2^{-vm\beta}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C||\vec{b}||_{\wedge_\beta}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C\xi\lambda.
\end{aligned}$$

For J_2, let $\tau,\tau'\in\mathbf{N}$ such that $\tau+\tau'=m,$ and $\tau'\neq 0$, we get

$$\begin{aligned}
J_2 &\leqslant \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}\int_{B_{2^{v+1}\varepsilon}(x_0)\setminus B_{2^v\varepsilon}(x_0)}(\vec{b}(y)-\vec{b}(x_0))_{\sigma}\\
&\quad\times(Q_\gamma(x,y)-Q_\gamma(x_0,y)f(y)d\mu(y)|\\
&\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}|\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^{\theta}}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}}\\
&\quad\times\int_{B_{2^{v+1}\varepsilon}(x_0)}|(\vec{b}(y)-\vec{b}(x_0))_{\sigma}f(y)d\mu(y)|\\
&\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}|\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^{\theta}}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}}\\
&\quad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|(\vec{b}(y)-\vec{b}(x_0))_{\sigma}|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C\sum_{\tau+\tau'=m}||\vec{b}_{\sigma^c}||_{\wedge_\beta}\mu(\tilde{B}_k)^{\tau\beta+\theta}\sum_{\nu=1}^{\infty}\mu(B_{2^{\nu+1}\varepsilon}(x_0))^{-1+\gamma-\theta+\tau'\beta+1/p'+1/p-m\beta-\gamma}\\
&\quad\times||\vec{b}_\sigma||_{\wedge_\beta}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C\sum_{\tau+\tau'=m}||\vec{b}||_{\wedge_\beta}\mu(\tilde{B}_k)^{\tau\beta+\theta}\sum_{\nu=1}^{\infty}\mu(B_{2^{\nu+1}\varepsilon}(x_0))^{-\theta-\tau\beta}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C||\vec{b}||_{\wedge_\beta}\sum_{\nu=1}^{\infty}2^{-\nu(\theta+\tau\beta)}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C||\vec{b}||_{\wedge_\beta}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C\xi\lambda.
\end{aligned}$$

Therefore $J\leqslant C\xi\lambda$. For JJ and JJJ, note that, for $y\in R(x)$,

$$\rho(x,y)\leqslant H\mathrm{diam}(\tilde{B}_k),$$

we get, by Lemma 2 and Hölder's inequality

$$\begin{aligned}
JJ&\leqslant C\int_{H\tilde{B}_k}|\prod_{j=1}^{m}b_j(x_0)-b_j(y)||Q_\gamma(x,y)||f(y)|d\mu(y)\\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1}\left(\int_{H\tilde{B}_k}|\prod_{j=1}^{m}b_j(x_0)-b_j(y)|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{H\tilde{B}_k}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1+m\beta+1/p'+1/p-m\beta-\gamma}||\vec{b}||_{\wedge_\beta}\left(\frac{1}{\mu(H\tilde{B}_k)^{1-(m\beta+\gamma)p}}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C||\vec{b}||_{\wedge_\beta}M_{m\beta+\gamma,p}(f)(z)\\
&\leqslant C\xi\lambda.
\end{aligned}$$

Similar $JJJ\leqslant C\xi\lambda$.
Thus $J+JJ+JJJ\leqslant C\xi\lambda$.
For $JJJJ$, since $x\notin\bigcup_k Q_k$, then $|\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f)(x_0)|\leqslant\lambda$. For $x\in B_j$,

$$\sup_{\varepsilon\approx\mathrm{diam}(\tilde{B}_k)}|\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)|\leqslant C\xi\lambda+\lambda.$$

Case 2. $\varepsilon>H\mathrm{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon=\varepsilon$. Similar to the proof of *Case 1*, we get

$$\sup_{\varepsilon>\mathrm{diam}(\tilde{B}_k)}|\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)|\leqslant C\xi\lambda+\lambda.$$

