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FURTHER EXTENSION OF FURUTA INEQUALITY

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Abstract. If $A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 \geqslant B \geqslant 0$, with $A_1 > 0$, $t_1, t_2, \cdots, t_{n-1}$, $t_n \in [0,1]$ and $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n. Then the following inequality holds for $r \geqslant t_n$

$$\begin{aligned} {A_{2n}}^{1-t_n+r} \geqslant \{&{A_{2n}}^{\frac{r}{2}}[{A_{2n-1}}^{\frac{-t_n}{2}}\{{A_{2(n-1)}}^{\frac{t_{n-1}}{2}} \cdots {A_4}^{\frac{t_2}{2}}[{A_3}^{\frac{-t_2}{2}}\{{A_2}^{\frac{t_1}{2}}({A_1}^{\frac{-t_1}{2}} B^{p_1} {A_1}^{\frac{-t_1}{2}})^{p_2} \\ &{A_2}^{\frac{t_1}{2}}\}^{p_3} {A_3}^{\frac{-t_2}{2}}]^{p_4} {A_4}^{\frac{t_2}{2}} \cdots {A_{2(n-1)}}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}} {A_{2n-1}}^{\frac{-t_n}{2}}]^{p_{2n}} {A_{2n}}^{\frac{r}{2}}\}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}}, \end{aligned}$$

where $\eth[2n] = \{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n$.

1. Introduction

A capital letter means a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$, and T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

THEOREM LH. (Löwner-Heinz inequality, denoted by **(LH)** briefly)

$$If\ A \geqslant B \geqslant 0\ holds,\ then\ A^{\alpha} \geqslant B^{\alpha}\ for\ any\ \alpha \in [0,1]. \qquad \textbf{(LH)}$$

This was originally proved in [18], [15], and then in [19]. Although **(LH)** asserts that $A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$, unfortunately $A^{\alpha} \geqslant B^{\alpha}$ does not always hold for $\alpha \geqslant 1$. The following result was been obtained from this point of view.

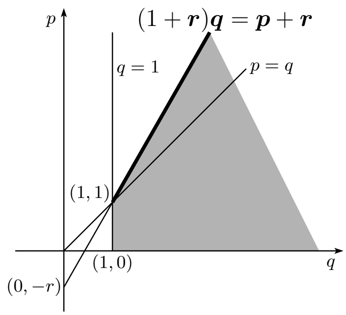
THEOREM F[11]. (Furuta inequality)
If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.



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The original proof of Theorem F is shown in [11], an elementary one-page proof is in [12] and alternative ones are in [2] and [17]. We remark that the domain of the parameters p, q and r in Theorem F is the best possible for the inequalities (i) and (ii) under the assumption $A \geqslant B \geqslant 0$, see [20].

LEMMA 1.1. [8] *Let A be a positive invertible operator and B be an invertible operator. For any real number s,*

$$(BAB^*)^s = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{s-1}A^{\frac{1}{2}}B^*,$$

especially in the case $s \geqslant 1$, the equality holds without invertibility of A and B.

By using this Lemma 1.1 and preceding Theorems in [8], Furuta gives the following Theorem's original proof. An elementary one-page proof is in [9].

THEOREM GF. (Generalized Furuta inequality, denoted by **(GF)** briefly).
If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$, $p \geqslant 1$,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}. \qquad \textbf{(GF)}$$

holds for $r \geqslant t$ and $s \geqslant 1$.

An alternative one is in [3]. We mention that further extensions of Theorem GF and related results to Theorem F are in [4], [13], [14] and etc. It is originally shown in [21] that the exponent value $\frac{1-t+r}{(p-t)s+r}$ of the right hand of (GF) is best possible and alternative ones are in [5], [23]. It is known that the operator inequality (GF) interpolates Theorem F and an inequality equivalent to the main result of Ando-Hiai log majorization [1] by the parameter $t \in [0,1]$.

