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HOMOGENEOUS MEANS GENERATED BY A MEAN–VALUE THEOREM

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Dedicated to Professor Peter Kahlig
on the occasion of his seventieth birthday

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Abstract. A variant of Cauchy mean-value theorem is presented. Applying a counterpart of the Lagrange mean-value theorem, we introduce new type of means. The functions generating the homogeneous means of this type are determined. In some special cases the effective formulas for these means are presented. Mean-value-theorems which do not lead to any mean are also mentioned.

1. Introduction

The classical Lagrange and Cauchy mean-value theorems, in a natural and straightforward way, lead to the important classes of means (cf. Bullen, Mitrinović, Vasić [2], and Bullen [3]).

In section 2 we present a variant of the Cauchy mean-value theorem. As a special case we obtain a counterpart of the Lagrange mean-value theorem (see Corollary 1). For a given real differentiable function, it allows to define, at least in an implicit way, a new type of mean. In general it is not symmetric. If it symmetric, then it coincides with the Lagrange mean (Corollary 2). Contrary to the case of a Lagrange mean, it is difficult to get its effective formula. Even the conditions for uniqueness of the mean are not obvious. In section 3, assuming the continuous differentiability of the function (a generator of the mean), we show that uniqueness of the mean implies its continuity as a function of two variables. Applying the implicit function theorem, we prove that twice differentiability of the generator with non-vanishing first derivative guarantee local uniqueness of the mean in a neighbourhood of the diagonal. Moreover, some sufficient conditions for the global uniqueness of the mean are presented (Theorem 3).

In section 4, under the regularity conditions mentioned above, we show that the considered mean is homogeneous iff either its generator is a power function or logarithmic function (Theorem 4). We denote by $\mathscr{F}^{[p]}$ the homogeneous mean generated by the power function $f(x) = x^p$ for $p \neq 0$ and by $\mathscr{F}^{[0]}$ the mean generated by $f = \log$. We show that $\lim_{p\to 0} \mathscr{F}^{[p]} = \mathscr{F}^{[0]}$ pointwise. Moreover, for $p \in \left\{3, 2, 1, \frac{1}{2}, -1, -2\right\}$ we give the effective forms of the mean $\mathscr{F}^{[p]}$.

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In section 5 some inequalities between the considered means and the arithmetic mean are given.

At the end of our paper we present some Flett type mean-value theorems with a comment explaining why these results are not useful in introducing the relevant means.

2. Some mean-value theorems and means

In this section we prove the following counterpart of the Cauchy mean-value theorem:

THEOREM 1. *If $f,g:[a,b]\to\mathbb{R}$ are differentiable on (a,b) and continuous at the points a and b, then there exists a point $\eta\in(a,b)$ such that*

$$g'(\eta)\,[f(\eta)-f(a)] = f'(\eta)\,[g(b)-g(\eta)]\,.$$

If moreover $g'(x)\neq 0$ for all $x\in(a,b)$, then there exists a point $\eta\in(a,b)$ such that

$$\frac{f(\eta)-f(a)}{g(b)-g(\eta)} = \frac{f'(\eta)}{g'(\eta)}.$$

Proof. The function $\varphi:[a,b]\to\mathbb{R}$,

$$\varphi(t) := [g(b)-g(t)]\,[f(t)-f(a)]\,, \qquad t\in[a,b],$$

is continuous in $[a,b]$. Since $\varphi(a)=0=\varphi(b)$, by the Rolle theorem there exists a point $\eta\in(a,b)$ such that $\varphi'(\eta)=0$. As

$$\varphi'(t) = -g'(t)\,[f(t)-f(a)] + [g(b)-g(t)]f'(t), \qquad t\in[a,b],$$

we hence get

$$-g'(\eta)\,[f(\eta)-f(a)] + [g(b)-g(\eta)]f'(\eta) = 0,$$

which completes the proof of the first assertion. Since the moreover part is obvious, the proof is complete. □