Thus, we have shown that for $x\in B_k$,

$$\mathscr{I}_\star^{\gamma\vec{b}}(f_2)(x)\leqslant C\xi\lambda+\lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned}
&\mu\left(\left\{x \in B_k : I_\star^{\gamma\vec{b}}(f)(x) > 3\lambda, ||\vec{b}||_{\wedge_\beta} M_{m\beta+\gamma,p}(f)(x) \leqslant \xi\lambda\right\}\right)\\
&\quad\leqslant \mu(\{x \in B_k : I_\star^{\gamma\vec{b}}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : I_\star^{\gamma\vec{b}}(f_2)(x) > \lambda + C\xi\lambda\})\\
&\quad\leqslant \mu(\{x \in B_k : I_\star^{\gamma\vec{b}}(f_1)(x) > \lambda\}) \leqslant \xi^r \mu(B_k).
\end{aligned}$$

This completes the proof of Theorem 1(a). (b) follows from (a) and Lemma 3. □

Proof of Theorem 2(a). When $\mathbf{m} = \mathbf{1}$, by the Whitney decomposition, $\{x \in X : \mathscr{I}_\star^{\gamma b_1}(f)(x) > \lambda\}$ may be written as a union of balls $\{B_k\}$ with mutually disjoint interiors and with distance from each to $X \setminus \bigcup_k B_k$ comparable to the diameter of B_k. It suffices to prove the good λ estimate for each B_k. There exists a constant $C = C(n)$ such that for each k, the cube $\tilde{B}_k$ intersects $X \setminus \bigcup_k B_k$, where $\tilde{B}_k$ denotes the ball with the same center as B_k and with the diam $\tilde{B}_k = C$ diam B_k. Then, for each k, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathscr{I}_\star^{\gamma b_1}(f)(x_0) \leqslant \lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$||b_1||_{BMO} M_{\gamma,p}(f)(z) \leqslant \xi\lambda.$$

Set $\overline{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{B}_k}$ and $f_2 = f\chi_{X\setminus\overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1. For $x \in B_k$, $1/r = 1/p + 1/q < 1$

$$\begin{aligned}
||\mathscr{I}_\star^{\gamma b_1}(f_1)||_{L^r} &\leqslant C\left(\int_{\rho(x,y)>\varepsilon} \left|(b_1(x) - b_1(y))Q_\gamma(x,y)f_1(y)\right|^r d\mu(y)\right)^{1/r}\\
&\leqslant C\mu(\overline{B}_k)^{\gamma-1}\left(\int_{\overline{B}_k} |b_1(x) - b_1(y)|^q d\mu(y)\right)^{1/q}\left(\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C\mu(\overline{B}_k)^{\gamma-1}\mu(\overline{B}_k)^{1/q}||b_1||_{BMO}\mu(\overline{B}_k)^{1/p-\gamma}\\
&\quad\times\left(\frac{1}{\mu(\overline{B}_k)^{1-\gamma p}}\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C\mu(\overline{B}_k)^{1/r-1}||b_1||_{BMO} M_{\gamma,p}(f)(z).
\end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned}
\mu(\{x \in X : \mathscr{I}_\star^{\gamma b_1}(f_1)(x) > \eta\lambda\}) &\leqslant C(\eta\lambda)^{-r}||\mathscr{I}_\star^{\gamma b_1}(f_1)||_{L^r}^r\\
&\leqslant C(\eta\lambda)^{-r}[||b_1||_{BMO} M_{\gamma,p}(f)(z)]^r \mu(\overline{B}_k)^{1-1/r}\\
&\leqslant C(\eta\lambda)^{-r}(\xi\lambda)^r \mu(\overline{B}_k)\\
&\leqslant C(\xi/\eta)^r \mu(B_k).
\end{aligned}$$

