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ADAPTED QUADRATIC APPROXIMATION FOR SINGULAR INTEGRALS

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Abstract. The goal of this work is to present an adapted modification to the parabolic approximation of the density function for singular integrals of Cauchy type. This approximation serves to eliminate the singularity of the integral and gives the help to obtain the numerical solution of singular integral equations with Cauchy type kernel on an oriented smooth contour.

1. Introduction

Many problems of mathematical physics, engineering and contact problems in the theory of elasticity lead to singular integral equations with Cauchy type kernel

$$a(t_0)\varphi(t_0)+\frac{b(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt+\int_\Gamma k(t,t_0)\varphi(t)dt=f(t_0), \tag{1}$$

where Γ designates an oriented smooth contour, the points t and t_0 are on Γ. This equation plays an important role in modern numerical computations in the applied sciences, in particular in the applied mathematics.

Our schemes describe the quadrature method for the approximation of singular integral operator with Cauchy kernel

$$F(t_0)=\frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt, \quad t,t_0\in\Gamma, \tag{2}$$

by a sequence of numerical integration operators.

Noting that, for the existence of the principal value of this integral for a given density $\varphi(t)$, we will need more than mere continuity. In other words, the density $\varphi(t)$ has to satisfy the Hölder condition $H(\mu)$[2].

The function $\varphi(t)$ will be said to satisfy a Hölder condition on Γ, if for any two points t_1 and t_2 of Γ

$$|\varphi(t_2)-\varphi(t_1)|\leqslant A\,|t_2-t_1|^\mu \quad 0<\mu\leqslant 1,$$

where A is a positive constant, called the Hölder constant and μ the Hölder index.

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2. The Quadrature

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s) = x(s) + iy(s), \;\; a \leqslant s \leqslant b,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a,b]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null. Let N be an arbitrary natural number, generally we take it large enough and divide the interval $[a,b]$ into N equal subintervals $I_1, I_2, ..., I_N$ by the points

$$s_\sigma = a + \sigma \frac{l}{N}\,,\; l = b - a\,,\; \sigma = 0,1,2,....,N.$$

Further, we fix a natural number $M > 1$, and divide each of segments $[s_\sigma, s_{\sigma+1}]$ by the equidistant points

$$s_{\sigma k} = s_\sigma + k\frac{h}{2M},\;\; h = \frac{l}{N},\;\; k = 0,1,...,2M.$$

In other words, we have for each subinterval $[s_\sigma, s_{\sigma+1}]$ the following subdivision

$$[s_\sigma, s_{\sigma+1}] = \{s_\sigma = s_{\sigma 0} < s_{\sigma 1} < ..... < s_{\sigma 2M} = s_{\sigma+1}\}.$$

We introduce the notation

$$t_\sigma = t(s_\sigma),\; t_{\sigma k} = t(s_{\sigma k});\quad \sigma = 0,1,2,...,N;\;\; k = 0,1,....,2M.$$

Assuming that, for the indices $\sigma, \nu = 0,1,2,....,N-1$, the points t and t_0 belong respectively to the arcs $\widehat{t_\sigma t_{\sigma+1}}$ and $\widehat{t_\nu t_{\nu+1}}$ where $\widehat{t_\alpha t_{\alpha+1}}$ designates the smallest arc with ends t_α and $t_{\alpha+1}$ [3], [5], [6] and [7].

For an arbitrary number $\sigma = 0,1,2,...,N-1$, we define the piecewice quadratic Lagrange interpolation polynomial $S_2(\varphi; t, \sigma)$ dependent on φ, t and σ which represents the quadratic approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. As we know, the interval $[t_\sigma, t_{\sigma+1}]$ is divided into subintervals $[t_{\sigma k}, t_{\sigma(k+2)}]$ of length $(t_{\sigma(k+2)} - t_{\sigma k})$, $k = 2i$, $i = 0,1,...,M-1$. We interpolate the function density $\varphi(t)$ with respect to the values $\varphi(t_{\sigma k}), \varphi(t_{\sigma(k+1)})$ and $\varphi(t_{\sigma(k+2)})$ at the points $t_{\sigma k}, t_{\sigma(k+1)}$ and $t_{\sigma(k+2)}$ respectively with a quadratic polynomial, given by the following formula.