DEFINITION 1.1. [7] Let $A > 0$, $B \geqslant 0$, $t \in [0,1]$ and $p_1, p_2, \cdots, p_n, \cdots, p_{2n} \geqslant 1$ for a natural number n. Let $C_{A,B}[2n]$ be defined by

$$\begin{aligned} C_{A,B}[2n] &= C_{A,B}[2n; p_1, p_2, \cdots, p_{2(n-1)}, p_{2n-1}, p_{2n}] \\ &= \underbrace{A^{\frac{t}{2}}\{A^{\frac{-t}{2}}[A^{\frac{t}{2}}\cdots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}\cdots A^{\frac{t}{2}}]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}}}_{A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}\rightarrow} \end{aligned} \qquad (1.1)$$

Let $q[2n]$ be defined by

$$\begin{aligned} q[2n] &= q[2n; p_1, p_2, \cdots, p_{2(n-1)}, p_{2n-1}, p_{2n}] \\ &= \underbrace{\{\cdots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\cdots-t\}p_{2n}+t}_{-t \text{ and } t \text{ alternately } n \text{ times } appear}. \end{aligned} \qquad (1.2)$$

For the sake of convenience, we define

$$C_{A,B}[0] = B \quad \text{and} \quad q[0] = 1. \qquad (1.3)$$

THEOREM H. [7] *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \cdots, p_{2n} \geqslant 1$ for natural number n. Then the following inequality holds for $r \geqslant t$,*

$$A^{1-t+r} \geqslant A^{\frac{r}{2}} [\underbrace{A^{\frac{-t}{2}} \{A^{\frac{t}{2}} \cdots [A^{\frac{-t}{2}} \{A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ } n \text{ times and } A^{\frac{t}{2}} \text{ } n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{t}{2}} \cdots A^{\frac{-t}{2}}\}^{p_{2n}}}_{A^{\frac{-t}{2}} \text{ } n \text{ times and } A^{\frac{t}{2}} \text{ } n-1 \text{ times by turns}\rightarrow} A^{\frac{r}{2}}]^{\frac{1-t+r}{\varphi[2n;r,t]}},$$

where $\varphi[2n;r,t] = q[2n] + r - t$.

2. Definitions of $\mathbb{G}_{A_i,B}[2n]$ and $\eth[2n]$

DEFINITION 2.1. Let $A_{2n-1}, A_{2(n-1)}, \cdots, A_2, A_1 > 0$, $A_{2n}, B \geqslant 0$, $t_1, t_2, \cdots, t_{n-1}, t_n \in [0,1]$ and $p_1, p_2, \cdots, p_n, \cdots, p_{2(n-1)}, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n.

Let $\mathbb{G}_{A_i,B}[2n]$ be defined by

$$\begin{aligned}\mathbb{G}_{A_i,B}[2n] &= \mathbb{G}_{A_i,B}[2n; p_1, p_2, \cdots, p_{2(n-1)}, p_{2n-1}, p_{2n}] \\ &= A_{2n}{}^{\frac{t_n}{2}} [A_{2n-1}{}^{\frac{-t_n}{2}} \{A_{2(n-1)}{}^{\frac{t_{n-1}}{2}} \cdots A_4{}^{\frac{t_2}{2}} [A_3{}^{\frac{-t_2}{2}} \{A_2{}^{\frac{t_1}{2}} (A_1{}^{\frac{-t_1}{2}} B^{p_1} A_1{}^{\frac{-t_1}{2}})^{p_2} A_2{}^{\frac{t_1}{2}}\}^{p_3} \\ &\quad A_3{}^{\frac{-t_2}{2}}]^{p_4} A_4{}^{\frac{t_2}{2}} \cdots A_{2(n-1)}{}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}} A_{2n-1}{}^{\frac{-t_n}{2}}]^{p_{2n}} A_{2n}{}^{\frac{t_n}{2}}\end{aligned} \tag{2.1}$$

