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BOUNDS FOR EIGENVALUES OF A GRAPH

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Abstract. New lower bounds for eigenvalues of a simple graph are derived. Upper and lower bounds for eigenvalues of bipartite graphs are presented in terms of traces and degree of vertices. Finally a non-trivial lower bound for the algebraic connectivity of a connected graph is given.

1. Introduction

Let $\mathbf{G}=(\mathbf{V},\mathbf{E})$ be a simple graph with vertex set $\mathbf{V} = \{v_1,v_2,\dots,v_n\}$ and edge set $\mathbf{E}$ of cardinality e. We assume d_i is the degree of the vertex v_i, $1\leqslant i\leqslant n$. For $v\in\mathbf{V}$, the set of its neighbours is denoted by N_v. Let $|N_v|$ denote the cardinality of N_v and $c_{ij}=|N_{v_i}\cap N_{v_j}|$.

Let $\mathbf{A}$ be a real or complex matrix of order n with real eigenvalues λ_i, $1\leqslant i\leqslant n$. When $\mathbf{A}$ is the adjacency matrix of a simple graph $\mathbf{G}$, let $\lambda_i(\mathbf{G})\equiv\lambda_i$, $1\leqslant i\leqslant n$ denote the eigenvalues of graph $\mathbf{G}$. We always assume that these are arranged in decreasing order,

$$\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n.$$

Recall that the spread of $\mathbf{A}$ is, $\operatorname{spr}\mathbf{A}=\lambda_1-\lambda_n$. Also the Laplacian $\mathbf{L}$,

$$\mathbf{L}=\mathbf{D}-\mathbf{A} \tag{1}$$

is the difference of the diagonal matrix, $\mathbf{D}=\operatorname{diag}(d_1,\dots,d_n)$ and the adjacency matrix $\mathbf{A}$ of graph $\mathbf{G}$. It is easily seen that $\mathbf{L}$ is a positive semidefinite matrix with each row sum equal to zero. Further zero is an eigenvalue of $\mathbf{L}$ with eigenvector $(1,1,\dots,1)^t$. We will denote the algebraic connectivity of $\mathbf{G}$-the second smallest eigenvalue of $\mathbf{L}$ by, $\alpha(\mathbf{G})$. It is well known that $\alpha(\mathbf{G})$ is greater than zero if and only if $\mathbf{G}$ is a connected graph(e.g. see[3]).

Given a square matrix $\mathbf{B}$ of order n, $\mathbf{B}^t$ will denote its transpose, and $\operatorname{tr}\mathbf{B}$ the trace of $\mathbf{B}$. Let $r_i(\mathbf{B})$ denote the 2-norm of the i^{th} row of the matrix $\mathbf{B}$ and $r_{min}(\mathbf{B})$ denote the minimum of the 2-norms of all rows of $\mathbf{B}$. Further by $\|.\|$ we will denote the 2-norm and by $\lfloor x\rfloor$ the greatest integer function.

In section 2 we give new lower bounds for the largest eigenvalue $\lambda_1(\mathbf{G})$, in terms of degree and/or the number of common neighbours of vertices, namely c_{ij}'s . In section 3 we give more results when $\mathbf{G}$ is a bipartite graph, in terms of traces of $\mathbf{A}$ and degree of vertices. In section 4, we present a non-trivial lower bound for $\alpha(\mathbf{G})$.

Below we state four results, that will be used in our study in the subsequent sections. The first result follows from Mirsky [6]:

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THEOREM A. *Let* $\mathbf{B}$ *be a real symmetric square matrix of order n. Then,*

$$\operatorname{spr}\mathbf{A} = 2\max\{|x^*\mathbf{A}y| \mid x,y \in C^n, \|x\| = \|y\| = 1, x^*y = 0\}.$$

Next two theorems are proved in Wolkowicz and Styan [9].