COROLLARY 1. *If $f:[a,b]\to\mathbb{R}$ is differentiable on (a,b) and continuous at a and b, then there exists a point $\eta\in(a,b)$ such that*

$$f(\eta)-f(a) = f'(\eta)(b-\eta).$$

Let $I\subset\mathbb{R}$ be an interval. Recall that a function $M:I^2\to I$ is called a *strict mean in I*, if

$$\min(x,y) < M(x,y) < \max(x,y), \quad (x,y)\in I^2,\ x\neq y; \quad M(x,x)=x \ \text{ for } \ x\in I.$$

THEOREM 2. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $f,g : I \to \mathbb{R}$ are differentiable and $g'(x) \neq 0$ for $x \in I$. Then there exists a strict mean $M : I^2 \to I$ such that*

$$\frac{f(M(x,y)) - f(x)}{g(y) - g(M(x,y))} = \frac{f'(M(x,y))}{g'(M(x,y))}, \quad x,y \in I, \ x \neq y. \tag{1}$$

Moreover, if $\frac{f'}{g'}$ is one-to-one and M is symmetric, that is,

$$M(x,y) = M(y,x), \quad x,y \in I,$$

then $M^{[f,g]} := M$ is unique, and

$$M^{[f,g]} = C^{[f,g]},$$

where $C^{[f,g]} : I^2 \to I$ is the Cauchy mean of the generators f and g, given by

$$C^{[f,g]}(x,y) := \left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(x) - f(y)}{g(x) - g(y)}\right), \quad x \neq y.$$

Proof. Take arbitrary $x,y \in I$, $x \neq y$. Applying the previous theorem with $a := \min(x,y)$, $b := \max(x,y)$ we obtain the existence of a strict mean M such that (1) holds true. If M is symmetric then, from (1),

$$\frac{f(M(x,y)) - f(x)}{g(y) - g(M(x,y))} = \frac{f(M(x,y)) - f(y)}{g(x) - g(M(x,y))},$$

whence

$$f(M(x,y)) = \frac{f(x)g(x) - f(y)g(y) - g(M(x,y))[f(x) - f(y)]}{g(x) - g(y)}.$$

Now, making again use of (1), we get

$$\frac{\frac{f(x)g(x) - f(y)g(y) - g(M(x,y))[f(x) - f(y)]}{g(x) - g(y)} - f(x)}{g(y) - g(M(x,y))} = \frac{f'(M(x,y))}{g'(M(x,y))},$$

which, after obvious simplifications, reduces to the equality

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(M(x,y))}{g'(M(x,y))}.$$

As $\frac{f'}{g'}$ is one-to-one, we hence get

$$M(x,y) = \left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(x) - f(y)}{g(x) - g(y)}\right).$$

Thus $M = M^{[f,g]}$ is unique and $M^{[f,g]} = C^{[f,g]}$. This completes the proof. □

COROLLARY 2. *Let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is differentiable, then there exists a strict mean $M : I^2 \to I$ such that*

$$\frac{f(M(x,y)) - f(x)}{y - M(x,y))} = f'(M(x,y)), \quad x,y \in I, \ x \neq y.$$

Moreover, if f' is one-to-one and M symmetric, that is,

$$M(x,y) = M(y,x), \quad x,y \in I,$$

then $M^{[f]} := M$ is unique, and

$$M^{[f]} = L^{[f]},$$

where $L^{[f]} : I^2 \to I$ is the Lagrange mean (of a generator f) given by

$$L^{[f]}(x,y) := \left(f'\right)^{-1}\left(\frac{f(x) - f(y)}{x - y}\right), \quad x \neq y.$$

REMARK 1. Take $I = (0,\infty)$. For $f(x) = x^p$ where $p \in \mathbb{R}$, $0 \neq p \neq 1$, we put $\mathscr{L}^{[p]} := L^{[f]}$, that is

$$\mathscr{L}^{[p]}(x,y) = \left(\frac{x^p - y^p}{p(x-y)}\right)^{1/(p-1)}, \quad x \neq y.$$

The means $\{\mathscr{L}^{[p]} : p \in \mathbb{R}\}$ where

$$\mathscr{L}^{[0]}(x,y) := \frac{x - y}{\log x - \log y}, \qquad \mathscr{L}^{[1]}(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}, \qquad x \neq y.$$

form the so called logarithmic family of means (cf. [2], p. 345). It is well known (and easy to verify) that for fixed $x,y > 0$, $x \neq y$, the function $p \to \mathscr{L}^{[p]}(x,y)$ is continuous and strictly increasing.