For examples,

$$\mathbb{G}_{A_i,B}[2] = A_2{}^{\frac{t_1}{2}} (A_1{}^{\frac{-t_1}{2}} B^{p_1} A_1{}^{\frac{-t_1}{2}})^{p_2} A_2{}^{\frac{t_1}{2}}$$

and

$$\mathbb{G}_{A_i,B}[4] = A_4{}^{\frac{t_2}{2}} [A_3{}^{\frac{-t_2}{2}} \{A_2{}^{\frac{t_1}{2}} (A_1{}^{\frac{-t_1}{2}} B^{p_1} A_1{}^{\frac{-t_1}{2}})^{p_2} A_2{}^{\frac{t_1}{2}}\}^{p_3} A_3{}^{\frac{-t_2}{2}}]^{p_4} A_4{}^{\frac{t_2}{2}}$$

Let $\eth[2n]$ be defined by

$$\begin{aligned}\eth[2n] &= \eth[2n; p_1, p_2, \cdots, p_{2(n-1)}, p_{2n-1}, p_{2n}] \\ &= \{\cdots [\{[(p_1 - t_1)p_2 + t_1]p_3 - t_2\}p_4 + t_2]p_5 - \cdots - t_n\}p_{2n} + t_n.\end{aligned} \tag{2.2}$$

For examples,

$$\eth[2] = (p_1 - t_1)p_2 + t_1$$

and

$$\eth[4] = \{[(p_1 - t_1)p_2 + t_1]p_3 - t_2\}p_4 + t_2$$

For the sake of convenience, we define

$$\mathbb{G}_{A_i,B}[0] = B \quad \text{and} \quad \eth[0] = 1. \tag{2.3}$$

The following Lemma is easily shown by (2.1) and (2.2).

LEMMA 2.1. *For $A_{2n+1}, A_{2n}, \cdots, A_2, A_1 > 0$ and $A_{2(n+1)}, B \geqslant 0$ for any natural number n,*
(i) $\mathbb{G}_{A_i,B}[2(n+1)] = A_{2(n+1)}^{\frac{t_{n+1}}{2}}(A_{2n+1}^{\frac{-t_{n+1}}{2}}(\mathbb{G}_{A_i,B}[2n])^{p_{2n+1}}A_{2n+1}^{\frac{-t_{n+1}}{2}})^{p_{2(n+1)}}A_{2(n+1)}^{\frac{t_{n+1}}{2}}$
(ii) $\eth[2(n+1)] = (\eth[2n]p_{2n+1} - t_{n+1})p_{2(n+1)} + t_{n+1}$.

Also we remark that (2.2) easily implies

$$\begin{aligned}\eth[2n] &= \{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n \\ &= \prod_{i=1}^{2n}p_i + \sum_{j=1}^{n-1}(t_j\prod_{i=2j+1}^{2n}p_i) + t_n - \sum_{j=1}^{n}(t_j\prod_{i=2j}^{2n}p_i)\end{aligned}$$

holds for any natural number n.

3. Statement of results

THEOREM 3.1. *If $A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 \geqslant B \geqslant 0$, with $A_1 > 0$, $t_1, t_2, \cdots, t_{n-1}, t_n \in [0,1]$ and $p_1, p_2, \cdots, p_n, \cdots, p_{2(n-1)}, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n. Then the following inequality holds*

$$\begin{aligned}A_{2n} \geqslant &\{A_{2n}^{\frac{t_n}{2}}[A_{2n-1}^{\frac{-t_n}{2}}\{A_{2(n-1)}^{\frac{t_{n-1}}{2}}\cdots A_4^{\frac{t_2}{2}}[A_3^{\frac{-t_2}{2}}\{A_2^{\frac{t_1}{2}}(A_1^{\frac{-t_1}{2}}B^{p_1}A_1^{\frac{-t_1}{2}})^{p_2}A_2^{\frac{t_1}{2}}\}^{p_3} \\ &A_3^{\frac{-t_2}{2}}]^{p_4}A_4^{\frac{t_2}{2}}\cdots A_{2(n-1)}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n-1}^{\frac{-t_n}{2}}]^{p_{2n}}A_{2n}^{\frac{t_n}{2}}\}^{\frac{1}{\eth[2n]}} \\ = &(\mathbb{G}_{A_i,B}[2n])^{\frac{1}{\eth[2n]}}\end{aligned} \tag{3.1}$$

where $\eth[2n]$ is in (2.2).

