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ON HADAMARD'S INEQUALITIES FOR THE CONVEX MAPPINGS DEFINED IN TOPOLOGICAL GROUPS AND A CONNECTED RESULT

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Abstract. In this paper, we study the Hadamard's inequality for midconvex and quasi-midconvex functions in topological groups. A mapping naturally connected with this inequality and a related result is also pointed out.

1. Introduction

Let $f : I \longrightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hadamard's inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f.

In the paper [4] (see also [5] and [6]) is considered the following mapping naturally connected with Hadamard's results:

$$H : [0,1] \longrightarrow \mathbb{R}, \quad H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx.$$

The following properties of H hold:

(i) H is convex and monotonic nondecreasing.

(ii) One has the bounds

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right).$$

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Another mapping also closely connected with Hadamard's inequality is the following one [5] (see also [6]):

$$F : [0,1] \longrightarrow \mathbb{R}, \quad F(t) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f(tx+(1-t)y)dxdy.$$

The properties of this mapping are the following ones:

(i) F is convex and monotonic nonincreasing on $[0,\frac{1}{2}]$ and nondecreasing on $[\frac{1}{2},1]$;

(ii) F is symmetric relative to the element $\frac{1}{2}$. That is,

$$F(t) = F(1-t) \qquad \text{for all } t \in [0,1];$$

(iii) One has the bounds

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right)dxdy \geqslant f\left(\frac{a+b}{2}\right)$$

(iv) The following inequality holds:

$$F(t) \geqslant \max\{H(t), H(1-t)\} \quad \text{for all } t \in [0,1].$$

Generalization of (1) for quasi convex functions defined on the real line is also well-known. It was established in [2] that for a quasi convex function f defined on $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{2}{b-a}\int_a^b f(x)dx.$$

In this paper we shall study generalizations of the left side of (1) inequality for some convex functions defined on an open subset of a topological group G.

2. A Secondary Result

Generalization of the left side of (1) for convex functions defined on a convex subset of $\mathbb{R}^n$ is well-known. For example, if $X \subset \mathbb{R}^n$ is a convex bounded symmetrical set (the latter means that $x \in X \Longrightarrow -x \in X$), then (c.f. [8])

$$f(0) \leqslant \frac{1}{\mu(X)}\int_X f(x)dx \tag{2}$$

for each lower semicontinuous convex function $f : X \longrightarrow \mathbb{R}$, where $\mu(X)$ is the volume of the set X. To show (2), consider the transformation of the $\mathbb{R}^n$ in itself given by:

$$h : \mathbb{R}^n \to \mathbb{R}^n, \quad h = (h_1, h_2, \cdots, h_n),$$

and

$$h_i(x_1, x_2, \cdots, x_n) = -x_i, \qquad i = 1, 2, \cdots, n.$$

Then $h(X) = X$ and since

$$\frac{D(h_1, h_2, \cdots, h_n)}{D(x_1, x_2, \cdots, x_n)} = \begin{vmatrix} -1 & 0 & 0 \cdots & 0 \\ 0 & -1 & 0 \cdots & 0 \\ 0 & 0 & -1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & -1 \end{vmatrix} = (-1)^n.$$

we have the change of variable:

$$\begin{aligned} &\int_X f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n \\ &= \int_X f\left(h_1(x_1, x_2, \cdots, x_n), \cdots, h_n(x_1, x_2, \cdots, x_n)\right) \left| \frac{D(h_1, h_2, \cdots, h_n)}{D(x_1, x_2, \cdots, x_n)} \right| dx_1 dx_2 \cdots dx_n \\ &= \int_X f(-x_1, -x_2, \cdots, -x_n) dx_1 dx_2 \cdots dx_n. \end{aligned}$$

