

## AN ANALYTIC SOLUTION FOR SOME SEPARABLE CONVEX QUADRATIC PROGRAMMING PROBLEMS WITH EQUALITY AND INEQUALITY CONSTRAINTS

L. BAYÓN, J. M. GRAU, M. M. RUIZ AND P. M. SUÁREZ

(Communicated by A. Prékopa)

*Abstract.* In this paper we provide a complete analytic solution to a particular separable convex quadratic programming problem with bound and equality constraints. This study constitutes the generalization of prior papers in which additional simplifications were considered. We present an algorithm that leads to determination of the analytic optimal solution. We demonstrate that our algorithm is able to deal with large-scale QP problems of this type. Finally, we present an very important application: the classical problem of economic dispatch.

### 1. Introduction

Quadratic Programming (QP) is the problem of minimizing a convex quadratic function in  $n$  variables, subject to  $m$  linear (in)equality constraints over the variables. In addition, the variables may have to lie between prespecified bounds. In this general formulation, QP can be written as:

$$\begin{aligned} \text{minimize: } & \frac{1}{2}x^T Hx + g^T x \\ \text{subject to: } & Ax \stackrel{\leq}{=} b \\ & l \leq x \leq u \end{aligned}$$

Here,  $H$  is a positive semi-definite  $n \times n$ -matrix,  $g$  an  $n$ -vector,  $A$  an  $m \times n$ -matrix,  $b$  an  $m$ -vector, and  $l, u$  are  $n$ -vectors of bounds (values  $-\infty, \infty$  may occur). The symbol  $\stackrel{\leq}{=}$  indicates that any of the  $m$  order relations it stands for can independently be ' $<$ ', ' $=$ ' or ' $>$ '. If  $H = 0$ , we obtain a linear program as a special case of QP.

QP problems have long been a subject of interest in the scientific community. Hundreds (and thousands) of papers [11] have been published that deal with applying QP algorithms to diverse problems. There is likewise a vast array of software packages for solving QP problems numerically, such as: BQPD, CGAL, CPLEX, KNITRO, LINDO, LOQO, LSSOL, MINQ, MOSEK, QPOPT or QUADPROG. Links to these (and other) QP codes can be found in [12], and in Hans Mittelmann's list [17] of QP solvers.

*Mathematics subject classification* (2010): 90C20.

*Keywords and phrases:* Quadratic programming, bound and equality constraints, equivalent thermal unit.

The solution to an unconstrained QP problem with  $H$  positive definite is obvious:  $x^* = -H^{-1}g$ . The solution of an equality constrained QP problem with  $H$  positive definite can be found by looking at the Lagrangian function:

$$L = \frac{1}{2}x^T Hx + g^T x + \lambda^T (Ax - b)$$

If the QP includes inequality constraints, then there are two main approaches to our problem. In the first, denoted pegging algorithms, an optimal solution is built up from solutions to relaxations of the problem wherein the bound constraints are relaxed. It is a recursive algorithm wherein at each iteration some variables will receive their optimal values. We refer to this as a primal algorithm (see, for example: [18], [19], [20], [21]). The second class of algorithms utilizes the simple form of the KKT conditions and/or the Lagrangian dual problem which has only one variable. We refer to this class of algorithms as a dual one (see, for example: [7], [13], [15]).

Readers are referred to [11], which constitutes an excellent list of over 1000 of the published works on QP. Within this extremely wide-ranging field of research, some authors (see [5], [10], [1], [14]) have sought the analytic solution for certain particular cases of QP problems. In this paper we provide a complete analytic solution to a particular QP problem: the separable convex quadratic programming problem consisting in

$$\begin{aligned} \text{minimize: } & \sum_{i=1}^N F_i(x_i) = \sum_{i=1}^N (\alpha_i + \beta_i x_i + \gamma_i x_i^2) \\ \text{subject to: } & \sum_{i=1}^N x_i = \xi \\ & m_i \leq x_i \leq M_i, \quad \forall i = 1, \dots, N \end{aligned}$$

being  $\gamma_i > 0$ . Several optimal algorithms have been presented for this bound and equality constrained QP problem [8], [9]. In this paper we present an algorithm that leads to determination of the analytic optimal solution. This study constitutes the generalization of prior papers ([16], [6], [2], [4]) in which additional simplifications were considered, such as only including constraints of the type  $x_i \geq 0$ , or imposing certain conditions in the bounds of the form:  $F'_i(m_i) < F'_j(M_j)$ ,  $\forall i, j$ .

In this paper we consider the above separable box and equality constrained QP problem, without any simplification. The type of constraints considered allow a hierarchy to be established among these constraints. This hierarchy is independent of the equality constraint (i.e. of  $\xi$ ), such that, of the  $3^N$  possible states of activity of the constraints, only  $2N + 1$  are theoretically feasible. On the basis of this idea, we propose an algorithm that determines the  $2N + 1$  feasible possibilities, as well as allowing the building of intervals, within each of which the set of active constraints remains constant and independent of the value of  $\xi$ . Considering  $\Psi(\xi)$  to be the solution of the problem and  $\Psi_i(\xi)$  the solution for each  $x_i$ , we establish their analytic expressions and prove that  $\Psi$  is piecewise quadratic, continuous and, under certain conditions, belongs to class  $C^1$ .

The paper is organized as follows. In the next section we present results of previous papers necessary for our approach. In Section 3 we provide some basic definitions and preparatory results. Section 4 presents the description of the algorithm that leads

to determination of the optimal solution. The results of numerical experiments are discussed in Section 5. We present two numerical examples: first, a classical problem of electrical engineering, the thermal equivalent plant, and then proceed to demonstrate that the analytic solution obtained with our algorithm is able to deal with large-scale QP problems of this type. Finally, Section 6 summarizes the main conclusions of our research.