COROLLARY 3.1. *If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n, Then the following inequality holds,*

$$A \geqslant \{A^{\frac{t}{2}}[A^{\frac{-t}{2}}(C_{A,B}[2(n-1)])^{p_{2n-1}}A^{\frac{-t}{2}}]^{p_{2n}}A^{\frac{t}{2}}\}^{\frac{1}{q[2n]}} \tag{3.2}$$

In order to prove the preceding conclusions, we need the following results.

LEMMA 3.1. (F. Hansen[16]) *If X and A are bounded linear operators on a Hilbert space with $X \geqslant 0$, $\|A\| \leqslant 1$, and f is an operator monotone function on interval $[0,+\infty)$. then*

$$A^*f(X)A \leqslant f(A^*XA).$$

THEOREM A. [10, 22] *If $A \geqslant B \geqslant C \geqslant 0$ with $B > 0$, then for $t \in [0,1]$, $p \geqslant 1$, $s \geqslant 1$ and $r \geqslant t$*

(i) $A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$,

(ii) $C^{1-t+r} \leqslant \{C^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$.

Put $r = t$ in (i) and (ii) of Theorem A, we have *(iii)* $A \geqslant \{A^{\frac{t}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$,

(iv) $C \leqslant \{C^{\frac{t}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$.

REMARK 3.1. (i) and (ii) in Theorem A can be obtained by (iii) and (iv) of Theorem A and Theorem F respectively. Now we shall give a simple proof of (iii) of Theorem A which is different from ones in [10] and [22] as follows:

Without loss of generality, we can assume C is invertible. Firstly, Let's prove the following inequality to hold for $p \geqslant 1,\ s \geqslant 1$ and $t \in [0,1]$,

$$\{A^{\frac{-t}{2}}(B^{\frac{t}{2}}C^{-p}B^{\frac{t}{2}})^sA^{\frac{-t}{2}}\}^{\frac{1}{(p-t)s+t}} \geqslant A^{-1}$$

Since $A \geqslant B > 0,\ t \in [0,1]$, we have $A^t \geqslant B^t$ by **LH**, and yields $A^{\frac{-t}{2}}B^tA^{\frac{-t}{2}} \leqslant I$, because $x^{\frac{1}{(p-t)s+t}}$ is an operator monotone function, by Lemma 3.1 and Theorem GF, we have

$$\begin{aligned}\{A^{\frac{-t}{2}}(B^{\frac{t}{2}}C^{-p}B^{\frac{t}{2}})^sA^{\frac{-t}{2}}\}^{\frac{1}{(p-t)s+t}} &= \{A^{\frac{-t}{2}}B^{\frac{t}{2}}B^{\frac{-t}{2}}(B^{\frac{t}{2}}C^{-p}B^{\frac{t}{2}})^sB^{\frac{-t}{2}}B^{\frac{t}{2}}A^{\frac{-t}{2}}\}^{\frac{1}{(p-t)s+t}}\\ &\geqslant A^{\frac{-t}{2}}B^{\frac{t}{2}}(B^{\frac{-t}{2}}(B^{\frac{t}{2}}C^{-p}B^{\frac{t}{2}})^sB^{\frac{-t}{2}})^{\frac{1}{(p-t)s+t}}B^{\frac{t}{2}}A^{\frac{-t}{2}}\\ &\geqslant A^{\frac{-t}{2}}B^{t-1}A^{\frac{-t}{2}} \geqslant A^{-1}\end{aligned}$$

this is

$$A \geqslant \{A^{\frac{t}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}.$$

Proof of Theorem 3.1.

