

NEW BOUNDS ON CERTAIN FUNDAMENTAL INTEGRAL INEQUALITIES

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Abstract. In the present note we derive new explicit upper bounds on some fundamental integral inequalities which can be used as tools in the study of a class of Volterra-Fredholm type integral equations. Some applications are also given to illustrate the usefulness of one of our results.

1. Introduction

In the study of parabolic equations which describe diffusion or heat transfer phenomena, the integral equations of the form

$$u(x,t) = h(x,t) + \int_0^t \int_B F(x,t,y,s,u(y,s)) dy ds, \quad (1.1)$$

occur in a natural way, see [1, p.18], [5, Chapter VI] and also [2-4]. The equation (1.1) appears to be Volterra type in t , and of Fredholm type with respect to x . We can view it as a mixed Volterra-Fredholm type integral equation. For the existence and uniqueness of solutions of equation (1.1), see [6]. It is relevant to mention the fact that the explicit bounds provided by the integral inequalities available in the literature are not directly applicable to study the qualitative behavior of solutions of equations of the form (1.1), see [7,8,10]. It is desirable to find explicit bounds on certain fundamental integral inequalities which will be equally important to achieve a diversity of desired goals. The main aim of the present note is to establish new explicit bounds on certain fundamental integral inequalities which can be used as powerful tools in handling the equations of the form (1.1). Some applications are also given to convey the importance of one of our results.

2. Statement of Results

Let R denote the set of real numbers, $R_+ = [0, \infty)$, $R_1 = [1, \infty)$ be the given subsets of R and B be a bounded domain in R^n , the n -dimensional Euclidean space defined

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by $B = \prod_{i=1}^n [a_i, b_i]$ ($a_i < b_i$). Let $x = (x_1, \dots, x_n)$ ($x_i \in \mathbf{R}$) is a variable point in B , $dx = dx_1 \dots dx_n$ and $'$ the derivative of a function with respect to $t \in \mathbf{R}_+$. For any continuous function $z: B \rightarrow \mathbf{R}$, we denote by $\int_B z(x) dx$ the n -fold integral $\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} z(x_1, \dots, x_n) dx_1 \dots dx_n$. Let $\Delta = B \times \mathbf{R}_+$ and denote by $C(S_1, S_2)$ the class of continuous functions from the set S_1 to the set S_2 .

Our main results are given in the following theorems.

THEOREM 1. Let $u, p, q, f, g \in C(\Delta, \mathbf{R}_+)$.

(a₁) If

$$u(x, t) \leq p(x, t) + q(x, t) \int_0^t \int_B [f(y, s)u(y, s) + g(y, s)] dy ds, \quad (2.1)$$

for $(x, t) \in \Delta$, then

$$\begin{aligned} u(x, t) &\leq p(x, t) + q(x, t) \int_0^t \int_B [f(y, s)p(y, s) + g(y, s)] \\ &\quad \times \exp\left(\int_s^t \int_B f(z, \tau)q(z, \tau) dz d\tau\right) dy ds, \end{aligned} \quad (2.2)$$

for $(x, t) \in \Delta$.

(a₂) Let $c \geq 0$ and $0 < \alpha < 1$ be real constants. If

$$u(x, t) \leq c + \int_0^t \int_B [f(y, s)u(y, s) + g(y, s)u^\alpha(y, s)] dy ds, \quad (2.3)$$

for $(x, t) \in \Delta$, then

$$\begin{aligned} u(x, t) &\leq \exp\left(\int_0^t \int_B f(z, \tau) dz d\tau\right) \\ &\quad \times \left[c^{1-\alpha} + (1-\alpha) \int_0^t \int_B g(y, s) \exp\left(- (1-\alpha) \int_0^s \int_B f(z, \tau) dz d\tau\right) dy ds \right]^{\frac{1}{1-\alpha}}, \end{aligned} \quad (2.4)$$

for $(x, t) \in \Delta$.

REMARK 1. If we take $g = 0$ in (2.1), then the bound obtained in (2.2) reduces to

$$u(x, t) \leq p(x, t) + q(x, t) \int_0^t \int_B f(y, s) p(y, s) \times \exp \left(\int_s^t \int_B f(z, \tau) q(z, \tau) dz d\tau \right) dy ds, \tag{2.5}$$

for $(x, t) \in \Delta$. In this case, we observe that the obtained result is a new variant of the inequality in Corollary 4.3.1 given in [7, p. 329]. We note that the bound on the unknown function $u(x, t)$ involved in (2.3) when $\alpha \neq 1, 1 < \alpha < \infty$ can be obtained by closely looking at the proof of inequality given in [7, Theorem 1.7.4, p. 153]. By taking (i) $g = 0$ and (ii) $f = 0$ in (2.3), it is easy to see that the bound obtained in (2.4) reduces respectively to

$$u(x, t) \leq c \exp \left(\int_0^t \int_B f(z, \tau) dz d\tau \right), \tag{2.6}$$

and

$$u(x, t) \leq \left[c^{1-\alpha} + (1-\alpha) \int_0^t \int_B g(y, s) dy ds \right]^{\frac{1}{1-\alpha}}, \tag{2.7}$$

for $(x, t) \in \Delta$.