The estimates on f_2. Let $H = H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k) \leqslant \varepsilon \leqslant H\mathrm{diam}(\tilde{B}_k)$. Set $V(x,y) = (b_1(x) - b_1(y))Q_\gamma(x,y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$
\begin{aligned}
|\mathscr{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leqslant & \left|\int_{\rho(x,y)>\varepsilon} [V(x,y) - V(x_0,y)]f_2(y)d\mu(y)\right| + \int_{R(x)} |V(x_0,y)f(y)|d\mu(y) \\
& + \int_{R(x_0)} |V(x_0,y)f(y)|d\mu(y) + |\mathscr{I}_\varepsilon^{\gamma b_1}(f)(x_0)| \\
= & I' + II' + III' + IV',
\end{aligned}
$$

where $R(u) = \{y \in X : \mathrm{diam}(\tilde{B}_k) < \rho(u,y) \leqslant H\mathrm{diam}(\tilde{B}_k)\}$. Now let us treat I', II' and III' respectively. For I', we write

$$
\begin{aligned}
&|V(x,y) - V(x_0,y)| \\
&= |(b_1(x) - b_1(y))Q_\gamma(x,y)f_2(y) - (b_1(x_0) - b_1(y))Q_\gamma(x_0,y)f_2(y)| \\
&= |b_1(x)Q_\gamma(x,y)f_2(y) - b_1(y)Q_\gamma(x,y)f_2(y) - b_1(x_0)Q_\gamma(x_0,y)f_2(y) \\
&\quad + b_1(y)Q_\gamma(x_0,y)f_2(y) - b_1(x_0)Q_\gamma(x,y)f_2(y) + b_1(x_0)Q_\gamma(x,y)f_2(y)| \\
&\leqslant |(b_1(x) - b_1(x_0))Q_\gamma(x,y)f_2(y)| + |(b_1(x_0) - b_1(y))(Q_\gamma(x,y) - Q_\gamma(x_0,y))f_2(y)| \\
&= I_1' + I_2'.
\end{aligned}
$$

For I_1', by Hölder's inequality and the following inequality, for $b \in BMO(X)$

$$
|b(x) - b_B| \leqslant \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x) \leqslant ||b||_{BMO},
$$

we have

$$
\begin{aligned}
\int_{\rho(x,y)>\varepsilon} I_1 d\mu(y) &\leqslant C \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |b_1(x) - b_1(x_0)||Q_\gamma(x,y)||f(y)|d\mu(y) \\
&\leqslant C||b_1||_{BMO} \sum_{v=1}^{\infty} \mu(B_{2^v\varepsilon}(x_0))^{\gamma-1} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y)\right)^{1/p} \\
&\quad \times \mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p} \\
&\leqslant C||b_1||_{BMO} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-\gamma} \\
&\quad \times \left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-\gamma p}} \int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y)\right)^{1/p} \\
&\leqslant C||b_1||_{BMO} M_{\gamma,p}(f)(z) \\
&\leqslant C\xi\lambda.
\end{aligned}
$$

For I_2', similar to I_2, we obtain

$$\begin{aligned}
\int_{\rho(x,y)>\varepsilon} I_2 d\mu(y) &\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)\backslash B_{2^v\varepsilon}(x_0)} |(b_1(x_0)-b_1(y))(Q_\gamma(x,y)-Q_\gamma(x_0,y))f(y)|d\mu(y)\\
&\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0)-b_1(y)|\frac{|\eta(x_0,y)^{1-\gamma}-\eta(x,y)^{1-\gamma}|}{|\eta(x,y)^{1-\gamma}\eta(x_0,y)^{1-\gamma}|}|f(y)|d\mu(y)\\
&\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0)-b_1(y)|\frac{|\eta(x_0,x)^{1-\gamma}|}{|\eta(x_0,y)^{2(1-\gamma)}|}|f(y)|d\mu(y)\\
&\leqslant C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0)-b_1(y)|\frac{\eta(x_0,x)^{\theta}}{\eta(x_0,y)^{1-\gamma+\theta}}|f(y)|d\mu(y)\\
&\leqslant C\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^\theta}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-\gamma+\theta}}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'}\\
&\quad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C\sum_{v=1}^{\infty}\mu(\tilde{B}_k)^\theta\mu(B_{2^{v+1}\varepsilon})(x_0)^{-1+\gamma-\theta+1/p'+1/p-\gamma}||b_1||_{\wedge_\beta}M_{\gamma,p}(f)(z)\\
&\leqslant C\sum_{v=1}^{\infty}2^{-v\theta}||b_1||_{BMO}M_{\gamma,p}(f)(z)\\
&\leqslant C||b_1||_{BMO}M_{\gamma,p}(f)(z)\\
&\leqslant C\xi\lambda.
\end{aligned}$$