### 2. Previous results

In this section we summarize the main results obtained by Bayon et al. in previous papers, which we consider necessary for a better understanding of the present paper.

In [2] we considered the case where the cost functions are second-order polynomials

$$F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2, \quad \forall i = 1, \dots, N$$

where  $\gamma_i > 0$  and, moreover, we imposed the natural restriction of positivity of the thermal power:  $x_i \geq 0$ . Without loss of generality, we assumed that  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_N$ . We denote by  $\Psi(\xi)$  the minimum value of  $\sum_{i=1}^N F_i(x_i)$  and  $(\Psi_1(\xi), \dots, \Psi_N(\xi))$  the vector where said minimum value is reached. Following the nomenclature employed in [2] and [4], we shall call  $\Psi$  the *equivalent minimizer* of  $\{F_i\}_{i=1}^N$  and each  $\Psi_i$  the *i-th distribution function*. We proved that the equivalent minimizer is a second-order polynomial with piece-wise constant coefficients:

$$\Psi(\xi) = \sum_{i=1}^N F_i(\Psi_i(\xi)) = \tilde{\alpha}_k + \tilde{\beta}_k \xi + \tilde{\gamma}_k \xi^2 \quad \text{if } \delta_k \leq \xi < \delta_{k+1}$$

with the coefficients

$$\delta_k = \frac{1}{2} \left[ \beta_k \sum_{i=1}^k \frac{1}{\gamma_i} - \sum_{i=1}^k \frac{\beta_i}{\gamma_i} \right]; \quad \tilde{\gamma}_k = \frac{1}{\sum_{i=1}^k \frac{1}{\gamma_i}}; \quad \tilde{\beta}_k = \tilde{\gamma}_k \sum_{i=1}^k \frac{\beta_i}{\gamma_i}; \quad \tilde{\alpha}_k = \sum_{i=1}^N \alpha_i + \frac{\tilde{\beta}_k^2}{4\tilde{\gamma}_k} - \sum_{i=1}^k \frac{\beta_i^2}{4\gamma_i}$$

We proved that for every  $k = 1, \dots, N$ , the  $k$ -th distribution function is

$$\Psi_k(\xi) = \begin{cases} \frac{\sum_{i=1}^j \frac{\beta_i}{\gamma_i} + 2\xi}{2\gamma_k \sum_{i=1}^j \frac{1}{\gamma_i}} - \frac{\beta_k}{2\gamma_k} & \text{if } \delta_k \leq \delta_j \leq \xi < \delta_{j+1} \\ 0 & \text{if } \xi < \delta_k \end{cases}$$

Moreover, we proved that  $\Psi(\xi)$  belongs to the class  $C^1$  and  $\Psi'(\delta_k) = \beta_k$  for  $i = 1, \dots, N$ .

In [4] we calculated the equivalent minimizer in the case where the cost functions are a general (non-quadratic) model. We assumed  $\{F_i\}_{i=1}^N \subset C^1[0, \infty)$ , with  $x_i \geq 0, F'_i$

strictly increasing, and with  $F'_i(0) \leq F'_{i+1}(0)$ ,  $(i = 1, \dots, N)$ . We proved that for every  $k = 1, \dots, N$ , the  $k$ -th distribution function is

$$\Psi_k(\xi) = \begin{cases} \left( \sum_{i=1}^j F_i'^{-1} \circ F_k' \right)^{-1} (\xi) & \text{if } \delta_k \leq \delta_j \leq \xi < \delta_{j+1} \\ 0 & \text{if } \xi \leq \delta_k \end{cases}$$

with  $\delta_k = \sum_{i=1}^k (F_i'^{-1} \circ F_k')(0)$ . Moreover, we analyzed the situation that arises when the thermal plants are constrained to more general restrictions of the type:  $P_{\min}^i \leq x_i \leq P_{\max}^i$ , while imposing certain conditions of the form:  $F'_i(P_{\min}^i) < F'_j(P_{\max}^j), \forall i, j$ . We proved that for every  $k = 1, \dots, N$ , the  $k$ -th distribution function is:

$$\Psi_k(\xi) = \begin{cases} P_{\min}^k & \text{if } \xi \leq \delta_k \\ \left( \sum_{i=1}^j F_i'^{-1} \circ F_k' \right)^{-1} \left( \xi - \sum_{i=j+1}^N P_{\min}^i \right) & \text{if } \delta_k \leq \delta_j \leq \xi < \delta_{j+1} \\ \left( \sum_{i=1}^N F_i'^{-1} \circ F_k' \right)^{-1} (\xi) & \text{if } \delta_N \leq \xi < \theta_1 \\ \left( \sum_{i=j+1}^N F_{\sigma(i)}'^{-1} \circ F_k' \right)^{-1} \left( \xi - \sum_{i=1}^j P_{\max}^{\sigma(i)} \right) & \text{if } \theta_j \leq \xi < \theta_{j+1} \\ P_{\max}^k & \text{if } \theta_{\sigma^{-1}(k)} \leq \xi \end{cases}$$

with  $\delta_k = \sum_{i=1}^k (F_i'^{-1} \circ F_k')(P_{\min}^k) + \sum_{i=k+1}^N P_{\min}^i$ ;  $\theta_k = \sum_{i=k}^N (F_{\sigma(i)}'^{-1} \circ F_{\sigma(k)}')(P_{\max}^{\sigma(k)}) + \sum_{i=1}^{k-1} P_{\max}^{\sigma(i)}$   
 $\sigma \in \Sigma_N$  the permutation such that  $F'_{\sigma(i)}(P_{\sigma(i)}^{\max}) \leq F'_{\sigma(i+1)}(P_{\sigma(i+1)}^{\max}), \forall i = 1, \dots, N - 1$

Moreover, we proved that  $\Psi(\xi)$  belongs to the class  $C^1$ .