For $t_{\sigma k} \leqslant t \leqslant t_{\sigma(k+2)}$,

$$\begin{aligned} S_2(\varphi; t, \sigma) = & \frac{(t - t_{\sigma(k+1)})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma k})}\varphi(t_{\sigma k}) \\ & - \frac{(t - t_{\sigma k})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)}) \\ & + \frac{(t - t_{\sigma k})(t - t_{\sigma(k+1)})}{(t_{\sigma(k+2)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)}). \end{aligned} \tag{3}$$

This piecewice quadratic interpolating polynomial exists and is unique.

We define for arbitrary numbers σ and ν, such that $0 \leqslant \sigma, \nu \leqslant N-1$, the following function $\beta_{\sigma\nu}(\varphi;t,t_0)$, dependent on φ, t and t_0

$$\beta_{\sigma\nu}(\varphi;t,t_0) = U(\varphi;t,\sigma) - V(\varphi;t_0,\sigma,\nu). \tag{4}$$

The function $U(\varphi;t,\sigma)$ represents a modified quadratic interpolation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ.

Indeed, for $t_{\sigma k} \leqslant t \leqslant t_{\sigma(k+2)}$ we put

$$\begin{aligned} U(\varphi;t,\sigma) = {} & \frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k})\frac{t-t_0}{t_{\sigma k}-t_0} \\ & - \frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)})\frac{t-t_0}{t_{\sigma(k+1)}-t_0} \\ & + \frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)})\frac{t-t_0}{t_{\sigma(k+2)}-t_0}, \end{aligned}$$

and the function $V(\varphi;t_0,\sigma,\nu)$ is given by

$$\begin{aligned} V(\varphi;t_0,\sigma,\nu) = {} & \frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma k}-t_0)(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+1)}-t_{\sigma k})} \\ & - \frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+1)}-t_{\sigma k})} \\ & + \frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+2)}-t_{\sigma k})} \end{aligned}$$

Denoting by $\psi_{\sigma\nu}(\varphi;t,t_0)$, $0 \leqslant \sigma, \nu \leqslant N-1$ the cubic approximation of the density $\varphi(t)$ at the point $t \in [t_\sigma, t_{\sigma+1}]$, for all $t_0 \in [t_\nu, t_{\nu+1}]$ we write

$$\psi_{\sigma\nu}(\varphi;t,t_0) = \varphi(t_0) + \beta_{\sigma\nu}(\varphi;t,t_0). \tag{5}$$

We replace the density $\varphi(t)$ by expansion (5) in the singular integral (2)

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt,$$

and obtain the following approximation

$$S(\varphi,t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\psi_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt = \varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{\beta_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt. \tag{6}$$

3. Main results

THEOREM. *Let Γ be an oriented smooth contour and let φ be a density function defined on Γ and satisfying the Hölder condition $H(\mu)$ then, the following estimation*

$$| F(t_0) - S(\varphi;t_0) | \leqslant \max\left(\frac{C\ln(2MN)}{(2MN)^{\mu}}, \frac{C}{N^{\mu}}\right) \; N,\ M > 1$$

holds, where the constant C depends only on the contour Γ.

Proof. Taking the points $t \in [t_\sigma, t_{\sigma+1}]$ and $t_0 \in [t_\nu, t_{\nu+1}]$, we can write for $t_{\sigma k} \leqslant t \leqslant t_{\sigma(k+2)}$ and $t_{\nu k} \leqslant t_0 \leqslant t_{\nu(k+2)}$

$$\begin{aligned}
\varphi(t) - \psi_{\sigma\nu}(\varphi;t,t_0) = \varphi(t) - \varphi(t_0) & \\
- \Bigg\{ & \frac{(t - t_{\sigma(k+1)})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma k})} \varphi(t_{\sigma k}) \frac{t - t_0}{t_{\sigma k} - t_0} \\
& - \frac{(t - t_{\sigma k})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})} \varphi(t_{\sigma(k+1)}) \frac{t - t_0}{t_{\sigma(k+1)} - t_0} \\
& + \frac{(t - t_{\sigma k})(t - t_{\sigma(k+1)})}{(t_{\sigma(k+2)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})} \varphi(t_{\sigma(k+2)}) \frac{t - t_0}{t_{\sigma(k+2)} - t_0} \\
& - \frac{S_2(\varphi;t_0,\nu)(t - t_0)(t - t_{\sigma(k+1)})(t - t_{\sigma(k+2)})}{(t_{\sigma k} - t_0)(t_{\sigma(k+2)} - t_{\sigma k})(t_{\sigma(k+1)} - t_{\sigma k})} \\
& + \frac{S_2(\varphi;t_0,\nu)(t - t_0)(t - t_{\sigma k)})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_0)(t_{\sigma(k+2)} - t_{\sigma(k+1)})(t_{\sigma(k+1)} - t_{\sigma k})} \\
& - \frac{S_2(\varphi;t_0,\nu)(t - t_0)(t - t_{\sigma k)})(t - t_{\sigma(k+1)})}{(t_{\sigma(k+2)} - t_0)(t_{\sigma(k+2)} - t_{\sigma(k+1)})(t_{\sigma(k+2)} - t_{\sigma k})} \Bigg\}.
\end{aligned} \tag{7}$$