Therefore $I'\leqslant C\xi\lambda$.

For II' and III', note that, for $y\in R(x)$,

$$\rho(x,y)\leqslant H\mathrm{diam}(\tilde{B}_k),$$

we get, by Hölder's inequality

$$\begin{aligned}
II' &\leqslant C\int_{H\tilde{B}_k}|b_1(x_0)-b_1(y)||Q_\gamma(x,y)||f(y)|d\mu(y)\\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1}\left(\int_{H\tilde{B}_k}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{H\tilde{B}_k}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1+1/p'+1/p-\gamma}||b_1||_{BMO}\left(\frac{1}{\mu(H\tilde{B}_k)^{1-\gamma p}}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C||b_1||_{BMO}M_{\gamma,p}(f)(z)\\
&\leqslant C\xi\lambda.
\end{aligned}$$

Similar $III'\leqslant C\xi\lambda$.

Thus $I'+II'+III'\leqslant C\xi\lambda$.

For IV', since $x \notin \bigcup_k Q_k$, then $|\mathscr{I}_\varepsilon^{\gamma b_1}(f)(x_0)| \leqslant \lambda$. For $x \in B_j$,

$$\sup_{\varepsilon \approx \mathrm{diam}(\tilde{B}_k)} |\mathscr{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leqslant C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H\mathrm{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of *Case 1*, we get

$$\sup_{\varepsilon > \mathrm{diam}(\tilde{B}_k)} |\mathscr{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leqslant C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$\mathscr{I}_\star^{\gamma b_1}(f_2)(x) \leqslant C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned}
&\mu\left(\left\{x \in B_k : I_\star^{\gamma b_1}(f)(x) > 3\lambda, ||b_1||_{BMO} M_{\gamma,p}(f)(x) \leqslant \xi\lambda\right\}\right)\\
&\quad \leqslant \mu(\{x \in B_k : I_\star^{\gamma b_1}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : I_\star^{\gamma b_1}(f_2)(x) > \lambda + C\xi\lambda\})\\
&\quad \leqslant \mu(\{x \in B_k : I_\star^{\gamma b_1}(f_1)(x) > \lambda\}) \leqslant \xi^r \mu(B_k).
\end{aligned}$$

When $\mathbf{m} > \mathbf{1}$, similar to the case $\mathbf{m} = \mathbf{1}$, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathscr{I}_\star^{\gamma\vec{b}}(f)(x_0) \leqslant \lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$||\vec{b}||_{BMO} M_{\gamma,p}(f)(z) \leqslant \xi\lambda.$$

Set $\overline{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{B}_k}$ and $f_2 = f\chi_{X\setminus\overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1. For $x \in B_k$, $1/r = 1/p + 1/q_j < 1$, $j = 1, \cdots, m$