THEOREM B. *Let* $\mathbf{B}$ *be an* $n\times n$ *complex or real matrix with real eigenvalues and let*

$$m = \operatorname{tr}\mathbf{B}/n, \qquad s^2 = \operatorname{tr}\mathbf{B}^2/n - m^2. \tag{2}$$

Then

$$m - s(n-1)^{1/2} \leqslant \lambda_n \leqslant m - s/(n-1)^{1/2},$$
$$m + s/(n-1)^{1/2} \leqslant \lambda_1 \leqslant m + s(n-1)^{1/2}.$$

THEOREM C. *Let* $\mathbf{B}$*, m and* s^2 *be defined as in Theorem B. Then for* $2 \leqslant k \leqslant n-1$,

$$m - s\left(\frac{k-1}{n-k+1}\right)^{1/2} \leqslant \lambda_k \leqslant m + s\left(\frac{n-k}{k}\right)^{1/2}. \tag{3}$$

The final theorem is from Hong and Pan [5]:

THEOREM D. *Let* $\mathbf{B}$ *be a Hermitian positive semidefinite square matrix of order n. Then*

$$\lambda_n \geqslant \left(\frac{n-1}{n}\right)^{\frac{n-1}{2}} |\det\mathbf{B}| \frac{r_{min}(\mathbf{B})}{\prod_{k=1}^n r_k(\mathbf{B})}$$

2. Simple Graphs

Now we obtain lower bounds for the spectral radius for a simple graph $\mathbf{G}$.The following result ([2]) is always at least as good as the known inequality

$$\lambda_1(\mathbf{G}) \geqslant \max_i \sqrt{d_i}.$$

THEOREM 1. *Let* $\mathbf{A}$ *be the adjacency matrix of order* $n \geqslant 2$ *of a simple graph* $\mathbf{G}$*. Then*

$$\lambda_1(\mathbf{G}) \geqslant \frac{1}{\sqrt{2}} \max_{j<i} \sqrt{d_i + d_j + \sqrt{(d_i-d_j)^2 + 4c_{ij}^2}}. \tag{4}$$

Proof. The largest eigenvalue of $\mathbf{A}$ is greater than or equal to the largest eigenvalue of any principal submatrix of $\mathbf{A}$. Any principal submatrix of order two of $\mathbf{A^2}$ is,

$$\begin{pmatrix} \mathbf{a_i^t a_i} & \mathbf{a_i^t a_j} \\ \mathbf{a_j^t a_i} & \mathbf{a_j^t a_j} \end{pmatrix},$$

where $\mathbf{a_i}$ is the i^{th} column vector of the matrix $\mathbf{A}$. The square root of its largest eigenvalue is

$$\frac{1}{\sqrt{2}} \sqrt{d_i + d_j + \sqrt{(d_i-d_j)^2 + 4c_{ij}^2}}.$$

Now the inequality follows.

THEOREM 2. *When* $\mathbf{A}$ *is the adjacency matrix of order n of a simple graph* $\mathbf{G}$, *we have*

$$\left[\frac{1}{n}\sum_{i=1}^{n}(d_i+2\sum_{j>i}c_{ij})^2\right]^{1/4} \leqslant \lambda_1(\mathbf{G}). \tag{5}$$

Proof. For a real symmetric $\mathbf{B}$, using Rayleigh quotient, with $u=(1,\ldots,1)^t$, we have,

$$\lambda_1^2 \geqslant \frac{\mathbf{u^tB^2u}}{n} = \frac{(\mathbf{Bu})^t(\mathbf{Bu})}{n} = \sum_{i=1}^{n} r_i^2/n, \tag{6}$$

where, r_i is the i^{th} row sum of $\mathbf{B}$.

Also $\mathbf{A^2}=(c_{ij})$, with, $c_{ii}=d_i$. The inequality (5) follows by applying (6) to $\mathbf{B}=\mathbf{A^2}$.

The result below generalizes the inequality derived in Johnson et al. [7].

THEOREM 3. *When* $\mathbf{A}$ *is the adjacency matrix of order n of a simple graph* $\mathbf{G}$, *we have*

$$\operatorname{spr}\mathbf{A} \geqslant 2\frac{\sum_{i=1}^{n} d_i^{1+\alpha} - \sum_{i=1}^{n} d_i d}{n\sqrt{\frac{\sum_{i=1}^{n} d_i^{2\alpha}}{n} - d^2}},$$

where, $d=\frac{\sum_{i=1}^{i=n} d_i^{\alpha}}{n}$ *and* $\alpha \geqslant 1$.