### 3. Basic definitions and preparatory results

Let  $A = \{1, \dots, N\}$  and  $\{F_i\}_{i \in A}$  be a family of strictly convex quadratic functions:

$$F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2$$

We denote by  $\text{Pr}^A(\xi)$  the separable QP Problem consisting in:

$$\begin{aligned} &\text{minimize: } \sum_{i \in A} F_i(x_i) \\ &\text{subject to: } \sum_{i \in A} x_i = \xi \\ & \quad m_i \leq x_i \leq M_i, \forall i \in A \end{aligned}$$

The compactness of the set defined by the constraints guarantees that  $\text{Pr}^A(\xi)$  has a solution  $\forall \xi \in [\sum_{i \in A} m_i, \sum_{i \in A} M_i]$ , and the strict convexity of each  $F_i$ , that it is unique.

PROPOSITION 1.  $(a_1, \dots, a_N) \in \prod_{i=1}^N [m_i, M_i]$  is a solution of  $Pr^A(\xi) \iff \exists K \in \mathbb{R}$  satisfying  $\forall i = 1, \dots, N$ :

- i)  $F'_i(a_i) = K$  if  $m_i < a_i < M_i$
- ii)  $K \leq F'_i(a_i)$  if  $a_i = m_i$
- iii)  $K \geq F'_i(a_i)$  if  $a_i = M_i$

*Proof.*

$\implies$ ) Let  $l \in A$  such that  $m_l < a_l < M_l$  and let us make  $K := F'_l(a_l)$ . It is easily proven, using theory of Lagrange multipliers, that if  $m_j < a_j < M_j$ , then  $F'_l(a_l) = F'_j(a_j) = K$ . Let us now assume that  $a_k = m_k$  and  $K > F'_k(m_k)$ , and we consider

$$\Phi(\varepsilon) = \sum_{i \in A - \{l, k\}} F_i(a_i) + F_l(a_l - \varepsilon) + F_k(a_k + \varepsilon)$$

Note that  $\Phi(0) = \sum_{i \in A} F_i(a_i)$  constitutes the minimum value of  $\sum_{i \in A} F_i(x_i)$  constrained to

$$\sum_{i \in A} x_i = \xi, \text{ and } m_i \leq x_i \leq M_i, \forall i \in A$$

However,  $\Phi'(0) = -F'_l(a_l) + F'_k(a_k) = -K + F'_k(m_k) < 0$ , which contradicts the minimal nature of  $(a_1, \dots, a_N)$  since, for some  $\varepsilon > 0$ , we shall have

$$\sum_{i \in A} F_i(x_i) > \sum_{i \in A - \{l, k\}} F_i(a_i) + F_l(a_l - \varepsilon) + F_k(a_k + \varepsilon)$$

the vector  $(a_1, \dots, a_l - \varepsilon, \dots, a_k + \varepsilon, \dots, a_N)$  satisfying the problem constraints.

Assuming that  $a_k = M_k$  and  $K < F'_k(M_k)$  leads to the identical contradiction.

$\impliedby$ ) It suffices to note that the Gâteaux derivative in  $(a_1, \dots, a_N)$  with respect to any admissible direction is greater than or equal to zero. In fact, every admissible direction  $\omega = (\omega_1, \dots, \omega_n)$  in the vector  $(a_1, \dots, a_N)$  satisfies the following:

$$\omega_i \geq 0 \text{ if } a_i = m_i, \omega_i \leq 0 \text{ if } a_i = M_i \text{ and } \sum_{i \in A} \omega_i = 0$$

Let us now consider

$$A_0 = \{\forall i \in A | m_i < a_i < M_i\}, A_- = \{\forall i \in A | m_i = a_i\} \text{ and } A_+ = \{\forall i \in A | M_i = a_i\}$$

We have that the Gâteaux derivative in  $(a_1, \dots, a_N)$  in the direction of the vector  $\omega$  is:

$$\delta F(v, \omega) = \lim_{\varepsilon \rightarrow 0} \frac{F(v + \varepsilon \omega) - F(v)}{\varepsilon} = \sum_{i \in A_0} F'_i(a_i) \omega_i + \sum_{i \in A_-} F'_i(m_i) \omega_i + \sum_{i \in A_+} F'_i(M_i) \omega_i \geq 0$$

□

COROLLARY 1. *The  $i$ -th distribution functions  $\Psi_i$  are not decreasing.*

*Proof.* If  $\Psi_i(\xi) = m_i$ , it is obvious that  $\Psi_i$  is not decreasing in  $\xi$ . If  $m_i < \Psi_i(\xi) \leq M_i$ , in virtue of Proposition 1, it is verified that  $F'_i(\Psi_i(\xi)) \leq K_\xi$ . Let us now assume that  $\Psi_i(\xi + \varepsilon) < \Psi_i(\xi)$ , hence  $F'_i(\Psi_i(\xi + \varepsilon)) < F'_i(\Psi_i(\xi)) \leq K_\xi$ . Hence,  $\Psi_i(\xi + \varepsilon) = M_i$ , which contradicts the assumption  $\Psi_i(\xi + \varepsilon) < \Psi_i(\xi) \leq M_i$ .  $\square$