3. Uniqueness and continuity of the mean $M^{[f]}$

THEOREM 3. *Suppose that $I \subset \mathbb{R}$ is an open interval, $f : I \to \mathbb{R}$ is continuously differentiable and $f'(x) \neq 0$ for $x \in I$. Then*

(i) if for every $x,y \in I$ there is a unique mean-value $M(x,y) = M^{[f]}(x,y)$ such that

$$f(M(x,y)) - f(x) = f'(M(x,y))(y - M(x,y)), \qquad x,y \in I, \tag{1}$$

then $M^{[f]}$ is continuous;

(ii) if f is twice continuously differentiable, then for each $x_0 \in I$ there exist $\delta > 0$ and a unique function $M :\ (x_0 - \delta, x_0 + \delta)^2 \to I$ satisfying the equation

$$f(M(x,y)) - f(x) = f'(M(x,y))(y - M(x,y)), \qquad x,y \in (x_0 - \delta, x_0 + \delta);$$

moreover M is a strict and continuously differentiable mean in $(x_0 - \delta, x_0 + \delta)^2$.

(iii) if f is twice differentiable and $f'f'' \leqslant 0$ in I, then there exists a unique strict mean $M : I^2 \to I$ such that (1) holds true.

Proof. Ad (i). The inequality

$$\min(x,y) \leqslant M(x,y) \leqslant \max(x,y), \qquad x,y>0,$$

implies that M is continuous at each point (x,x). Take arbitrary point (x,y), $x \neq y$, and a sequence (x_n,y_n), $x_n \neq y_n$, $x_n,y_n>0$ for $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty}(x_n,y_n)=(x,y).$$

Of course we have

$$\frac{f(M(x_n,y_n))-f(x_n)}{y_n-M(x_n,y_n)}=f'(M(x_n,y_n)), \qquad n\in\mathbb{N},$$

and the sequence $(M(x_n,y_n))$ is bounded. Choose an arbitrary convergent subsequence of $(M(x_n,y_n))$. Without any loss of generality we can assume that $(M(x_n,y_n))$ is convergent and denote by ξ its limit. Letting $n\to\infty$ in the above equality and taking into account the continuity of f and f', we obtain

$$\frac{f(\xi)-f(x)}{y-\xi}=f'(\xi).$$

The assumed uniqueness of the mean value implies that $\xi=M(x,y)$. It follows that

$$\lim_{n\to\infty}M(x_n,y_n)=M(x,y),$$

which proves the continuity of M.

Ad (ii). Suppose that f is twice continuously differentiable in I and $f'(x)\neq 0$ for $x\in I$. Define $F: I^3\to\mathbb{R}$ by

$$F(x,y,M):=f(M)-f(x)-f'(M)(y-M), \quad x,y,M\in I,$$

and fix $x_0\in I$, $y_0=x_0$ and $M_0=x_0$. Note that

$$\frac{\partial F}{\partial M}=f'(M)-f''(M)(y-M)+f'(M)=2f'(M)-f''(M)(y-M)$$

and

$$\frac{\partial F}{\partial M}(x_0,x_0,M_0)=2f'(M_0)-f''(x_0)(x_0-M_0)=2f'(x_0)\neq 0.$$

Since

$$F(x_0,x_0,M_0)=f(M_0)-f(x_0)-f'(M_0)(x_0-M_0)=0,$$

by the Implicit Function Theorem there is $\delta>0$ and a unique continuous function $M:(x_0-\delta,x_0+\delta)^2\to I$ such that

$$F(x,y,M(x,y))=0, \quad x,y\in(x_0-\delta,x_0+\delta).$$

Moreover M is of the class C^1 on $(x_0-\delta,x_0+\delta)^2$. By Theorem 1, the function must coincide with the mean M on $(x_0-\delta,x_0+\delta)^2$. This proves the local uniqueness of M in a neighborhood of (x_0,x_0).