Proof. Let $\mathbf{y}=\frac{1}{\sqrt{n}}(1,1,\ldots,1)^t$ and vector u with $u_i=\frac{d_i^{\alpha}}{d}-1, 1\leqslant i\leqslant n$, where $d=\frac{\sum_{i=1}^{n} d_i^{\alpha}}{n}$. Then with $x=\frac{u}{\|u\|}$, in Theorem A, we get

$$\operatorname{spr}\mathbf{A} \geqslant \frac{2}{\sqrt{n}\|u\|}\left|\frac{\sum_{i=1}^{n} d_i^{1+\alpha}}{d} - \sum_{i=1}^{n} d_i\right|.$$

Furthermore

$$\frac{\sum_{i=1}^{n} d_i^{1+\alpha}}{d} - \sum_{i=1}^{n} d_i$$

is positive for $\alpha=1$ and is an increasing function of α, for $\alpha\geqslant 1$. The proof is now clear.

3. Bipartite Graphs

3.1. More eigenvalue bounds

When $\mathbf{A}$ is an adjacency matrix of order n of a simple graph $\mathbf{G}$, with n vertices, $\operatorname{tr}\mathbf{A}^2=2e$ and $\operatorname{tr}\mathbf{A}^4=2(e+\sum_{j\neq i}c_{ij}+4c_4(\mathbf{G}))$, where $c_4(\mathbf{G})$ is number of 4-cycles in $\mathbf{G}$ (Harary [4]). Below we utilize the theorems of Wolkowicz and Styan [9] (see Section 1) to get bounds in terms of traces of powers of $\mathbf{A}$, when $\mathbf{G}$ a bipartite graph.

For the adjacency matrix $\mathbf{A}$ of a bipartite graph $\mathbf{G}$, with $M = \operatorname{tr}\mathbf{A^4}, p = \lfloor \frac{n}{2} \rfloor$, we redefine

$$m = e/p \text{ and } s^2 = \frac{M}{2p} - m^2. \tag{7}$$

The following result now follows from Theorem B:

THEOREM 4. *Let* $\mathbf{A}$ *be the adjacency matrix of order n of bipartite graph* $\mathbf{G}$. *Then with m and s as in (7),*

$$\max(0, \frac{e}{p} - s\sqrt{p-1}) \leqslant \lambda_p^2(\mathbf{G}) \leqslant \frac{e}{p} - \frac{s}{\sqrt{p-1}},$$

$$\frac{e}{p} + \frac{s}{\sqrt{p-1}} \leqslant \lambda_1^2(\mathbf{G}) \leqslant \frac{e}{p} + s\sqrt{p-1}. \tag{8}$$

Proof. When $\mathbf{G}$ is a bipartite graph eigenvalues of the symmetric matrix $\mathbf{A}$ are symmetric about origin. Thus $\mathbf{A^2}$ has at most p distinct, positive eigenvalues. Now the inequalities (8) follow when we apply Theorem B to the matrix $\mathbf{A^2}$.

For the remaining eigenvalues of a bipartite graph we have the following bounds:

THEOREM 5. *Let* $\mathbf{A}$ *be the adjacency matrix of a bipartite graph* $\mathbf{G}$. *Then with m and s as in (7) and for* $2 \leqslant k \leqslant p-1$,

$$\max\left(0, \frac{e}{p} - s\sqrt{\frac{k-1}{p-k+1}}\right) \leqslant \lambda_k^2(\mathbf{G}) \leqslant \frac{e}{p} + s\sqrt{\frac{p-k}{k}}. \tag{9}$$

Proof. The argument is the same as in above theorem except that we now use Theorem C.

More generally define,

$$m_l = \frac{\operatorname{tr}\mathbf{A^{2^{l-1}}}}{2p}, M_l = \operatorname{tr}\mathbf{A^{2^l}}, \text{ and } s_l^2 = \frac{M_l}{2p} - m_l{}^2, \tag{10}$$

where p = $\lfloor \frac{n}{2} \rfloor$ and l is any positive integer, $l \geqslant 1$. We note that matrix $\mathbf{A}$ has at most p distinct, positive eigenvalues.