$$\begin{aligned}
||\mathscr{I}_\star^{\gamma\vec{b}}(f_1)||_{L^r} &\leqslant C\left(\int_{\rho(x,y)>\varepsilon} \left|\prod_{j=1}^m (b_j(x) - b_j(y)) Q_\gamma(x,y) f_1(y)\right|^r d\mu(y)\right)^{1/r}\\
&\leqslant C\mu(\overline{B}_k)^{\gamma-1}\left(\int_{\overline{B}_k} |\prod_{j=1}^m b_j(x) - b_j(y)|^{q_j} d\mu(y)\right)^{1/q_j}\left(\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p}\\
&\leqslant C\mu(\overline{B}_k)^{\gamma-1}\mu(\overline{B}_k)^{\sum_{j=1}^m 1/q_j} ||\vec{b}||_{BMO}\, \mu(\overline{B}_k)^{1/p-\gamma}\\
&\quad \times\left(\frac{1}{\mu(\overline{B}_k)^{1-\gamma p}}\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p}
\end{aligned}$$

$$\leqslant C\mu(\overline{B}_k)^{1/r-1}||\vec{b}||_{BMO}M_{\gamma,p}(f)(z).$$

Let $\eta > 0$, we have

$$\begin{aligned}\mu(\{x\in X:\mathscr{I}_\star^{\gamma\vec{b}}(f_1)(x)>\eta\lambda\}) &\leqslant C(\eta\lambda)^{-r}||\mathscr{I}_\star^{\gamma\vec{b}}(f_1)||_{L^r}^r\\ &\leqslant C(\eta\lambda)^{-r}[||\vec{b}||_{BMO}M_{\gamma,p}(f)(z)]^r\mu(\overline{B}_k)^{1-1/r}\\ &\leqslant C(\eta\lambda)^{-r}(\xi\lambda)^r\mu(\overline{B}_k)\\ &\leqslant C(\xi/\eta)^r\mu(B_k).\end{aligned}$$

The estimates on f_2. Let $H = H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k) \leqslant \varepsilon \leqslant H\mathrm{diam}(\tilde{B}_k)$. Set $U(x,y)=\prod_{j=1}^m(b_j(x)-b_j(y))Q_\gamma(x,y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X\setminus\bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned}|\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| \leqslant& \left|\int_{\rho(x,y)>\varepsilon}[U(x,y)-U(x_0,y)]f_2(y)d\mu(y)\right|+\int_{R(x)}|U(x_0,y)f(y)|d\mu(y)\\ &+\int_{R(x_0)}|U(x_0,y)f(y)|d\mu(y)+|\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f)(x_0)|\\ =& J'+JJ'+JJJ'+JJJJ',\end{aligned}$$

where $R(u) = \{y \in X : \mathrm{diam}(\tilde{B}_k) < \rho(u,y) \leqslant H\mathrm{diam}(\tilde{B}_k)\}$. Now let us treat J', JJ' and JJJ' respectively. For J', we write

$$\begin{aligned}J' \leqslant& |(b_1(x)-b_1(x_0))\cdots(b_m(x)-b_m(x_0))\int_{\rho(x,y)>\varepsilon}Q_\gamma(x,y)f_2(y)d\mu(y)|\\ &+\sum_{j=1}^m\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}\int_{\rho(x,y)>\varepsilon}(\vec{b}(y)-\vec{b}(x_0))_\sigma(Q_\gamma(x,y)-Q_\gamma(x_0,y)f_2(y)d\mu(y)|\\ =& J_1'+J_2'.\end{aligned}$$

For J_1', we have

$$\begin{aligned}J_1' \leqslant& \prod_{j=1}^m|b_j(x)-b_j(x_0)|\sum_{v=1}^\infty\int_{B_{2^{v+1}\varepsilon}(x_0)\setminus B_{2^v\varepsilon}(x_0)}|Q_\gamma(x,y)||f(y)|d\mu(y)\\ \leqslant& C||\vec{b}||_{BMO}\sum_{v=1}^\infty\mu(B_{2^v\varepsilon}(x_0))^{\gamma-1}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^pd\mu(y)\right)^{1/p}\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p}\\ \leqslant& C||\vec{b}||_{BMO}\sum_{v=1}^\infty\mu(B_{2^{v+1}\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-\gamma}\\ &\times\left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-\gamma p}}\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^pd\mu(y)\right)^{1/p}\\ \leqslant& C||\vec{b}||_{BMO}M_{\gamma,p}(f)(z)\\ \leqslant& C\xi\lambda.\end{aligned}$$