DEFINITION 1. Let us consider in the set  $A \times \{m, M\}$  the binary relation  $\preceq$  defined as follows:

$$\begin{aligned} (i, m) \preceq (j, m) &\iff F'_i(m_i) < F'_j(m_j) \text{ or } (F'_i(m_i) = F'_j(m_j) \text{ and } i \leq j) \\ (i, m) \preceq (j, M) &\iff F'_i(m_i) < F'_j(M_j) \text{ or } (F'_i(m_i) = F'_j(M_j) \text{ and } i \leq j) \\ (i, M) \preceq (j, m) &\iff F'_i(M_i) < F'_j(m_j) \text{ or } (F'_i(M_i) = F'_j(M_j) \text{ and } i \leq j) \\ (i, M) \preceq (j, M) &\iff F'_i(M_i) < F'_j(M_j) \text{ or } (F'_i(M_i) = F'_j(M_j) \text{ and } i \leq j) \end{aligned}$$

Obviously,  $\preceq$  is a total order relation and  $(A \times \{m, M\}, \preceq)$  is isomorphic with respect to  $(\{1, 2, \dots, 2N\}, \leq)$ . We denote by  $g$  the isomorphism  $g(i) := (g_1(i), g_2(i))$ ,  $g : (\{1, 2, \dots, 2N\}, \leq) \longrightarrow (A \times \{m, M\}, \preceq)$ , which at each natural number  $n \in \{1, 2, \dots, 2N\}$  corresponds to the  $n$ -th element of  $A \times \{m, M\}$  following the order established by  $\preceq$ .

PROPOSITION 2. *Let  $v = (a_1, \dots, a_N)$  be a solution of  $\text{Pr}^A(\xi)$ .*

- a) *If  $(i, m) \preceq (j, m)$  then  $a_j > m_j \Rightarrow a_i > m_i$  (or  $a_i = m_i \Rightarrow a_j = m_j$ ).*
- b) *If  $(i, M) \preceq (j, M)$  then  $a_i < M_i \Rightarrow a_j < M_j$  (or  $a_j = M_j \Rightarrow a_i = M_i$ ).*
- c) *If  $(i, m) \preceq (j, M)$  and  $F'_i(m_i) \neq F'_j(M_j)$  then  $a_j = M_j \Rightarrow a_i > m_i$  (or  $a_i = m_i \Rightarrow a_j < M_j$ ).*
- d) *If  $(i, M) \preceq (j, m)$  then  $a_i < M_i \Rightarrow a_j = m_j$  (or  $a_j > m_j \Rightarrow a_i = M_i$ ).*

*Proof.* a) If  $(i, m) \preceq (j, m)$ , then  $F'_i(m_i) \leq F'_j(m_j)$ . Assuming that  $a_i = m_i$  and  $a_j > m_j$  leads to the contradiction. Let us consider the function:

$$\Phi(\varepsilon) = \sum_{k \in A - \{i, j\}} F_k(a_k) + F_i(a_i + \varepsilon) + F_j(a_j - \varepsilon)$$

Hence  $\Phi'(0) = F'_i(m_i) - F'_j(a_j) < F'_i(m_i) - F'_j(m_j) \leq 0$ , which contradicts the minimal nature of  $(a_1, \dots, a_N)$ .

b), c) and d) By identical reasoning.  $\square$

This proposition allows us to interpret that the set  $A \times \{m, M\}$  symbolizes the  $2N$  possible states of activity/inactivity of the variable constraints. Accordingly,  $(i, m)$  symbolizes that the constraint  $x_i \geq m_i$  is *inactive* ( $x_i > m_i$ ) and  $(i, M)$  symbolizes that the constraint  $x_i \leq M_i$  is *active* ( $x_i = M_i$ ). Thus, the relation  $\preceq$  establishes a hierarchical order among these in the sense that a vector  $v = (a_1, \dots, a_N)$  which constitutes the solution of the problem  $\text{Pr}^A(\xi)$  and satisfies  $a_i = m_i$  will necessarily also have to satisfy  $a_k = m_k$  if  $(i, m) \preceq (k, m)$  and, likewise,  $a_k < M_k$  if  $(i, m) \preceq (k, M)$ . In other words, the activation of the minimal constraints and the activation of the maximal constraints

present an order of priority (Proposition 2) that the solution of the problem must necessarily respect. This fact, which is not exclusive to quadratic problems, is of extraordinary importance, since it allows the  $3^N$  possible combinations of activity/inactivity of the constraints to be reduced to only  $2N + 1$ .

DEFINITION 2. Let  $A = \{1, 2, \dots, N\}$ . We denote by  $3^A$  the set:

$$3^A := \{(A_1, A_2, A_3) \in P(A)^3 \mid A_1 \cup A_2 \cup A_3 = A \text{ and } A_i \cap A_j = \emptyset, \forall i \neq j\}$$

We can interpret each triad  $(A_1, A_2, A_3)$  as the representation of the state of activity of the constraints in the sense that the elements of  $A_1$  symbolize the variables whose lower constraint is active ( $x_i = m_i$ ),  $A_3$  the variables whose upper constraints are active ( $x_i = M_i$ ) and  $A_2$  the variables whose constraints are both inactive. We thus have a total of  $3^N$  possibilities, many of which may be ruled out in virtue of Proposition 2.