Ad (iii). Take arbitrary $x,y\in I$, $x\neq y$. Without any loss of generality we can assume that $x<y$. Define a function $\varphi:[x,y]\to\mathbb{R}$ by the formula

$$\varphi(t):=f(t)-f(x)-f'(t)(y-t),\qquad t\in[x,y],$$

and note that

$$\varphi'(t)=2f'(t)-f''(t)(y-t),\qquad t\in[x,y].$$

Since $y-t>0$ for all $t\in(x,y)$ and, by the assumption, $f'(t)f''(t)\leqslant 0$ for $t\in[x,y]$, the derivative φ' is either nonnegative or nonpositive in I. If there were a nontrivial subinterval $J\subset[x,y]$ such that $\varphi'(t)=0$ for $t\in J$, we would have

$$2f'(t)-f''(t)(y-t)=0,\qquad t\in J,$$

whence, as $t<y$,

$$\frac{f''(t)}{f'(t)}=\frac{2}{y-t}>0,\qquad t\in J,$$

which contradicts to the assumption that $f'f''\leqslant 0$ in I. It follows that φ is strictly monotonic in $[x,y]$. As, by the Lagrange mean-value theorem, $f(y)-f(x)=f'(\xi)(y-x)$ for some $\xi\in(x,y)$, we have

$$\varphi(x)\varphi(y)=-f'(x)(y-x)[f(y)-f(x)]=-f'(x)f'(\xi)(y-x)^2.$$

Since f' is of a constant sign in I, we infer that $\varphi(x)\varphi(y)<0$. It follows that there is a unique $M\in(x,y)$ such that $\varphi(M)=0$. This completes the proof. □

REMARK 2. In the case when the mean M yielded by Theorem 1 is unique, we denote it by $M^{[f]}$ and the function f is referred to as a generator of M.

REMARK 3. If f is a generator of $M^{[f]}$ and $g=af+b$ for some $a,b\in\mathbb{R}$, $a\neq 0$, then it easy to check that $M^{[g]}=M^{[f]}$.

4. Homogeneity of $M^{[f]}$

THEOREM 4. *Let $f:(0,\infty)\to\mathbb{R}$ be twice differentiable and $f'(x)\neq 0$ for $x>0$.*

(i) If $M:(0,\infty)^2\to(0,\infty)$ is a unique homogeneous mean (that is $M(tx,ty)=tM(x,y)$ for $t,x,y>0$)) and

$$f(M(x,y))-f(x)=f'(M(x,y))(y-M(x,y)),\qquad x,y>0,\tag{2}$$

then either

$$f(x)=ax^p+b,\qquad x>0,$$

for some $p,a,b\in\mathbb{R}$, $p\neq 0\neq a$, or

$$f(x)=a\log x+b,\qquad x>0,$$

for some $a,b\in\mathbb{R}$, $a\neq 0$.

(ii) If f has one of the above forms then there is a unique function $M:(0,\infty)^2\to(0,\infty)$ satisfying (2); moreover M is a strict homogeneous mean.

Proof. Ad (i). Suppose that $f'(x)\neq 0$ for $x\in(0,\infty)$ and $M^{[f]}$ is homogeneous. Then

$$\frac{f(tM(x,y))-f(tx)}{t[y-M(x,y)]}=f'(tM(x,y)),\qquad t,x,y\in(0,\infty),\ x\neq y,$$

and, of course,

$$\frac{y-M(x,y)}{f(M(x,y))-f(x)}=\frac{1}{f'(M(x,y))},\qquad x,y\in(0,\infty),\ x\neq y.$$