Now, by the convexity of f on X we also have:

$$\begin{aligned} f(0, 0, \cdots, 0) &= f\left(\frac{x_1 - x_1}{2}, \frac{x_2 - x_2}{2}, \cdots, \frac{x_n - x_n}{2}\right) \\ &= f\left(\frac{(x_1, x_2, \cdots, x_n) + (-x_1, -x_2, \cdots, -x_n)}{2}\right) \\ &\leqslant \frac{1}{2}\left[f(x_1, x_2, \cdots, x_n) + f(-x_1, -x_2, \cdots, -x_n)\right] \end{aligned}$$

which gives, by integration of f on X, that:

$$\begin{aligned} &\int_X f(0, 0, \cdots, 0) dx_1 dx_2 \cdots dx_n \\ &\leqslant \frac{1}{2}\left[\int_X f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n + \int_X f(-x_1, -x_2, \cdots, -x_n) dx_1 dx_2 \cdots dx_n\right] \\ &= \int_X f(x_1, x_2, \cdots, x_n). \end{aligned}$$

Consequently, we get

$$f(0, 0, \cdots, 0)\mu(X) \leqslant \int_X f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n$$

and thus

$$f(0, 0, \cdots, 0) \leqslant \frac{1}{\mu(X)} \int_X f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n.$$

3. Hadamard's Inequality

In this section we prove the Hadamard's inequality for midconvex and quasi-midconvex functions in topological groups.

Let G be a topological group, Ω a nonempty open subset of G and f a real-valued function on Ω. We say that f is *globally (right) midconvex* if

$$2f(a) \leqslant f(az) + f(az^{-1}) \tag{3}$$

for all $a, z \in G$ such that $a, az, az^{-1} \in \Omega$. Also, we say that f is *locally (right) midconvex in* $a \in \Omega$ if there exists an open symmetric set $V = V^{-1}$ from e such that

$$2f(a) \leqslant f(az) + f(az^{-1}) \tag{4}$$

for all $z \in G$ such that $az, az^{-1} \in \Omega$ [1]. Also, f is called *quasi-(right)midconvex*, if

$$f(az) \leqslant \max\{f(a), f(az^2)\} \tag{5}$$

for every $a, z \in G$ so that $a, az, az^2 \in \Omega$ [7]. Note that a is midpoint of az^{-1} and az, and az is midpoint of a and az^2.

DEFINITION 1. Let Ω be an open subset of topological group G, and $a \in G$. Ω is said to be *symmetric relative to a*, if $a^{-1}\Omega$ is symmetric and $e \in a^{-1}\Omega$.

DEFINITION 2. Let G be a topological group and $\Omega \subset G$ an open set. A function $\omega : \Omega \longrightarrow \mathbb{R}$ is called *symmetric relative to* $a \in G$, if

$$\forall z \in G; \quad az, az^{-1} \in \Omega \qquad \omega(az) = \omega(az^{-1}).$$

The following theorems hold:

THEOREM 1. *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. If $f : \Omega \longrightarrow \mathbb{R}$ is measurable and locally midconvex in a and also, $f \in L_1(\Omega)$, and $\omega : \Omega \longrightarrow \mathbb{R}$ is nonnegative and symmetric to a and $\omega \in L_1(\Omega)$ such that $f\omega \in L_1(\Omega)$, then*

$$f(a)\int_\Omega \omega(az)d\mu(z) \leqslant \int_\Omega f(az)\omega(az)d\mu(z),$$

where μ is the Haar measure.

Proof. Since f is locally midconvex in a, so

$$2f(a) \leqslant f(az) + f(az^{-1})$$

for any $z \in \Omega$, by (4). Since ω is nonnegative and symmetric relative to a, thus

$$\begin{aligned}2f(a)\omega(az) &\leqslant f(az)\omega(az) + f(az^{-1}) + \omega(az)\\ &= f(az)\omega(az) + f(az^{-1}) + \omega(az^{-1}).\end{aligned}$$

Integrating this inequality on Ω, we get

$$\begin{aligned}
\int_\Omega 2f(a)\omega(az)d\mu(z) &\leqslant \int_\Omega f(az)\omega(az)d\mu(z) + \int_\Omega f(az^{-1})\omega(az^{-1})d\mu(z) \\
&= \int_{a^{-1}\Omega} f(z)\omega(z)d\mu(z) + \int_{a^{-1}\Omega} f(z^{-1})\omega(z^{-1})d\mu(z) \\
&= \int_G f(z)\omega(z)\chi_{a^{-1}\Omega}(z)d\mu(z) + \int_G f(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z)d\mu(z) \\
&= \int_G f(z)\omega(z)\chi_{a^{-1}\Omega}(z)d\mu(z) + \int_G f(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z^{-1})d\mu(z) \\
&= 2\int_G f(z)\omega(z)\chi_{a^{-1}\Omega}(z)d\mu(z). \\
&= 2\int_{a^{-1}\Omega} f(z)\omega(z)d\mu(z).
\end{aligned}$$