First step. (3.1) for $n=1$ is shown by (iii) of Theorem A, that is, if $A_2 \geqslant A_1 \geqslant B \geqslant 0$ with $A_1 > 0$, then for $t_1 \in [0,1]$ and $p_1 \geqslant 1, p_2 \geqslant 1$

$$A_2 \geqslant \{A_2^{\frac{t_1}{2}}(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}})^{p_2}A_2^{\frac{t_1}{2}}\}^{\frac{1}{(p_1-t_1)p_2+t_1}} \tag{3.3}$$

Second step. Assume (3.1) holds for n, that is, if $A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 \geqslant B \geqslant 0$, with $A_1 > 0, t_1, t_2, \cdots, t_n \in [0,1]$ and $p_1, p_2, \cdots p_n, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n,

$$A_{2n} \geqslant \mathbb{G}_{A,B}[2n]^{\frac{1}{\eth[2n]}}. \tag{3.1}$$

Then we shall show (3.1) for $n+1$ by Induction as follows. Put $D = A_{2(n+1)}, E = A_{2n+1}$ and $F = \mathbb{G}_{A,B}[2n]^{\frac{1}{\eth[2n]}}$. The hypothesis for $n+1$ and (3.1) imply

$$D \geqslant E \geqslant F \geqslant 0 \ \text{ with } \ E > 0. \tag{3.4}$$

(3.4) yields the following (3.5) by (iii) of Theorem A, for $t \in [0,1], p \geqslant 1$ and $s \geqslant 1$:

$$D \geqslant \{D^{\frac{t}{2}}(E^{-\frac{t}{2}}F^pE^{-\frac{t}{2}})^sD^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}. \tag{3.5}$$

Put $t = t_{n+1} \in [0,1], p = \eth[2n]p_{2n+1} \geqslant 1$ for $p_{2n+1} \geqslant 1$ and $s = p_{2(n+1)} \geqslant 1$ in (3.5). Then by (ii) of Lemma 2.1, (3.5) implies

$$(p-t)s+t = (\eth[2n]p_{2n+1} - t_{n+1})p_{2(n+1)} + t_{n+1} = \eth[2(n+1)] \tag{3.6}$$

$$A_{2(n+1)} \geqslant \{A_{2(n+1)}^{\frac{t_{n+1}}{2}}(A_{2n+1}^{\frac{-t_{n+1}}{2}}\mathbb{G}_{A,B}[2n]^{p_{2n+1}}A_{2n+1}^{\frac{-t_{n+1}}{2}})^{p_{2(n+1)}}A_{2(n+1)}^{\frac{t_{n+1}}{2}}\}^{\frac{1}{\eth[2(n+1)]}}$$
$$= \mathbb{G}_{A,B}[2(n+1)]^{\frac{1}{\eth[2(n+1)]}}, \qquad \text{by (i) of Lemma 2.1 and (3.6)} \tag{3.7}$$

and (3.7) means that (3.1) holds for $n+1$. Whence the proof is complete. □

Proof of Corollary 3.1. We have only to put $A = A_1 = A_2 = A_3 = \cdots = A_{2(n-1)} = A_{2n-1} = A_{2n}$, $t = t_1 = t_2 = \cdots = t_{n-1} = t_n$ in Theorem 3.1. □

THEOREM 3.2. *If $B \geqslant C_1 \geqslant C_2 \geqslant \cdots \geqslant C_{n-1} \geqslant C_n \geqslant C_{n+1} \geqslant 0$ with $C_n > 0$, $t_1, t_2, \cdots, t_{n-1}, t_n \in [0,1]$ and $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n. Then the following inequality holds*

$$C_{2n} \leqslant \{C_{2n}{}^{\frac{t_n}{2}}[C_{2n-1}{}^{\frac{-t_n}{2}}\{C_{2(n-1)}{}^{\frac{t_{n-1}}{2}}\cdots C_4{}^{\frac{t_2}{2}}[C_3{}^{\frac{-t_2}{2}}\{C_2{}^{\frac{t_1}{2}}(C_1{}^{\frac{-t_1}{2}}B^{p_1}C_1{}^{\frac{-t_1}{2}})^{p_2}C_2{}^{\frac{t_1}{2}}\}^{p_3}$$
$$C_3{}^{\frac{-t_2}{2}}]^{p_4}C_4{}^{\frac{t_2}{2}}\cdots C_{2(n-1)}{}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}C_{2n-1}{}^{\frac{-t_n}{2}}]^{p_{2n}}C_{2n}{}^{\frac{t_n}{2}}\}^{\frac{1}{\eth[2n]}}$$
$$= (\mathbb{G}_{C_i,B}[2n])^{\frac{1}{\eth[2n]}} \tag{3.8}$$