Multiplying the respective sides of these equations we obtain

$$\frac{f(tM(x,y))-f(tx)}{t[f(M(x,y))-f(x)]}=\frac{f'(tM(x,y))}{f'(M(x,y))},\qquad t,x,y\in(0,\infty),\ x\neq y.$$

In view of Theorem 3 the mean M is continuous. Therefore, for every $x\in(0,\infty)$, the set $J_x:=M(x,(0,\infty))=\{M(x,y):y\in(0,\infty)\}$ is an interval and $x\in\operatorname{Int}J_x$. Setting $u=M(x,y)$, we have

$$\frac{f(tu)-f(tx)}{t[f(u)-f(x)]}=\frac{f'(tu)}{f'(u)},\qquad t\in(0,\infty),\ u\in J_x,\ u\neq x.$$

Since the left hand-side is symmetric with respect to x and $u\in J_x$, and the right hand-side does not depend on x, we hence get

$$\frac{f(tx)-f(tu)}{t[f(x)-f(u)]}=\frac{f'(tx)}{f'(x)},\qquad t\in(0,\infty),\ x\in J_u,\ x\neq u.$$

Both equations imply that

$$\frac{f'(tu)}{f'(u)}=\frac{f'(tx)}{f'(x)},\qquad t\in(0,\infty),\ x,u\in J_x\cap J_u,\ x\neq u.$$

It follows that the set of all (t,x,y) satisfying this equality is open in $(0,\infty)^3$. The continuity of f' implies that it is also closed in $(0,\infty)^3$. Consequently,

$$\frac{f'(tu)}{f'(u)}=\frac{f'(tx)}{f'(x)},\qquad t,x,u>0.$$

Since the function on the left hand-side does not depend on u, it follows that for every $t>0$ there is an $m(t)$ such that

$$f'(tu)=m(t)f'(u),\qquad t,u>0.$$

For $u=1$ we hence get

$$f'(t)=f'(1)m(t),\qquad t>0,$$

whence, $f'(1) \neq 0$,

$$m(tu) = m(t)m(u), \qquad t,u > 0,$$

that is m is multiplicative. By the continuity of f', the function m is continuous. Therefore (cf. J. Aczél [1], p. 41, or M. Kuczma [6], p. 311) there is $p \in \mathbb{R}$ such that

$$m(x) = x^{p-1}, \qquad x > 0.$$

Consequently, if $p \neq 0$

$$f(x) = ax^p + b, \qquad x > 0,$$

and

$$f(x) = a \log x + b, \qquad x > 0.$$

where, in both cases, $a = f'(1) \neq 0$ and $b \in \mathbb{R}$. This completes the proof of part (i).

Ad (ii). First suppose that $f(x) = ax^p + b$ $(t > 0)$ for some $a, b \in \mathbb{R}$, $a \neq 0 \neq p$. Then we have $f'(x)f''(x) = a^2p^2(p-1)x^{2p-3}$ for all $x > 0$. Now part (iii) of Theorem 3 implies the uniqueness of the mean M if $p \leqslant 1$.

To prove the uniqueness of the mean M in the case $p > 1$ let us fix arbitrary $x, y > 0$, $x < y$ and, similarly as in the proof of the previous theorem, consider the function $\varphi : [x,y] \to \mathbb{R}$ with $f(x) = ax^p + b$, that is

$$\varphi(t) := at^p - ax^p - ap(y-t)t^{p-1}, \qquad t \in [x,y].$$

Since

$$\varphi(t) = ax^p \left(\left(\frac{t}{x}\right)^p - 1 - p\left(\frac{y}{x} - \frac{t}{x}\right)\left(\frac{t}{x}\right)^{p-1} \right),$$

setting

$$u := \frac{t}{x}, \qquad z := \frac{y}{x}$$

we get

$$\varphi(t) = ax^p \left[(p+1)u^p - pzu^{p-1} - 1 \right], \qquad u \in [1,z].$$