Consequently, we have

$$f(a)\int_\Omega \omega(az)d\mu(z) \leqslant \int_\Omega f(az)\omega(az)d\mu(z). \qquad \square$$

REMARK 1. If in the above theorem, $a = e$ and $\omega \equiv 1$ on Ω, we have

$$f(e) \leqslant \frac{1}{\mu(\Omega)}\int_\Omega f(z)d\mu(z).$$

This result is similar to the result of section 2.

THEOREM 2. *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$ and $e \in \Omega$. If f is measurable and quasi-midconvex real-valued function on Ω such that $f \in L_2(\Omega)$ and also, $\omega : \Omega \longrightarrow \mathbb{R}$ is a nonnegative and symetric to a and $\omega \in L_2(\Omega)$, then*

$$f(a)\int_\Omega \omega(az)d\mu(z) \leqslant \int_\Omega f(az)\omega(az)d\mu(z) + I(a) \tag{6}$$

where

$$I(a) = \frac{1}{2}\int_\Omega |f(az) - f(az^{-1})|\omega(az)d\mu(z).$$

Furthermore, $I(a)$ satisfies the inequalities:

$$0 \leqslant I(a) \leqslant \min\left\{\begin{array}{c} \int_\Omega |f(az)|\omega(az)d\mu(z), \\ \frac{1}{\sqrt{2}}\left(\int_\Omega f^2(az)d\mu(z) - \int_\Omega f(az)f(az^{-1})d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \end{array}\right\}. \tag{7}$$

Proof. Since Ω is a symmetric set relative to a, thus for z in G, by (5), we have

$$f(a) \leqslant \max\{f(az), f(az^{-1})\}$$

where $\max\{f(az), f(az^{-1})\} = \frac{f(az)+f(az^{-1})+|f(az)-f(az^{-1})|}{2}$ and since ω is nonnegative and symmetric relative to a, therefore

$$\int_\Omega f(a)\omega(az)d\mu(z) \leqslant \frac{1}{2}\int_\Omega f(az)\omega(az)d\mu(z) + \frac{1}{2}\int_\Omega f(az^{-1})\omega(az^{-1})d\mu(z) + \frac{1}{2}\int_\Omega |f(az)-f(az^{-1})|\omega(az)d\mu(z).$$

So,

$$f(a)\int_\Omega \omega(az)d\mu(z) \leqslant \int_\Omega f(az)\omega(az)d\mu(z) + I(a)$$

and the inequality (6) is proved.

We now observe that, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
0 \leqslant I(a) &= \frac{1}{2}\int_\Omega |f(az)-f(az^{-1})|\omega(az)d\mu(z)\\
&\leqslant \frac{1}{2}\left(\int_\Omega (f(az)-f(az^{-1}))^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\left(\int_\Omega \left[f^2(az)-2f(az)f(az^{-1})+f^2(az^{-1})\right]d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\left(2\int_\Omega [f^2(az)-f(az)f(az^{-1})]d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}}\\
&= \frac{\sqrt{2}}{2}\left(\int_\Omega f^2(az)d\mu(z) - \int_\Omega f(az)f(az^{-1})d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
I(a) &\leqslant \frac{1}{2}\left(\int_\Omega |f(az)|\omega(az)d\mu(z) + \int_\Omega |f(az^{-1})|\omega(az)d\mu(z)\right)\\
&= \int_\Omega |f(az)|\omega(az)d\mu(z)
\end{aligned}$$

and the inequality (7) is proved. □

DEFINITION 3. The function $f:\Omega \longrightarrow \mathbb{R}$ is said to be a *P-function* in Ω, if

$$f(a) \leqslant f(az) + f(az^{-1}),$$

for all $a \in \Omega$ and $z \in G$ such that $az, az^{-1} \in \Omega$.

