ON BERGSTROM INEQUALITY FOR COMMUTING GRAMIAN NORMAL OPERATORS

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Abstract. We hereby present a variant of Bergstrom inequality for gramian normal operators on Loynes spaces and Hilbert spaces, as well as several consequences.

1. Introduction

The aim of this paper is to present some equalities and inequalities for operators on Loynes spaces. In fact, in Theorem 5 (ii), Bergstrom's classical inequality for commuting gramian normal and bounded operators is given. Moreover, in Consequence 2 the two inequalities presented in [2] are improved. Similar results are given in [8, 9].

In order to fulfill this goal, we first recall some equalities for complex numbers from [2] and [10].

Lema 1. If \( z_1, z_2, \ldots, z_n, (n \geq 2) \) is a sequence of complex numbers, then

\[
\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 = (n - 2) \sum_{k=1}^{n} |z_k|^2 + \sum_{k=1}^{n} |z_k|^2.
\]

Lema 2. If \( z_1, z_2, \ldots, z_n, (n \geq 2) \) is a sequence of complex numbers, then

\[
\sum_{1 \leq i < j \leq n} |z_i - z_j|^2 = n \sum_{k=1}^{n} |z_k|^2 - \sum_{k=1}^{n} |z_k|^2.
\]

As a generalization of the well-known identity,

\[
\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} - \frac{|z_1 + z_2|^2}{a_1 + a_2} = \frac{|a_1 z_2 - a_2 z_1|^2}{a_1 a_2 (a_1 + a_2)},
\]

when \( a_1, a_2 \) are real numbers with \( a_1, a_2 \neq 0, a_1 + a_2 \neq 0 \), in [10], the following was proven.


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THEOREM 1. If \( n \in \mathbb{N}, n \geq 2, z_1, z_2, \ldots, z_n \in \mathbb{C} \) and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( \sum_{k=1}^{n} a_k \neq 0 \), then we have

\[
\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} + \ldots + \frac{|z_n|^2}{a_n} = \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j}.
\]

If we replace \( z_2 \) by \( -z_2 \) in (*) we also obtain

\[
\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} - \frac{|z_1 - z_2|^2}{a_1 + a_2} = \frac{|a_1 z_2 + a_2 z_1|^2}{a_1 a_2 (a_1 + a_2)},
\]

where \( a_1, a_2 \) are real numbers with \( a_1, a_2 \neq 0, a_1 + a_2 \neq 0 \).

Note that the identity from Theorem 1 reduces to the identity from Lemma 2 for \( a_i = 1, \forall i \in \{1, 2, \ldots, n\} \).

Also, we need a refinement of the classical Cauchy-Buniakowsky-Schwarz inequality, which was proven in [1] by N. G. de Bruijn.

THEOREM 2. If \( a_1, a_2, \ldots, a_n \) is a sequence of real numbers and \( z_1, z_2, \ldots, z_n \) is a sequence of complex numbers, then

\[
\left| \sum_{k=1}^{n} a_k z_k \right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 \left[ \sum_{k=1}^{n} |z_k|^2 + \sum_{k=1}^{n} |z_k^2| \right].
\]

The equality holds when there is a complex number \( \alpha \) such that \( \alpha^2 \sum_{k=1}^{n} z_k^2 \geq 0 \) and \( a_k = \text{Re}(\alpha z_k) \) for all \( k \).

In what follows, we mention some basic definitions such as the definition of the admissible spaces, or of Loynes spaces (called pseudo-Hilbert spaces in the papers of S.A. Chobanyan and A. Weron, see [3], [12] and [13]) in order to state and prove the results in the next section, and also several properties for operators in these spaces.

DEFINITION 1. ([4, 7]) A locally convex space \( Z \) is called admissible in the Loynes sense if the following conditions are satisfied:

(A.1) \( Z \) is complete;

(A.2) there is a closed convex cone in \( Z \), denoted \( Z_+ \), that defines an order relation on \( Z \) (that is \( z_1 \leq z_2 \) if \( z_2 - z_1 \in Z_+ \));

(A.3) there is an involution in \( Z \), \( Z \ni z \rightarrow z^* \in Z \) (that is \( z^{**} = z \), \( (\alpha z)^* = \overline{\alpha} z^* \), \( (z_1 + z_2)^* = z_1^* + z_2^* \)), so that \( z \in Z_+ \) implies \( z^* = z \);

(A.4) the topology of \( Z \) is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);

(A.5) any monotonously decreasing sequence in \( Z_+ \) is convergent.
Observation 1. ([4]) A set $C \subseteq \mathbb{Z}$ is called solid if $0 \leq z' \leq z''$ and $z'' \in C$ implies $z' \in C$.