Define $\gamma : [0,\infty) \to \mathbb{R}$ by

$$\gamma(u) := (p+1)u^p - pzu^{p-1} - 1, \qquad u > 0.$$

We have

$$\gamma'(u) = pu^{p-2}\left[(p+1)u - (p-1)z\right],$$

whence $\gamma'(u) < 0$ for $u < \frac{p-1}{p+1}z$ and $\gamma'(u) > 0$ for $u > \frac{p-1}{p+1}z$. Consequently, γ is strictly decreasing in the interval $\left[0, \frac{p-1}{p+1}z\right]$ and strictly increasing in $\left[\frac{p-1}{p+1}z, \infty\right)$. Since $\gamma(0) = -1$ and $\lim_{u\to\infty} \gamma(u) = \infty$, it follows that there is exactly one $u > 0$ such that $\gamma(u) = 0$. This proves the uniqueness of $M(x,y)$.

Now assume that $f(x) = a \log x + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$. Since $f'(x)f''(x) = -a^2/x^3 < 0$, the uniqueness of M follows from part (iii) of Theorem 3. This completes the proof. $\square$

NOTATION 1. Setting $f(x) = ax^p + b \quad (x > 0)$ for some $a, b \in \mathbb{R}$, $p \neq 0 \neq a$, in (2) we get

$$M(x,y)^p - x^p = pM(x,y)^{p-1}(y - M(x,y)), \qquad x,y > 0. \tag{3}$$

Denote the unique mean M satisfying this equality by $\mathscr{F}^{[p]}$.

Setting $f(x) = a\log x + b \quad (x > 0)$ for some $a, b \in \mathbb{R}$, $a \neq 0$, in (2) we get

$$\log M(x,y) - \log x = \frac{y - M(x,y)}{M(x,y)}, \qquad x,y > 0. \tag{4}$$

Denote the unique mean M satisfying this equality by $\mathscr{F}^{[0]}$.

PROPOSITION 1. *For any* $x, y > 0$,

$$\lim_{p\to 0} \mathscr{F}^{[p]}(x,y) = \mathscr{F}^{[0]}(x,y)$$

and

$$\log \mathscr{F}^{[0]}(x,y) - \log x = \frac{y - \mathscr{F}^{[0]}(x,y)}{\mathscr{F}^{[0]}(x,y)}.$$

Proof. Applying the implicit function theorem to equality (3) we infer that, for any $x, y > 0$, the function $p \to \mathscr{F}^{[p]}(x,y)$ is continuous. So $\lim_{p\to 0} \mathscr{F}^{[p]}(x,y)$ exists. Let us fix $x, y > 0$, $x \neq y$. From (3) and the from definition of the mean $\mathscr{F}^{[p]}$ for $p \neq 0$ we have

$$(p+1)\left[\mathscr{F}^{[p]}(x,y)\right]^p - py\left[\mathscr{F}^{[p]}(x,y)\right]^{p-1} - x^p = 0,$$

whence

$$\frac{(p+1)\left[\mathscr{F}^{[p]}(x,y)\right]^p - x^p}{p} = y\left[\mathscr{F}^{[p]}(x,y)\right]^{p-1}. \tag{5}$$

Since

$$\lim_{p\to 0} \frac{(p+1)u^p - x^p}{p} = \lim_{p\to 0} \left[u^p + (p+1)u^p \log u - x^p \log x\right] = 1 + \log u - \log x,$$

letting $p \to 0$ in (5) we get

$$1 + \log \lim_{p\to 0} \mathscr{F}^{[p]}(x,y) - \log x = y\left[\lim_{p\to 0} \mathscr{F}^{[p]}(x,y)\right]^{-1}$$

whence

$$\log \lim_{p\to 0} \mathscr{F}^{[p]}(x,y) - \log x = \frac{y - \lim_{p\to 0} \mathscr{F}^{[p]}(x,y)}{\lim_{p\to 0} \mathscr{F}^{[p]}(x,y)}.$$

Now (4) and the uniqueness of $M^{[\log]}$ imply that $\lim_{p\to 0} \mathscr{F}^{[p]}(x,y) = \mathscr{F}^{[0]}(x,y)$. $\square$

REMARK 4. It easy to see that

$$\mathscr{F}^{[1]}(x,y)=\frac{x+y}{2},\qquad \mathscr{F}^{[-1]}=\sqrt{xy},\qquad x,y>0.$$

so $\mathscr{F}^{[1]}$ and $\mathscr{F}^{[-1]}$ are symmetric.