For J_2', we get

$$\begin{aligned}
J_2' &\leqslant \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}\int_{B_{2^{v+1}\varepsilon}(x_0)\setminus B_{2^v\varepsilon}(x_0)}(\vec{b}(y)-\vec{b}(x_0))_\sigma \\
&\quad\times(Q_\gamma(x,y)-Q_\gamma(x_0,y)f(y)d\mu(y)| \\
&\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}|\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^\theta}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}} \\
&\quad\times\int_{B_{2^{v+1}\varepsilon}(x_0)}|(\vec{b}(y)-\vec{b}(x_0))_\sigma f(y)d\mu(y)| \\
&\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}|\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^\theta}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}} \\
&\quad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|(\vec{b}(y)-\vec{b}(x_0))_\sigma|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leqslant C||\vec{b}_{\sigma^c}||_{BMO}\mu(\tilde{B}_k)^\theta\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+\gamma-\theta+1/p'+1/p-\gamma}||\vec{b}_\sigma||_{BMO}M_{\gamma,p}(f)(z) \\
&\leqslant C||\vec{b}||_{BMO}\mu(\tilde{B}_k)^\theta\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-\theta}M_{\gamma,p}(f)(z) \\
&\leqslant C||\vec{b}||_{BMO}\sum_{v=1}^{\infty}2^{-v\theta}M_{\gamma,p}(f)(z) \\
&\leqslant C||\vec{b}||_{BMO}M_{\gamma,p}(f)(z) \\
&\leqslant C\xi\lambda.
\end{aligned}$$

Therefore $J'\leqslant C\xi\lambda$. For JJ' and JJJ', note that, for $y\in R(x)$,

$$\rho(x,y)\leqslant H\mathrm{diam}(\tilde{B}_k),$$

we get, by Hölder's inequality, for $1/p_1'+\cdots 1/p_m'+1/p=1$

$$\begin{aligned}
JJ' &\leqslant C\int_{H\tilde{B}_k}|\prod_{j=1}^{m}b_j(x_0)-b_j(y)||Q_\gamma(x,y)||f(y)|d\mu(y) \\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1}\left(\int_{H\tilde{B}_k}|\prod_{j=1}^{m}b_j(x_0)-b_j(y)|^{p_j'}d\mu(y)\right)^{1/p_j'}\left(\int_{H\tilde{B}_k}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leqslant C\mu(H\tilde{B}_k)^{\gamma-1+1/p_1'+\cdots 1/p_m'+1/p-\gamma}||\vec{b}||_{BMO}\left(\frac{1}{\mu(H\tilde{B}_k)^{1-\gamma p}}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leqslant C||\vec{b}||_{BMO}M_{\gamma,p}(f)(z) \\
&\leqslant C\xi\lambda.
\end{aligned}$$

Similar $JJJ'\leqslant C\xi\lambda$.

Thus $J'+JJ'+JJJ' \leqslant C\xi\lambda$.

For $JJJJ'$, since $x \notin \bigcup_k Q_k$, then $|\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f)(x_0)| \leqslant \lambda$. For $x \in B_j$,

$$\sup_{\varepsilon \approx \operatorname{diam}(\tilde{B}_k)} |\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| \leqslant C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H\operatorname{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of *Case 1*, we get

$$\sup_{\varepsilon > \operatorname{diam}(\tilde{B}_k)} |\mathscr{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| \leqslant C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$\mathscr{I}_\star^{\gamma\vec{b}}(f_2)(x) \leqslant C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned}
&\mu\left(\left\{x \in B_k : I_\star^{\gamma\vec{b}}(f)(x) > 3\lambda, ||\vec{b}||_{BMO} M_{\gamma,p}(f)(x) \leqslant \xi\lambda\right\}\right)\\
&\leqslant \mu(\{x \in B_k : I_\star^{\gamma\vec{b}}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : I_\star^{\gamma\vec{b}}(f_2)(x) > \lambda + C\xi\lambda\})\\
&\leqslant \mu(\{x \in B_k : I_\star^{\gamma\vec{b}}(f_1)(x) > \lambda\}) \leqslant \xi^r \mu(B_k).
\end{aligned}$$