Proceeding as above we deduce the following results:

THEOREM 6. *Let* $\mathbf{A}$ *be the adjacency matrix of a bipartite graph* $\mathbf{G}$. *Then with* m_l *and* s_l *as in (10),*

$$\max(0, m_l - s_l\sqrt{p-1}) \leqslant \lambda_p^{2^{l-1}}(\mathbf{G}) \leqslant m_l - \frac{s_l}{\sqrt{p-1}},$$

$$m_l + \frac{s_l}{\sqrt{p-1}} \leqslant \lambda_1^{2^{l-1}}(\mathbf{G}) \leqslant m_l + s_l\sqrt{p-1}. \tag{11}$$

THEOREM 7. *Let* $\mathbf{A}$ *be the adjacency matrix of a bipartite graph* $\mathbf{G}$*, then with* m_l *and* s_l *as in (10),*

$$\max\left(0, m_l - s_l\sqrt{\frac{k-1}{p-k+1}}\right) \leqslant \lambda_k^{2^{l-1}}(\mathbf{G}) \leqslant m_l + s_l\sqrt{\frac{p-k}{k}}, \quad 2 \leqslant k \leqslant p-1. \quad (12)$$

Further Theorem 3 yields the following result:

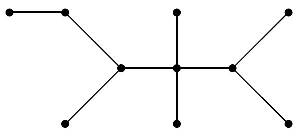
THEOREM 8. *When* $\mathbf{A}$ *is the adjacency matrix of a bipartite graph* $\mathbf{G}$*,*

$$\lambda_1 \geqslant \frac{\sum_{i=1}^n d_i^{1+\alpha} - \sum_{i=1}^n d_i d}{n\sqrt{\frac{\sum_{i=1}^n d_i^{2\alpha}}{n} - d^2}}, \quad (13)$$

where, $\alpha \geqslant 1$ *and* $d = \frac{\sum_{i=1}^{i=n} d_i^\alpha}{n}$.

3.2. Examples

We now compare our bounds for the largest eigenvalue, for two trees. Both the examples are taken from Cvetković et al. [1]. All numerical eigenvalues are given approximately. We recall that a tree is a bipartite graph. The first graph (2.139, [1]) is:

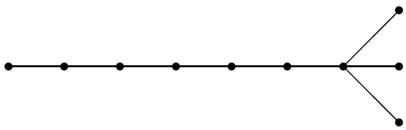


The actual value of the largest eigenvalue is 2.307. The lower bounds for this eigenvalue are:

inequality	(4)	(5)	(8)
bound value	2.101	2.240	1.664

The upper bound of inequality (8) is 2.383.

The second graph (2.161, [1]) is:



The actual value of the largest eigenvalue is 2.119.The lower bounds for this eigenvalue are:

inequality	(4)	(5)	(8)
bound value	2.101	2.030	1.631

The upper bound of inequality (8) is 2.289.

We conclude that lower bounds given by (4) and (5), as well as the upper bound of inequality (8) are good for both the examples considered.

4. Algebraic Connectivity

In this section we obtain a lower bound for the algebraic connectivity of a connected graph $\mathbf{G}$.

THEOREM 9. *Let $\mathbf{G}$ be a simple connected graph and $\mathbf{L}$ be its Laplacian matrix. Then*

$$\alpha(\mathbf{G}) \geqslant \left(\frac{n-2}{n-1}\right)^{\frac{n-2}{2}} \tau(\mathbf{G}) \max_{1\leqslant i\leqslant n-1} \frac{r_{min}(\mathbf{L_i})}{\prod_{l=1}^{n-1} r_l(\mathbf{L_i})} > 0 \tag{14}$$

where $\mathbf{L_i}$ is a principal submatrix of $\mathbf{L}$ obtained after deleting i^{th} row and column of $\mathbf{L}$ and $\tau(\mathbf{G})$ is the number of spanning trees.

Proof. Employing Matrix Tree theorem (e.g. see West [8]) we get, that determinants of all principal minors are positive and equal $\tau(\mathbf{G})$. The first inequality in (14) now follows from Theorem D. The second inequality follows noticing $\mathbf{G}$ is connected if and only if rank of $\mathbf{L}$ is $n-1$.

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