After some calculations one gets, for $x,y>0$,

$$\mathscr{F}^{[1/2]}(x,y)=\frac{2x+3y+2\sqrt{x^2+3xy}}{9},\ \mathscr{F}^{[2]}(x,y)=\frac{\sqrt{3x^2+y^2}+y}{3},$$

$$\mathscr{F}^{[-2]}(x,y)=\frac{1}{3}\sqrt[3]{27x^2y+3\sqrt{81x^4y^2+3x^6}}-\frac{x^2}{\sqrt[3]{27x^2y+3\sqrt{81x^4y^2+3x^6}}},$$

$$\mathscr{F}^{[3]}(x,y)=\frac{1}{4}\sqrt[3]{8x^3+y^3+4\sqrt{x^3y^3+4x^6}}+\frac{1}{4}\frac{y^2}{\sqrt[3]{8x^3+y^3+4\sqrt{x^3y^3+4x^6}}}+\frac{1}{4}y.$$

PROPOSITION 2. *A mean $\mathscr{F}^{[p]}$ ($p\in\mathbb{R}$) is symmetric if, and only if, $p=1$ or $p=-1$.*

Proof. For an indirect argument assume that $\mathscr{F}^{[p]}$ is symmetric for some $p\notin\{-1,0,1\}$. From (3) and the definition of the mean $\mathscr{F}^{[p]}$ we have

$$(p+1)\left[\mathscr{F}^{[p]}(x,y)\right]^p-py\left[\mathscr{F}^{[p]}(x,y)\right]^{p-1}-x^p=0$$

for all $x,y>0$. Now, interchanging x and y we get

$$(p+1)\left[\mathscr{F}^{[p]}(y,x)\right]^p-px\left[\mathscr{F}^{[p]}(y,x)\right]^{p-1}-y^p=0.$$

Since $\mathscr{F}^{[p]}(y,x)=\mathscr{F}^{[p]}(x,y)$, subtracting the suitable sides of the above equations, we obtain

$$y^p-x^p=p\left[\mathscr{F}^{[p]}(x,y)\right]^{p-1}(y-x),$$

whence, taking into account the definition of $\mathscr{L}^{[p]}$ in Remark 1,

$$\mathscr{F}^{[p]}(x,y)=\left(\frac{y^p-x^p}{p(y-x)}\right)^{1/(p-1)}=\mathscr{L}^{[p]}(x,y),\qquad x,y>0,\ x\neq y,$$

Setting this into the first of the above two equations we obtain

$$(p+1)\left[\mathscr{F}^{[p]}(x,y)\right]^p=\frac{y^{p+1}-x^{p+1}}{y-x},$$

whence

$$\mathscr{F}^{[p]}(x,y)=\left(\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right)^{1/p}=\mathscr{L}^{[p+1]}(x,y),\qquad x,y>0,\ x\neq y.$$

Consequently, $\mathscr{L}^{[p+1]} = \mathscr{F}^{[p]} = \mathscr{L}^{[p]}$, which is a contradiction.

Now assume that $M := \mathscr{F}^{[0]}$ is symmetric. By Proposition 1 we would have

$$\log M(x,y) - \log x = \frac{y - M(x,y)}{M(x,y)}, \qquad \log M(x,y) - \log y = \frac{x - M(x,y)}{M(x,y)}, \qquad x,y > 0,$$

whence, by subtracting the respective sides of these equalities,

$$M(x,y) = \frac{x-y}{\log x - \log y}, \quad x,y > 0,$$

that is, $M = \mathscr{L}^{[0]}$. Since $M = \mathscr{L}^{[0]}$ does not satisfies any of these equalities, the proof is completed. □