Proof. We can assume $C_{2n} > 0$ without loss of generality, then $C_{2n}{}^{-1} \geqslant C_{2n-1}{}^{-1} \geqslant \cdots \geqslant C_2{}^{-1} \geqslant C_1{}^{-1} \geqslant B^{-1} \geqslant 0$.

Since (3.1) of Theorem 3.1 holds for $t_1, t_2, \cdots, t_{n-1}, t_n \in [0,1]$ and $p_1, p_2, \cdots, p_{2n} \geqslant 1$ for a natural number n and by putting $A_{2n} = C_{2n}{}^{-1}$, $A_{2n-1} = C_{2n-1}{}^{-1}, \cdots, A_2 = C_2{}^{-1}$, $A_1 = C_1{}^{-1}$, $B = B^{-1}$ in (3.1), we have

$$C_{2n}{}^{-1} \geqslant \{C_{2n}{}^{\frac{-t_n}{2}}[C_{2n-1}{}^{\frac{t_n}{2}}\{C_{2(n-1)}{}^{\frac{-t_{n-1}}{2}}\cdots C_4{}^{\frac{-t_2}{2}}[C_3{}^{\frac{t_2}{2}}\{C_2{}^{\frac{-t_1}{2}}(C_1{}^{\frac{t_1}{2}}B^{-p_1}C_1{}^{\frac{t_1}{2}})^{p_2}C_2{}^{\frac{-t_1}{2}}\}^{p_3}$$
$$C_3{}^{\frac{t_2}{2}}]^{p_4}C_4{}^{\frac{-t_2}{2}}\cdots C_{2(n-1)}{}^{\frac{-t_{n-1}}{2}}\}^{p_{2n-1}}C_{2n-1}{}^{\frac{t_n}{2}}]^{p_{2n}}C_{2n}{}^{\frac{-t_n}{2}}\}^{\frac{1}{\eth[2n]}}$$

or equivalently

$$C_{2n} \leqslant \{C_{2n}{}^{\frac{t_n}{2}}[C_{2n-1}{}^{\frac{-t_n}{2}}\{C_{2(n-1)}{}^{\frac{t_{n-1}}{2}}\cdots\cdots C_4{}^{\frac{t_2}{2}}[C_3{}^{\frac{-t_2}{2}}\{C_2{}^{\frac{t_1}{2}}(C_1{}^{\frac{-t_1}{2}}B^{p_1}C_1{}^{\frac{-t_1}{2}})^{p_2}C_2{}^{\frac{t_1}{2}}\}^{p_3}$$
$$C_3{}^{\frac{-t_2}{2}}]^{p_4}C_4{}^{\frac{t_2}{2}}\cdots C_{2(n-1)}{}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}C_{2n-1}{}^{\frac{-t_n}{2}}]^{p_{2n}}C_{2n}{}^{\frac{t_n}{2}}\}^{\frac{1}{\eth[2n]}}$$
$$= (\mathbb{G}_{C_i,B}[2n])^{\frac{1}{\eth[2n]}} \qquad \square$$

THEOREM 3.3. *If $A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 \geqslant B \geqslant 0$, with $A_1 > 0$, $t_1, t_2, \cdots, t_{n-1}, t_n \in [0,1]$ and $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n. Then the following inequality holds for $r \geqslant t_n$*