### 4. Algorithm

In this section we present the optimization algorithm that leads to determination of the optimal solution. The algorithm generates all the feasible states of activity/inactivity of the constraints on the solution of the problem (which do not contradict Proposition 2). We build a sequence  $(\Omega_n, \Theta_n, \Xi_n) \subset 3^A$  starting with the triad  $(A, \emptyset, \emptyset)$ , which represents the fact that all the constraints on minimum are active and ending with the triad  $(\emptyset, \emptyset, A)$ , which represents the fact that all the constraints on maximum are active. Each step of the process consists in decreasing by one unit the number of active constraints on minimum or increasing by one unit the number of active constraints on maximum, following the order established by the relation  $\preceq$ . Specifically, the constraint on minimum that is deactivated or the constraint on maximum that is activated in the  $n$ -th step is symbolized by  $g(n)$ . In the  $n$ -th step,  $g(n) = (i, M)$  (resp.  $g(n) = (i, m)$ ) consists to activate (resp. deactivate) the constraint:  $x_i \leq M_i$  (resp.  $x_i \geq m_i$ ).

Let us consider the following recurrent sequence  $X_n := (\Omega_n, \Theta_n, \Xi_n) \in 3^A$ , ( $n = 0, \dots, 2N$ ):

$$\begin{aligned} & \Omega_0 = A & \Theta_0 = \emptyset & \Xi_0 = \emptyset \\ \text{If } g_2(n) = M \implies & \Omega_n := \Omega_{n-1} & \Theta_n := \Theta_{n-1} - \{g_1(n)\} & \Xi_n := \Xi_{n-1} \cup \{g_1(n)\} \\ \text{If } g_2(n) = m \implies & \Omega_n := \Omega_{n-1} - \{g_1(n)\} & \Theta_n := \Theta_{n-1} \cup \{g_1(n)\} & \Xi_n := \Xi_{n-1} \end{aligned}$$

PROPOSITION 3.  $v = (a_1, \dots, a_N) \in \prod_{i=1}^N [m_i, M_i]$  is the solution of  $Pr^A(\xi) \Leftrightarrow \exists K \in \mathbb{R}$ , and  $n \in A$  satisfying:

- i)  $F'_i(a_i) = K, \forall i \in \Theta_n$
- ii)  $K \leq F'_i(m_i), \forall i \in \Omega_n$
- iii)  $K \geq F'_i(M_i), \forall i \in \Xi_n$

*Proof.*

$\Leftarrow$ ) It is an immediate consequence of Proposition 1.

⇒) It is likewise an immediate consequence of Proposition 1, bearing in mind that the sequence  $X_n$  contains the  $2N + 1$  possible states of activity/inactivity of the constraints, which is compatible with the fact that  $(a_1, \dots, a_N)$  is the solution to the problem  $\text{Pr}^A(\xi)$ . □

PROPOSITION 4. *There exist  $\{\phi_i\}_{i=1}^{2N} \subset \mathbb{R}$ ,  $\sum_{i=1}^N m_i = \phi_1 \leq \dots \leq \phi_{2N} = \sum_{i=1}^N M_i$  such that if  $\phi_n < \xi < \phi_{n+1}$ , the solution of the problem  $v = (\Psi_1(\xi), \dots, \Psi_N(\xi))$  satisfies:*

$$\Psi_k(\xi) = \begin{cases} \frac{\xi - \sum_{i \in \Omega_n} m_i - \sum_{j \in \Xi_n} M_j + \sum_{i \in \Theta_n} \frac{\beta_i}{2\gamma_i}}{\gamma_k \sum_{i \in \Theta_n} \frac{1}{\gamma_i}} - \frac{\beta_k}{2\gamma_k} & \text{if } k \in \Omega_n \\ m_k & \text{if } k \in \Theta_n \\ M_k & \text{if } k \in \Xi_n \end{cases}$$

being

$$\phi_n = \begin{cases} \frac{1}{2} \sum_{i \in \Omega_n} \frac{F'_{g_1(n)}(m_{g_1(n)}) - \beta_i}{\gamma_i} + \sum_{i \in \Omega_n} m_i + \sum_{j \in \Xi_n} M_j & \text{if } g_2(n) = m \\ \frac{1}{2} \sum_{i \in \Theta_n} \frac{F'_{g_1(n)}(M_{g_1(n)}) - \beta_i}{\gamma_i} + \sum_{i \in \Omega_n} m_i + \sum_{j \in \Xi_n} M_j & \text{if } g_2(n) = M \end{cases}$$

*Proof.* Since the distribution functions  $\Psi_i$  are not decreasing, if the solution for the problem  $\text{Pr}^A(\xi)$  presents an inactive constraint on minimum (respectively active constraint on maximum), it will likewise do so for greater values than  $\xi$ . It is therefore obvious that there exists a collection of real numbers  $\{\phi_i\}_{i=1}^{2N}$  such that in each interval  $(\phi_n, \phi_{n+1}]$  the corresponding problem  $(\phi_n < \xi \leq \phi_{n+1})$  has identical active constraints on maximum and inactive constraints on minimum and one less inactive constraint on minimum or one more active constraint on maximum in the following interval  $(\phi_{n+1}, \phi_{n+2}]$ . Specifically, the active constraints on minimum in each interval  $(\phi_n, \phi_{n+1}]$  will be those represented by the set  $\Omega_n$  and the active constraints on maximum those represented by  $\Xi_n$ .