THEOREM 3. *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. If f is measurable and a P-function real-valued on Ω such that $f \in L_1(\Omega)$ and also, $\omega:\Omega \longrightarrow \mathbb{R}$ is nonnegative and symmetric to a and $\omega \in L_1(\Omega)$ such that $f\omega \in L_1(\Omega)$, then*

$$f(a)\int_\Omega \omega(az)d\mu(z) \leqslant 2\int_\Omega f(az)\omega(az)d\mu(z).$$

Proof. Since f is P-function, we have

$$f(a)\omega(az) \leqslant f(az)\omega(az) + f(az^{-1})\omega(az).$$

Integrating this inequality on Ω, we get

$$\begin{aligned}\int_\Omega f(a)\omega(az)d\mu(z) &\leqslant \int_\Omega f(az)\omega(az)d\mu(z) + \int_\Omega f(az^{-1})\omega(az)d\mu(z)\\ &= \int_\Omega f(az)\omega(az)d\mu(z) + \int_\Omega f(az^{-1})\omega(az^{-1})d\mu(z)\\ &= 2\int_\Omega f(az)\omega(az)d\mu(z)\end{aligned}$$

Thus, $f(a)\int_\Omega \omega(az)d\mu(z) \leqslant 2\int_\Omega f(az)\omega(az)d\mu(z)$. □

Now, for a globally midconvex function $f : G \longrightarrow \mathbb{R}$ that $e \in \Omega$, we can define the mapping $H : \Omega \longrightarrow \mathbb{R}$,

$$H(x) = \frac{1}{\mu(\Omega)}\int_\Omega f(xz)d\mu(z).$$

The properties of this mapping are embodied in the following theorem:

THEOREM 4. *Suppose that $f : G \longrightarrow \mathbb{R}$ is globally midconvex and Ω is an open symmetric subset of G such that $e \in \Omega$ and $0 < \mu(\Omega) < \infty$. Then*

(i) The mapping H is globally midconvex on G, if G is abelian.

(ii) $f(e) \leqslant H(e)$.

Proof. *(i)* Assume that $a,x,z \in G$ such that $ax, ax^{-1} \in \Omega$, so

$$\begin{aligned}H(ax) + H(ax^{-1}) &= \frac{1}{\mu(\Omega)}\int_\Omega f(axz)d\mu(z) + \frac{1}{\mu(\Omega)}\int_\Omega f(ax^{-1}z)d\mu(z)\\ &= \frac{1}{\mu(\Omega)}\int_\Omega \left[f(azx) + f(azx^{-1})\right]d\mu(z)\\ &\geqslant \frac{2}{\mu(\Omega)}\int_\Omega f(az)d\mu(z)\\ &= 2H(a)\end{aligned}$$

that is, H is globally midconvex.

(ii) Since f is midconvex, by Theorem 1, $f(e) \leqslant \frac{1}{\mu(\Omega)}\int_\Omega f(z)d\mu(z) = H(e)$. □

4. Applications

In this section, we study special cases of theorems 1, 2 and 3.

Set $G=\mathbb{R}$. Since $\mathbb{R}$ is an abelian additive group, thus, for all $a,z\in\mathbb{R}$, $a-z$ and $a+z$ are points for which a is the midpoint. Now, if $a-z=x$ and $a+z=y$, then $a=\frac{x+y}{2}$. Consequently, definitions of globally midconvex and quasi-midconvex functions are to be written as follows:

$$f\left(\frac{x+y}{2}\right)\leqslant\frac{f(x)+f(y)}{2}, \quad (f \text{ globally midconvex})$$

$$f\left(\frac{x+y}{2}\right)\leqslant\max\{f(x),f(y)\}. \quad (f \text{ quasi midconvex})$$

APPLICATION 1. If in the Theorem 1 let $G=\mathbb{R}$ and $\Omega=[-a,a]$, we have

$$f(0)\int_{-a}^{a}\omega(x)dx\leqslant\int_{-a}^{a}f(x)\omega(x)dx,$$

and if we set $\omega\equiv 1$ on $[-a,a]$, we get

$$f(0)\leqslant\frac{1}{2a}\int_{-a}^{a}f(x)dx$$

that is special case of (2), where $n=1$.

APPLICATION 2. If in the Theorem 1, $G=\mathbb{R}^n$ with an additive operation and $\Omega=X$ is an open bounded symmetric and convex subset of $\mathbb{R}^n$, then the result of section 2 holds.

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