THEOREM 2. Let $f, g \in C(\Delta, R_+)$ and $k \geq 0, c \geq 1, \beta > 1$ be real constants. (b₁) If $u \in C(\Delta, R_+)$ and

$$u^\beta(x, t) \leq k^\beta + \beta \int_0^t \int_B [f(y, s) u(y, s) + g(y, s) u^\beta(y, s)] dy ds, \tag{2.8}$$

for $(x, t) \in \Delta$, then

$$u(x, t) \leq \exp \left(\int_0^t \int_B g(z, \tau) dz d\tau \right) \times \left[k^{\beta-1} + (\beta-1) \int_0^t \int_B f(y, s) \exp \left(-(\beta-1) \int_0^s \int_B g(z, \tau) dz d\tau \right) dy ds \right]^{\frac{1}{\beta-1}}, \tag{2.9}$$

for $(x, t) \in \Delta$.

(b₂) If $u \in C(\Delta, R_1)$ and

$$u(x, t) \leq c + \int_0^t \int_B f(y, s) u(y, s) \log u(y, s) dy ds, \tag{2.10}$$

for $(x, t) \in \Delta$, then

$$u(x, t) \leq c \exp\left(\int_0^t \int_B f(y, s) dy ds\right), \quad (2.11)$$

for $(x, t) \in \Delta$.

REMARK 2. If we take $g = 0$ in (2.8), then the bound obtained in (2.9) reduces to

$$u(x, t) \leq \left[k^{\beta-1} + (\beta-1) \int_0^t \int_B f(y, s) dy ds \right]^{\frac{1}{\beta-1}}, \quad (2.12)$$

for $(x, t) \in \Delta$. By taking $g = 0$ and $\beta = 2$ in part (b_1) , we get a new variant of the inequality given in Theorem 5.8.1 in [7, p. 527].

3. Proofs of Theorems 1 and 2

Since the proofs resemble one another, we give the details for (a_1) and (b_1) only; the proofs of (a_2) and (b_2) can be completed by following the proofs of (a_1) and (b_1) and closely looking at the ideas used in the proofs of Theorems 2.7.4 and 3.8.1 given in [7]

(a_1) Introducing the notation

$$e(s) = \int_B [f(y, s)u(y, s) + g(y, s)] dy, \quad (3.1)$$

in (2.1) we get

$$u(x, t) \leq p(x, t) + q(x, t) \int_0^t e(s) ds, \quad (3.2)$$

for $(x, t) \in \Delta$. Define

$$m(t) = \int_0^t e(s) ds, \quad (3.3)$$

for $t \in R_+$, then $m(0) = 0$ and from (3.2) we have

$$u(x, t) \leq p(x, t) + q(x, t)m(t), \quad (3.4)$$

for $(x, t) \in \Delta$. From (3.3), (3.1) and (3.4) we observe that

$$\begin{aligned} m'(t) &= e(t) = \int_B [f(y, t)u(y, t) + g(y, t)] dy \\ &\leq \int_B [f(y, t) \{p(y, t) + q(y, t)m(t)\} + g(y, t)] dy \\ &= m(t) \int_B f(y, t) q(y, t) dy + \int_B [f(y, t) p(y, t) + g(y, t)] dy. \end{aligned} \quad (3.5)$$

The inequality (3.5) implies (see [7, Theorem 1.3.2])

$$m(t) \leq \int_0^t \int_B [f(y,s)p(y,s) + g(y,s)] \exp \left(\int_s^t \int_B f(z,\tau)q(z,\tau) dzd\tau \right) dyds, \tag{3.6}$$

for $(x,t) \in \Delta$. Using (3.6) in (3.4) we get the required inequality in (2.2).

(b₁) Introducing the notation

$$E(s) = \int_B [f(y,s)u(y,s) + g(y,s)u^\beta(y,s)] dy, \tag{3.7}$$

in (2.8) we get

$$u^\beta(x,t) \leq k^\beta + \beta \int_0^t E(s) ds. \tag{3.8}$$

Let $k > 0$ and define

$$z(t) = k^\beta + \beta \int_0^t E(s) ds, \tag{3.9}$$

then $z(0) = k^\beta$ and from (3.8) we have

$$u^\beta(x,t) \leq z(t), \tag{3.10}$$

for $(x,t) \in \Delta$. From (3.9), (3.7), (3.10) we observe that

$$\begin{aligned} z'(t) &= \beta E(t) = \beta \int_B [f(y,t)u(y,t) + g(y,t)u^\beta(y,t)] dy \\ &\leq \beta \int_B [f(y,t)(z(t))^{\frac{1}{\beta}} + g(y,t)z(t)] dy \\ &= \beta \left[z(t) \int_B g(y,t) dy + (z(t))^{\frac{1}{\beta}} \int_B f(y,t) dy \right]. \end{aligned} \tag{3.11}$$

The inequality (3.11) implies (see [7, Theorem 3.5.5])

$$\begin{aligned} z(t) &\leq \exp \left(\beta \int_0^t \int_B g(z,\tau) dzd\tau \right) \left\{ k^{\beta-1} + (\beta-1) \int_0^t \int_B f(y,s) \right. \\ &\quad \left. \times \exp \left(-(\beta-1) \int_0^s \int_B g(z,\tau) dzd\tau \right) dyds \right\}^{\frac{\beta}{\beta-1}}. \end{aligned} \tag{3.12}$$

Using (3.12) in (3.10) we get the required inequality in (2.9). If $k \geq 0$, we carry out the above procedure with $k + \varepsilon$ instead of k , where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \rightarrow 0$ to obtain (2.9).