Taking into account the expression (7) we get

$$\frac{1}{\pi i}\int_\Gamma \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi;t,t_0)}{t - t_0} dt = \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\int_{t_\sigma t_{\sigma+1}} \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi;t,t_0)}{t - t_0} dt, \tag{8}$$

hence

$$\begin{aligned}
F(t_0) - S(\varphi;t_0) = \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k} t_{\sigma(2k+2)}} & \frac{\varphi(t) - \varphi(t_0)}{t - t_0} \\
- \Bigg\{ & \frac{(t - t_{\sigma(k+1)})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma k})} \varphi(t_{\sigma k}) \frac{t - t_0}{t_{\sigma k} - t_0} \\
& - \frac{(t - t_{\sigma k})(t - t_{\sigma(k+2)})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+2)} - t_{\sigma(k+1)})} \varphi(t_{\sigma(k+1)}) \frac{t - t_0}{t_{\sigma(k+1)} - t_0}
\end{aligned}$$

$$+\frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)})\frac{t-t_0}{t_{\sigma(k+2)}-t_0}$$
$$-\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma k}-t_0)(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+1)}-t_{\sigma k})}$$
$$+\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+1)}-t_{\sigma k})}$$
$$\left.-\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+2)}-t_{\sigma k})}\right\}\frac{1}{t-t_0}dt.$$

We note that the equalities $t_{\sigma k}-t_0=0$, $t_{\sigma(k+1)}-t_0=0$ and $t_{\sigma(k+2)}-t_0=0$ are possible only when $\sigma=\nu-1,\nu+1$ and ν. For the two first cases the integral (8) exists when $t_{\sigma k}$ tends to t_0 or $t_{\sigma(k+2)}$ tends to t_0, in the other case, if $\sigma=\nu$ we can easily see that, the function $\beta_{\sigma\sigma}(\varphi;t,t_0)$ contains $(t_{\sigma k}-t_0)$, $(t_{\sigma(k+1)}-t_0)$ and $(t_{\sigma(k+2)}-t_0)$ as factor so, for the points $t,\ t_0\in[t_\sigma,t_{\sigma+1}]$, such that $t_{\sigma k}\leqslant t,\ t_0\leqslant t_{\sigma(k+2)}$, we write

$$\beta_{\sigma\sigma}(\varphi;t,t_0)=U(\varphi;t,\sigma)-V(\varphi;t_0,\sigma,\sigma),$$

hence

$$\begin{aligned}\beta_{\sigma\sigma}(\varphi;t,t_0)=&\frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})(t-t_0)}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma k}-t_0)}(\varphi(t_{\sigma k})-S_2(\varphi;t_0,\sigma))\\&-\frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})(t-t_0)}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+1)}-t_0)}(\varphi(t_{\sigma(k+1)})-S_2(\varphi;t_0,\sigma))\\&+\frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})(t-t_0)}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+2)}-t_0)}(\varphi(t_{\sigma(k+2)})-S_2(\varphi;t_0,\sigma)).\end{aligned}\tag{9}$$

Taking into account expressions $(7),(9),$ we obtain

$$\frac{1}{\pi i}\int_\Gamma\frac{\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt=\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}t_{\sigma(2k+2)}}\frac{\varphi(t)-\varphi(t_0)}{t-t_0}$$
$$-\left\{\frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k})\frac{t-t_0}{t_{\sigma k}-t_0}\right.$$
$$-\frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)})\frac{t-t_0}{t_{\sigma(k+1)}-t_0}$$
$$+\frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)})\frac{t-t_0}{t_{\sigma(k+2)}-t_0}$$
$$-\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma k}-t_0)(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+1)}-t_{\sigma k})}$$

$$+\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+1)}-t_{\sigma k})}$$
$$-\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+2)}-t_{\sigma k})}\Bigg\}\frac{1}{t-t_0}dt.$$