Bearing in mind that in each interval  $(\phi_n, \phi_{n+1}]$  the next constraint on minimum to become deactivated or the next constraint on maximum to become active (in accordance with the relation  $\preceq$ ) is that corresponding to  $g(n)$ , we obtain the expression for each  $\Psi_i$  by reasoning similarly as in [2] and [4]. □

PROPOSITION 5. *The function  $\Psi$  (equivalent minimizer) is piecewise quadratic, continuous and, if  $\Theta_n \neq \emptyset$ ,  $\forall 0 < i < 2N$ , then it also belongs to class  $C^1$ . Specifically, if  $\phi_n \leq \xi \leq \phi_{n+1}$*

$$\Psi(\xi) = \widehat{\alpha}_n + \widehat{\beta}_n(\xi - \mu_n) + \widehat{\gamma}_n(\xi - \mu_n)^2$$

$$\mu_n := \sum_{i \in \Omega_n} m_i + \sum_{j \in \Xi_n} M_j; \quad \widehat{\gamma}_n := \frac{1}{\sum_{i \in \Theta_n} \frac{1}{\gamma_i}}; \quad \widehat{\beta}_n := \widehat{\gamma}_n \sum_{i \in \Theta_n} \frac{\beta_i}{\gamma_i}$$



$$\widehat{\alpha}_n := \sum_{i \in \Theta_n} \alpha_i + \frac{\widehat{\beta}_n^2}{4\widehat{\gamma}_n} - \sum_{i \in \Theta_n} \frac{\beta_i^2}{4\gamma_i} + \sum_{i \in \Omega_n} F_i(m_i) + \sum_{i \in \Xi_n} F_i(M_i)$$

*Proof.* Both its piecewise quadratic nature and the values of the coefficients are easily established as in [2] and [4]. The continuity and the character  $C^1$ , (which can only be guaranteed when  $\Theta_n \neq \emptyset, \forall n = 1, \dots, 2N - 1$ ), are easily proven by simply using the technique employed in [2] and [4] for particular cases.  $\square$

### 5. Examples

#### 5.1. Equivalent Thermal unit

Quadratic programs are widely used in many hundreds of real-life applications, such us portfolio analysis, support vector machines, structural analysis, discrete-time stabilization, optimal and fuzzy control, finite impulse response design, optimal power and economic dispatch. In this section we present an example, embedded in the line of research entitled *Optimization of hydrothermal systems*, which constitutes a complicated problem that has attracted significant interest in recent decades: the equivalent thermal unit. In previous papers ([2], [4]) we prove that power plants can be substituted by a single one that behaves equivalently to the entire set. This supposes a significant simplification of hydrothermal models [3] and will also be useful for any method used to study the problem. It allows us to develop algorithms that are simpler, more reliable, and which require less time for their execution.

Table I: Coefficients of the thermal plants.

Plant $i$	$\alpha_i$	$\beta_i$	$\gamma_i$	$m_i$	$M_i$
1	841.75	50.431	0.1987	50	360
2	115.67	31.265	0.0166	60	543
3	116.58	66.915	0.1429	50	253
4	423.02	25.014	0.0248	34	350
5	300.00	1.5500	0.0111	30	250

A program that solves the optimization problem was developed using the Mathematica package and was then applied to an example of a thermal system made up of 5 thermal plants. For the thermal plants, the cost function  $F_i$  used is a quadratic model:  $F_i(x) = \alpha_i + \beta_i x + \gamma_i x^2$ . The data of the plants are summarized in Table I. The units for the coefficients are:  $\alpha_i$  in (euro/h),  $\beta_i$  in (euro/h.MW), and  $\gamma_i$  in (euro/h.MW<sup>2</sup>). We shall now apply the theory developed previously. Bearing in mind the values of  $F'_i(m_i)$  and  $F'_i(M_i)$ :

$F'_5(m_5)$	$F'_5(M_5)$	$F'_4(m_4)$	$F'_2(m_2)$	$F'_4(M_4)$	$F'_2(M_2)$	$F'_1(m_1)$	$F'_3(m_3)$	$F'_3(M_3)$	$F'_1(M_1)$
2.22	7.10	26.70	33.26	42.37	49.29	70.30	81.21	139.22	193.49

we have that the elements of  $(A \times \{m, M\})$ , in accordance with the order  $\preceq$  are:

$$\{5, m\} \preceq \{5, M\} \preceq \{4, m\} \preceq \{2, m\} \preceq \{4, M\} \preceq \{2, M\} \preceq \{1, m\} \preceq \{3, m\} \preceq \{3, M\} \preceq \{1, M\}$$

$$g(1) \quad g(2) \quad g(3) \quad g(4) \quad g(5) \quad g(6) \quad g(7) \quad g(8) \quad g(9) \quad g(10)$$

and the sequence  $X_n := (\Omega_n, \Theta_n, \Xi_n)$

$\Omega_n$	$\Theta_n$	$\Xi_n$
{1, 2, 3, 4, 5}	{}	{}
{1, 2, 3, 4}	{5}	{}
{1, 2, 3, 4}	{}	{5}
{1, 2, 3}	{4}	{5}
{1, 3}	{4, 2}	{5}
{1, 3}	{2}	{5, 4}
{1, 3}	{}	{5, 4, 2}
{3}	{1}	{5, 4, 2}
{}	{1, 3}	{5, 4, 2}
{}	{1}	{5, 4, 2, 3}
{}	{}	{1, 2, 3, 4, 5}