4. Some applications

In this section, we apply the inequality established in Theorem 1, part (a₁) to obtain explicit estimates on the solutions of equations of the form (1.1), which occur in a wide variety of applications (see [1-6]). For the existence and uniqueness of solutions of equation of the form (1.1), see [6].

The following theorem deals with the estimate on the solution of equation (1.1).

THEOREM 3. *Suppose that $h \in C(\Delta, R)$, $F \in C(\Delta^2 \times R, R)$ and*

$$|F(x, t, y, s, u)| \leq q(x, t) f(y, s) |u|, \quad (4.1)$$

where $q, f \in C(\Delta, R_+)$. If $u(x, t)$ is any solution of equation (1.1) on Δ , then

$$\begin{aligned} |u(x, t)| &\leq |h(x, t)| + q(x, t) \int_0^t \int_B f(y, s) |h(y, s)| \\ &\quad \times \exp \left(\int_s^t \int_B f(z, \tau) q(z, \tau) dz d\tau \right) dy ds, \end{aligned} \quad (4.2)$$

for $(x, t) \in \Delta$.

Proof. Let $u \in C(\Delta, R)$ be a solution of equation (1.1). Then from the hypotheses, we have

$$|u(x, t)| \leq |h(x, t)| + q(x, t) \int_0^t \int_B f(y, s) |u(y, s)| dy ds. \quad (4.3)$$

Now a suitable application of the inequality in Theorem 1, part (a₁) (when $g = 0$) to (4.3) gives the desired estimate in (4.2).

We next consider the following two mixed Volterra-Fredholm type integral equations

$$v(x, t) = h_1(x, t) + \int_0^t \int_B L(x, t, y, s, v(y, s)) dy ds, \quad (4.4)$$

$$w(x, t) = h_2(x, t) + \int_0^t \int_B M(x, t, y, s, w(y, s)) dy ds, \quad (4.5)$$

where $h_1, h_2 \in C(\Delta, R)$ and $L, M \in C(\Delta^2 \times R, R)$.

The following theorem holds.

THEOREM 4. *Suppose that the function L in equation (4.4) satisfies the condition*

$$|L(x, t, y, s, v) - L(x, t, y, s, w)| \leq q(x, t) f(y, s) |v - w|, \tag{4.6}$$

where $q, f \in C(\Delta, R_+)$. Then for every solution $w \in C(\Delta, R)$ of equation (4.5) and $v \in C(\Delta, R)$ a solution of equation (4.4), we have the estimation

$$|v(x, t) - w(x, t)| \leq [h(x, t) + r(x, t)] + q(x, t) \int_0^t \int_B f(y, s) [h(y, s) + r(y, s)] \\ \times \exp \left(\int_s^t \int_B f(z, \tau) q(z, \tau) dz d\tau \right) dy ds, \tag{4.7}$$

for $(x, t) \in \Delta$, in which

$$h(x, t) = |h_1(x, t) - h_2(x, t)|, \tag{4.8}$$

$$r(x, t) = \int_0^t \int_B |L(x, t, y, s, w(y, s)) - M(x, t, y, s, w(y, s))| dy ds, \tag{4.9}$$

for $(x, t) \in \Delta$.

Proof. Using the facts that $v(x, t)$ and $w(x, t)$ are respectively the solutions of equations (4.4) and (4.5) and hypotheses, we have

$$|v(x, t) - w(x, t)| \leq |h_1(x, t) - h_2(x, t)| \\ + \int_0^t \int_B |L(x, t, y, s, v(y, s)) - L(x, t, y, s, w(y, s))| dy ds \\ + \int_0^t \int_B |L(x, t, y, s, w(y, s)) - M(x, t, y, s, w(y, s))| dy ds \\ \leq [h(x, t) + r(x, t)] + q(x, t) \int_0^t \int_B f(y, s) |v(y, s) - w(y, s)| dy ds. \tag{4.10}$$

Now a suitable application of Theorem 1, part (a_1) (when $g = 0$) to (4.10) yields (4.7).

REMARK 3. We note that, Theorem 1, part (a_1) (when $g = 0$) can be used to establish the basic results on the uniqueness and continuous dependence of solutions of equation (1.1) by closely looking at the results recently given in [9]. Moreover, many generalizations, extensions, variants and applications of the inequalities given above are also possible. We leave it to the reader to fill in where needed. We hope that the results given here will encourage further research and widen the scope of their applications.

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