$$A_{2n}{}^{1-t_n+r} \geqslant \{A_{2n}{}^{\frac{r}{2}}[A_{2n-1}{}^{\frac{-t_n}{2}}\{A_{2(n-1)}{}^{\frac{t_{n-1}}{2}}\cdots\cdots A_4{}^{\frac{t_2}{2}}[A_3{}^{\frac{-t_2}{2}}\{A_2{}^{\frac{t_1}{2}}(A_1{}^{\frac{-t_1}{2}}B^{p_1}A_1{}^{\frac{-t_1}{2}})^{p_2}$$
$$A_2{}^{\frac{t_1}{2}}\}^{p_3}A_3{}^{\frac{-t_2}{2}}]^{p_4}A_4{}^{\frac{t_2}{2}}\cdots A_{2(n-1)}{}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n-1}{}^{\frac{-t_n}{2}}]^{p_{2n}}A_{2n}{}^{\frac{r}{2}}\}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}}$$
$$= \{A_{2n}{}^{\frac{r}{2}}[A_{2n-1}{}^{\frac{-t_n}{2}}(\mathbb{G}_{A_i,B}[2(n-1)])^{p_{2n-1}}A_{2n-1}{}^{\frac{-t_n}{2}}]^{p_{2n}}A_{2n}{}^{\frac{r}{2}}\}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}} \tag{3.9}$$

Proof. Put $D = A_{2n}$, $E = (\mathbb{G}_{A_i,B}[2n])^{\frac{1}{\eth[2n]}}$ in (3.1) of Theorem 3.1. Then $D \geqslant E$ by (3.1) for $t_1, t_2, \cdots, t_{n-1}, t_n \in [0,1]$ and $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n, by applying Theorem F,

$$D^{1+r_1} \geqslant (D^{\frac{r_1}{2}} E^{s_1} D^{\frac{r_1}{2}})^{\frac{1+r_1}{s_1+r_1}} \text{ holds for } s_1 \geqslant 1 \text{ and } r_1 \geqslant 0. \tag{3.10}$$

In (3.10) we have only to put $r_1 = r - t_n \geqslant 0$ and $s_1 = \eth[2n] \geqslant 1$ to obtain (3.9) since $s_1 + r_1 = \eth[2n] - t_n + r$.

So the proof is completed. □

REMARK 3.2. Theorem 3.3 becomes Theorem H, when $A_1 = A_2 = \cdots = A_{2n} = A$ and $t_1 = t_2 = \cdots = t_n = t$ hold.

THEOREM 3.4. *If $B \geqslant C_1 \geqslant C_2 \geqslant \cdots \geqslant C_{2(n-1)} \geqslant C_{2n-1} \geqslant C_{2n} \geqslant 0$ with $C_{2n-1} > 0$, $t_1, t_2, \cdots, t_{n-1}, t_n \in [0,1]$ and $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n. Then the following inequality holds for $r \geqslant t_n$*

$$\begin{aligned} {C_{2n}}^{1-t_n+r} \leqslant \{&{C_{2n}}^{\frac{r}{2}}[{C_{2n-1}}^{\frac{-t_n}{2}}\{{C_{2(n-1)}}^{\frac{t_{n-1}}{2}} \cdots {C_4}^{\frac{t_2}{2}}[{C_3}^{\frac{-t_2}{2}}\{{C_2}^{\frac{t_1}{2}}({C_1}^{\frac{-t_1}{2}} B^{p_1} {C_1}^{\frac{-t_1}{2}})^{p_2} \\ &{C_2}^{\frac{t_1}{2}}\}^{p_3} {C_3}^{\frac{-t_2}{2}}]^{p_4} {C_4}^{\frac{t_2}{2}} \cdots {C_{2(n-1)}}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}} {C_{2n-1}}^{\frac{-t_n}{2}}]^{p_{2n}} {C_{2n}}^{\frac{r}{2}}\}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}} \\ = \{&{C_{2n}}^{\frac{r}{2}}[{C_{2n-1}}^{\frac{-t_n}{2}}(\mathbb{G}_{C_i,B}[2(n-1)])^{p_{2n-1}} {C_{2n-1}}^{\frac{-t_n}{2}}]^{p_{2n}} {C_{2n}}^{\frac{r}{2}}\}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}} \end{aligned} \tag{3.11}$$

Proof. The proof is similar to the proof of Theorem 3.2. □

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