The family  $\{\phi_i\}_{i=1}^{10} \subset \mathbb{R}$ , where  $\sum_{i=1}^5 m_i = \phi_1 \leq \dots \leq \phi_{2N} = \sum_{i=1}^5 M_i$ , is:  $\phi_1 = 224.$ ;  $\phi_2 = 444.$ ;  $\phi_3 = 444.$ ;  $\phi_4 = 576.19$ ;  $\phi_5 = 1034.61$ ;  $\phi_6 = 1243.$ ;  $\phi_7 = 1243.$ ;  $\phi_8 = 1270.44$ ;  $\phi_9 = 1619.43$ ;  $\phi_{10} = 1756$ . The coincidences  $\phi_2 = \phi_3$  and  $\phi_6 = \phi_7$  are due to the fact that  $\Theta_2 = \emptyset = \Theta_6$ . The fact that, for  $\xi = \phi_2$ , the solution of  $\text{Pr}^A(\xi)$  has all its constraints active ( $\Theta_2 = \emptyset$ ) makes it impossible for this situation to be produced in any interval of the form  $[\phi_2, \phi_2 + \varepsilon)$  with  $\varepsilon > 0$ , and hence  $\phi_3$  must necessarily coincide with  $\phi_2$ . In this case, the equivalent minimizer presents angular points in  $\phi_2 = \phi_3 = 444$  and in  $\phi_6 = \phi_7 = 1243$  (see Figure 1-a). The equivalent plant of these functions,  $\Psi(\text{euro}/h)$  (with  $\xi$  in *MW*) is a second-order polynomial with piecewise constant coefficients:

$$\Psi(\xi) = \begin{cases} 11450.2 - 2.7568\xi + 0.0111\xi^2 & \text{if } \phi_1 \leq \xi \leq \phi_3 \\ 5448.39 + 4.678\xi + 0.0248\xi^2 & \text{if } \phi_3 \leq \xi \leq \phi_4 \\ 516.274 + 21.7978\xi + 0.00994\xi^2 & \text{if } \phi_4 \leq \xi \leq \phi_5 \\ 7640.99 + 8.025\xi + 0.0166\xi^2 & \text{if } \phi_5 \leq \xi \leq \phi_7 \\ 262881. - 423.667\xi + 0.1987\xi^2 & \text{if } \phi_7 \leq \xi \leq \phi_8 \\ 76335. - 129.996\xi + 0.08312\xi^2 & \text{if } \phi_8 \leq \xi \leq \phi_9 \\ 379447. - 504.339\xi + 0.1987\xi^2 & \text{if } \phi_9 \leq \xi \leq \phi_{10} \end{cases}$$

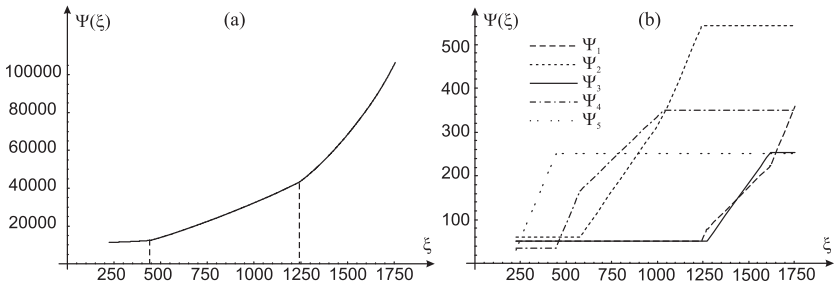


Fig. 1. (a) Equivalent Thermal Plant. (b) The distribution functions.

Figure 1-b shows the distribution functions. These are the power ratings that the

thermal plants must generate, for each power demand, for the overall cost to be minimum.

REMARK. Coincidences in the  $\phi_i$  may also arise without any  $\Theta_i$  being empty. In fact, this occurs whenever we have situations of the type:  $F'_i(m_i) = F'_j(m_j)$  or  $F'_i(m_i) = F'_j(M_j)$ . In these cases, however, the equivalent minimizer does not cease to belong to class  $C^1$ .

**5.2. Large-scale QP**

Our particular concern is for medium- to large-scale problems, i.e. those involving tens or hundreds of thousands of unknowns and/or constraints. There is already a vast literature concerned with methods appropriate for small problems (those involving hundreds or low thousands of variables/constraints), and a number of excellent software packages.

In this section we present an example of a large-scale QP problem. We shall generate an example, which is very easy to reproduce, considering the quadratic model:  $F_i(x) = \alpha_i + \beta_i x + \gamma_i x^2$ , with  $\xi = 50$ , generating the coefficients with the simple formulas:

$$\alpha_i = 0; \beta_i = i; \gamma_i = \frac{1}{2i}; m_i = \frac{1}{i}; M_i = \frac{1}{i} + 1, \quad i = 1, \dots, n$$

Table II shows that the Matlab solver QUADPROG cannot deal with this type of problem, giving erroneous solutions from  $n = 201$  onward.

Table II: Comparison of Costs.

$n$	201	202	203	300	500
Solution $\Psi(\xi)$	1200.67478	1201.45199	1202.23030	1281.96697	1459.51433
QUADPROG	1200.67478	1220.65661	1223.27776	2473.34456	6664.10737

Secondly, Table III presents the solution obtained and the CPU time (in seconds) used (measured on a Pentium IV, 3.4GHz PC) when  $n$  is large. We present the times that are consumed in each of the different phases of algorithm: Phase I: Construction of the sequence  $X_n$ ; Phase II: ordonation of the elements of  $(A \times \{m, M\})$ , in accordance with the order  $\preceq$ ; Phase III: Calculus of the  $n$  such that  $\phi_n < \xi < \phi_{n+1}$ ; Phase IV: calculus of the exact solution  $\Psi(\xi)$ .

Table III: CPU time.