This completes the proof of Theorem 2(a). (b) follows from (a) and Lemma 3. □

REFERENCES

[1] J. ALVAREZ, R.J.BABGY, D.S.KURTZ AND C.PÉREZ, *Weighted estimates for commutators of linear operators*, Studia Math., **104** (1993), 195–209.

[2] J. J. BETANCOR, *A commutator theorem for fractional integrals in spaces of homogeneous type*, Inter. J. Math. & Math. Sci., **6**, 24 (2000), 419–421.

[3] A. BERNARDIS, S. HARTZSTEIN AND G. PRADOLINI, *Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type*, J. Math. Anal. Appl., **322** (2006), 825–846.

[4] M. BRAMANTI AND M. C. CERUTTI, *Commutators of singular integrals and fractional integrals on homogeneous spaces*, Harmonic Analysis and Operator Theory, Contemp. Math., 189, Amer. Math. Soc., 1995, 81–94.

[5] S. CHANILLO, *A note on commutators*, Indiana Univ Math., J., **31** (1982), 7–16.

[6] W. G. CHEN, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica, **16** (2000), 613–626.

[7] W. CHEN AND E. SAWYER, *Endpoint estimates for commutators of fractional integrals on spaces of homogeneous type*, J. Math. Anal. Appl., **282** (2003), 553–566.

[8] J. COHEN AND J. GOSSELIN, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math., **30** (1986), 445–465.

[9] R. COIFMAN, R. ROCHBERG AND G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611–635.

[10] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Math., 16, Amsterdam, 1985.

[11] I. GENEBASHVILI, A. GOGATISHVILI, V. KOKILASHVILI AND M. KRBEC, *Weighted theory for integral transforms on spaces of homogeneous type*, Piman Monogr. and Surveys in Pure and Appl. Math., 92, Addison-Wesley/Longman, 1998.

[12] G. E. HU AND D. C. YANG, *A variant sharp estimate for multilinear singular integral operators*, Studia Math., **141** (2000), 25–42.

[13] S. JANSON, *Mean oscillation and commutators of singular integral operators*, Ark. Math., **16** (1978), 263–270.

[14] R. MACÍAS AND C. SEGOVIA, *Lipschitz functions on spaces of homogeneous type*, Adv. Math., **33** (1979), 257–270.

[15] M. PALUSZYNSKI, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochbeg and Weiss*, Indiana Univ. Math. J., **44** (1995), 1–17.

[16] C. PÉREZ, *Endpoint estimate for commutators of singular integral operators*, J. Func. Anal., **128** (1995), 163–185.

[17] C. PÉREZ AND G. PRADOLINI, *Sharp weighted endpoint estimates for commutators of singular integral operators*, Michigan Math. J., **49** (2001), 23–37.

[18] C. PÉREZ AND R. TRUJILLO-GONZALEZ, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc., **65** (2002), 672–692.

[19] G. PRADOLINI, O. SALINAS, *Commutators of singular integrals on space of homogeneous type*, preprint, available at `http://math.unl.edu.ar/preprints/`.

[20] E. M. STEIN, *Harmonic analysis: teal variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

(Received April 8, 2009)

Zhang Qian
Department of Mathematics
Changsha University of Science and Technology
Changsha, 410077
P. R. of China

Liu Lanzhe
Department of Mathematics
Changsha University of Science and Technology
Changsha, 410077
P. R. of China
e-mail: `lanzheliu@163.com`