5. Some inequalities

PROPOSITION 3. *Let $f : I \to \mathbb{R}$ be differentiable in an interval $I \subset \mathbb{R}$ and let $M : I^2 \to I$ be a strict mean such that*

$$\frac{f(M(x,y)) - f(x)}{y - M(x,y)} = f'(M(x,y)), \qquad x,y \in I. \tag{6}$$

(i) If f is convex (strictly convex) then, for all $x,y \in I$, $x < y$,

$$M(x,y) \geqslant \frac{x+y}{2} \qquad \left(\textit{respectively, } M(x,y) > \frac{x+y}{2}\right);$$

(ii) If f is concave (strictly concave) then, for all $x,y \in I$, $x < y$,

$$M(x,y) \leqslant \frac{x+y}{2} \qquad \left(\textit{respectively, } M(x,y) < \frac{x+y}{2}\right).$$

Proof. Take arbitrary $x,y \in I$, $x < y$, and note that (6) can be written in the form

$$\frac{f(M(x,y)) - f(x)}{M(x,y) - x} \frac{M(x,y) - x}{y - M(x,y)} = f'(M(x,y)).$$

Since $x < M(x,y) < y$, the convexity of f implies that

$$\frac{f(M(x,y)) - f(x)}{M(x,y) - x} \leqslant f'(M(x,y)).$$

Now the above equality yields the inequality

$$\frac{M(x,y) - x}{y - M(x,y)} \geqslant 1$$

whence

$$M(x,y) \geqslant \frac{x+y}{2}.$$

Obviously, if f is strictly convex then the above three inequalities are sharp. We omit a similar argument in the remaining case. □

6. Final remarks

Let us note the following mean-value results of Flett' type.

THEOREM 5. *Let $I \subset \mathbb{R}$ be a nontrivial interval with endpoints a and b. Suppose that $f : I \to \mathbb{R}$ is differentiable.*

(i) If $f'(a) = f'(b)$ then there exists $\eta \in \operatorname{int} I$ such that

$$f(\eta) - f(a) = f'(\eta)(b - \eta).$$

(ii) If

$$[f(a) - f(b)]\left[f(a) - f(b) + f'(b)(b-a)\right] < 0$$

then there exists $\eta \in \operatorname{int} I$ such that

$$f(b) - f(a) = f'(\eta)(\eta - a).$$

(iii) If

$$\left[f(a) - f(b) + f'(b)(b-a)\right] f'(a)(b-a) < 0$$

then there exists $\eta \in \operatorname{int} I$ such that

$$f(\eta) - f(a) = f'(\eta)(b - a).$$

Proof. Ad. (i). Let us define $\Phi : [a,b] \to \mathbb{R}$ by

$$\Phi(t) = (f(t) - f(a))(b - t).$$

By Lagrange Mean-Value Theorem there exists $\eta \in (a,b)$ such that $\Phi'(\eta) = \frac{\Phi(b)-\Phi(a)}{b-a}$. This concludes the proof.

Ad. (ii). Applying intermediate value property to the function $\Phi(t) = f'(t)(t - a) - f(b) + f(a)$, $t \in (a,b)$ we obtain $\eta \in (a,b)$ such that $\Phi(\eta) = 0$ which completes the proof.

Ad. (ii). The proof is the same as in (ii) applied to function $\Phi(t) = f(a) - f(t) + f'(t)(b-a)$. □

(For some other generalizations of the Flett theorem cf. [4], [7], [8]).

For obvious reason, the boundary condition $f'(a) = f'(b)$ excludes the Flett theorem as a tool in definining any means. Indeed, to determine a mean value $M(x,y)$ for arbitrary $x, y \in I$, $x \neq y$, with the aid of Flett's result, it would be necessary to assume that $f'(x) = f'(y)$ for all $x, y \in I$, $x \neq y$, then f would have to be an affine function.

Note that from this point of view, the inequality type conditions of parts 2 and 3, though strongly implicit, are less restrictive.

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