	$n$	1000	5000	10000
	Exact Solution $\Psi(\xi)$	1929.51391	5861.49621	10833.06937
CPU	Phase I: sequence $X_n$	0.188	3.791	15.304
CPU	Phase II: $(A \times \{m, M\}, \preceq)$	0.546	3.4	7.301
CPU	Phase III: $\phi_n < \xi < \phi_{n+1}$	0.206	0.678	2.066
CPU	Phase IV: $\Psi(\xi)$	0.281	6.849	27.831

## 6. Conclusions

In this paper we have provided a complete analytic solution to a particular separable convex quadratic programming problem with bound and equality constraints. This study constitutes the generalization of prior papers in which additional simplifications were considered. We have demonstrated that our algorithm is able to deal with large-scale QP problems. This study puts the finishing touches to the so-called economic dispatch problem in Electrical Engineering and may also be applied to problems in economics such as the maximization of consumer utility under budgetary restrictions with numerous goods whose utility functions are quadratic.

## REFERENCES

- [1] C. AUDET, P. HANSEN, S. LE DIGABEL, *Exact solution of three nonconvex quadratic programming problems*, In *Frontiers in Global Optimization*, Volume 20 of *Nonconvex Optimization and Applications*, Kluwer Acad. Publ., Dordrecht (2004), 25–45.
- [2] L. BAYÓN, J. M. GRAU, P. M. SUÁREZ, *A New Formulation of the Equivalent Thermal in Optimization of Hydrothermal Systems*, *Math. Probl. Eng.*, **8**, 3 (2002), 181–196.
- [3] L. BAYÓN, J. M. GRAU, M. M. RUIZ, P. M. SUÁREZ, *The First Weierstrass-Erdmann Condition in Variational Problems Involving Differential Inclusions*, *Math. Inequal. Appl.*, **7**, 3 (2004), 457–469.
- [4] L. BAYÓN, J. M. GRAU, M. M. RUIZ, P. M. SUÁREZ, *New developments on equivalent thermal in hydrothermal optimization: an algorithm of approximation*, *J. Comput. Appl. Math.*, **175**, 1 (2005), 63–75.
- [5] R. CAMBINI, *Soluzione ottima esplicita di un particolare problema di programmazione quadratica*, *Decisions in Economics and Finance*, Springer, Milan, **15**, 2 (1992), 73–86.
- [6] S. COSARES, D. S. HOCHBAUM, *Strongly polynomial algorithms for the quadratic transportation problem with a fixed number of sources*, *Math. Oper. Res.*, **19**, 1 (1994), 94–111.
- [7] K. DAHIYA, S. K. SUNEJA, V. VERMA, *Convex programming with single separable constraint and bounded variables*, *Computational Optimization and Applications. An International Journal*, **36**, 1 (2007), 67–82.
- [8] Z. DOSTAL, *Inexact semimonotonic augmented Lagrangians with optimal feasibility convergence for convex bound and equality constrained quadratic programming*, *SIAM J. Numer. Anal.*, **43**, 1 (2005), 96–115.
- [9] Z. DOSTAL, *An optimal algorithm for bound and equality constrained quadratic programming problems with bounded spectrum*, *Computing*, **78**, 4 (2006), 311–328.
- [10] C. J. GOH AND X. Q. YANG, *Analytic efficient solution set for multi-criteria quadratic programs*, *European J. Oper. Res.*, **92**, 1 (1996), 166–181.
- [11] N. I. M. GOULD, PH. L. TOINT, *A Quadratic Programming Bibliography*, [http://www.optimization-online.org/DB\\_HTML/2001/02/285.html](http://www.optimization-online.org/DB_HTML/2001/02/285.html), 2001.
- [12] N. I. M. GOULD, PH. L. TOINT, *A Quadratic Programming Page*, <http://www.numerical.rl.ac.uk/qp/qp.html>.
- [13] W. W. HAGER, S. PARK, *The gradient projection method with exact line search*, *Journal of Global Optimization*, **30**, 1 (2004), 103–118.
- [14] S. KIM, M. KOJIMA, *Exact Solutions of Some Nonconvex Quadratic Optimization Problems via SDP and SOCP Relaxations*, *Comput. Optim. Appl.*, **26**, 2 (2003), 143–154.
- [15] P. A. LOTITO, *Issues in the implementation of the DSD algorithm for the traffic assignment problem*, *European Journal of Operational Research*, **175**, 3 (2006), 1577–1587.
- [16] N. MEGIDDO, A. TAMIR, *Linear time algorithms for some separable quadratic programming problems*, *Oper. Res. Lett.*, **13** (1993), 203–211.
- [17] H. D. MITTELMANN, *Decision Tree for Optimization Software*, <http://plato.asu.edu/guide.html>.
- [18] S. M. STEFANOV, *Convex Separable Minimization Subject to Bounded Variables*, *Computational Optimization and Applications*, **18**, 1 (2001), 27–48.

- [19] S.M. STEFANOV, *Separable Programming. Theory and Methods*, Kluwer Academic Publishers: Dordrecht-Boston-London, 2001.
- [20] S.M. STEFANOV, *Convex quadratic minimization subject to a linear constraint and box constraints*, *Appl. Math. Res. eXpress*, **1** (2004), 17–42.
- [21] S.M. STEFANOV, *Polynomial algorithms for projecting a point onto a region defined by a linear constraint and box constraints in  $\mathbb{R}^n$* , *J. Appl. Math.*, **2004**, 5 (2004), 409–431.

(Received July 8, 2008)

*L. Bayón*  
*E.U.I.T.I. Department of Mathematics*  
*Campus of Viesques*  
*33203 Gijón*  
*Spain*  
*e-mail: bayon@uniovi.es*

*J. M. Grau*  
*University of Oviedo*  
*Spain*  
*e-mail: grau@uniovi.es*

*M. M. Ruiz*  
*University of Oviedo*  
*Spain*  
*e-mail: mruiz@uniovi.es*

*P. M. Suárez*  
*University of Oviedo*  
*Spain*  
*e-mail: pedrosr@uniovi.es*