Passing now to the estimation of the expression (8), for $t_0 \in t_\nu \widehat{t_{\nu+1}}$ and $\sigma \neq \nu-1,\ \nu+1$ and ν we have

$$\Bigg|\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}t_{\sigma(2k+2)}}\frac{\varphi(t)-\varphi(t_0)}{t-t_0}
-\Bigg\{\frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k})\frac{t-t_0}{t_{\sigma k}-t_0}
-\frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)})\frac{t-t_0}{t_{\sigma(k+1)}-t_0}
+\frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)})\frac{t-t_0}{t_{\sigma(k+2)}-t_0}
-\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma k}-t_0)(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+1)}-t_{\sigma k})}
+\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+1)}-t_{\sigma k})}
-\frac{S_2(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+2)}-t_{\sigma k})}\Bigg\}\frac{1}{t-t_0}dt.\Bigg| = O\left(\frac{\ln(2MN)}{(2MN)^\mu}\right).$$

Naturally, the estimation given above is obtained by using the density φ, as an element of the Hölder space $H(\mu)$[2], and the following natural estimation

$$\left|\frac{(t-t_0)(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma k}-t_0)(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+1)}-t_{\sigma k})}\right| = O(1),$$
$$\left|\frac{(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+1)}-t_{\sigma k})}\right| = O(1),$$
$$\left|\frac{(t-t_0)(t-t_{\sigma k)})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_0)(t_{\sigma(k+2)}-t_{\sigma(k+1)})(t_{\sigma(k+2)}-t_{\sigma k})}\right| = O(1).$$

Besides, it is easy to obtain

$$\max_{t_0\in t_\nu t_{\nu+1}}\left|O\left(\frac{1}{(2M)^\mu N^\mu}\right)\sum_{\substack{\sigma=0\\ \sigma\neq\nu}}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma(2k)}t_{\sigma(2k+2)}}\frac{dt}{t-t_0}\right| = O\left(\frac{\ln(2M)N}{(2M)^\mu N^\mu}\right).$$

Further, for the cases where $\sigma = \nu - 1,\ \nu + 1$ and ν, using the relation (9) and the smoothness of Γ with the condition of the function φ in the space $H(\mu)$, we get

$$\left| \int_{t_\nu t_{\nu+1}} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt \right| \leqslant A \int_{s_\nu}^{s_{\nu+1}} | s - s_0 |^{\mu - 1} \, ds = O(N^{-\mu}) \qquad \square$$

4. Numerical experiments

Using our approximation, we apply the algorithm to singular integrals and we present results concerning the accuracy of the calculations. In each table I represents the exact principal value of the singular integral and $\widetilde{I}$ corresponds to the approximate calculation produced by our approximation (6).

EXAMPLE 1. Consider the singular integral,

$$I = F(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0} dt,$$

where Γ designates the circle centred at the point 0 with a unit radius, t and t_0 are any points on Γ and the density function φ is given by the following expression

$$\varphi(t) = \frac{1}{t^2}$$

N	M	$\| I - \widetilde{I} \|_1$	$\| I - \widetilde{I} \|_2$	$\| I - \widetilde{I} \|_\infty$
10	2	7.8253150E-03	5.2587800E-03	4.9651256E-03
11	2	6.1442256E-03	3.8414600E-03	3.5070777E-03
12	2	3.7623644E-03	2.3593807E-03	2.1644831E-03

EXAMPLE 2. We take the singular integral,

$$I = F(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0} dt,$$

where the curve Γ designates the unit circle t and t_0 are any points on Γ and the density function φ is

$$\varphi(t) = \sin t^2 + \cos t$$

N	M	$\| I - \widetilde{I} \|_1$	$\| I - \widetilde{I} \|_2$	$\| I - \widetilde{I} \|_\infty$
10	2	2.1330118E-03	1.3611738E-03	1.2570620E-03
11	2	9.4902515E-04	5.7528692E-04	4.8601627E-04
12	2	4.9412251E-04	3.2134863E-04	2.9397011E-04

5. Conclusion

The proposed approximation can be used to remove singularity in Cauchy principal value integrals of the form (2). It was tested for the numerical calculus of many singular integrals of Cauchy type, where it gave good results.

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