Definition 2. ([4, 7]) Let $Z$ be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called pre-Loynes $Z$-space if it satisfies the following properties:

(L1) $\mathcal{H}$ is endowed with a $Z$-valued inner product (gramian), i.e. there exists an application $\mathcal{H} \times \mathcal{H} \ni (h,k) \rightarrow [h,k] \in Z$ having the following properties:

(G1) $[h,h] \geq 0$; $[h,h] = 0$ implies $h = 0$;

(G2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;

(G3) $[\lambda h, k] = \lambda [h, k]$;

(G4) $[h, k]^* = [k, h]$;

for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

(L2) The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \rightarrow [h, h] \in Z$ is continuous.

Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called Loynes $Z$-space.

An operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ is called gramian bounded (see [4]) if there exists a constant $\mu > 0$ so that in the sense of the order in $Z$

$$[Th, Th]_{\mathcal{H}} \leq \mu [h, h]_{\mathcal{H}}, \quad h \in \mathcal{H},$$

(**) holds.

We denote the class of such operators by $\mathcal{B}(\mathcal{H}, \mathcal{H})$, and $\mathcal{B}^*(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})' \cap \mathcal{L}^*(\mathcal{H}, \mathcal{H})'$. Considering $\mathcal{C}(\mathcal{H}, \mathcal{H})$ the class of linear and continuous operators from $\mathcal{H}$ to $\mathcal{H}$ we define $\mathcal{C}^*(\mathcal{H}, \mathcal{H}) := \mathcal{L}^*(\mathcal{H}, \mathcal{H}) \cap \mathcal{C}(\mathcal{H}, \mathcal{H})$. If $\mathcal{H} = \mathcal{H}$, then $\mathcal{C}^*(\mathcal{H}, \mathcal{H})$ is $\mathcal{C}^*(\mathcal{H})$.

We also denote the introduced norm by

$$\|T\| = \inf \{ \sqrt{\mu}, \mu > 0 \text{ and satisfies (**)} \}.$$

Corollary 1. ([4, 6, 7]) The space $\mathcal{B}^*(\mathcal{H}, \mathcal{H})$ is a Banach space, and its involution from $\mathcal{B}^*(\mathcal{H}, \mathcal{H})$ to $\mathcal{B}^*(\mathcal{H}, \mathcal{H})$ satisfies

$$\|T^*T\| = \|T\|^2, \quad T \in \mathcal{B}^*(\mathcal{H}, \mathcal{H}).$$

In particular, $\mathcal{B}^*(\mathcal{H})$ is a $\mathcal{C}^*$-algebra.

Let $\mathcal{H}$ be a Loynes $Z$-space, where $Z$ is an admissible space. Next, we recall the well-known results which will be used below in proving Theorem 4.

Proposition 1. [4] For an operator $N \in \mathcal{C}^*(\mathcal{H})$ the following assertions are equivalent:
\((i)\) \(N\) is gramian normal;
\((ii)\) \([Nh,Nh] = [N^*h,N^*h]\), \(h \in \mathcal{H}\);
\((iii)\) There exist two commuting gramian self-adjoint operators \(A, B \in \mathcal{L}^*_h(\mathcal{H})\), so that \(N = A + iB\).

**Theorem 3.** [4] Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be two Loynes \(Z\)-spaces and \(N_1 \in \mathcal{B}^*(\mathcal{H}_1)\), \(N_2 \in \mathcal{B}^*(\mathcal{H}_2)\) two gramian normal operators on \(\mathcal{H}_1\) and \(\mathcal{H}_2\) respectively. If there exists \(T \in \mathcal{B}^*(\mathcal{H}_1, \mathcal{H}_2)\) which inverts \(N_1\) and \(N_2\), that is \(TN_1 = N_2 T\), then \(T\) also inverts the adjoints \(N_1^*\) and \(N_2^*\), i.e. \(T N_1^* = N_2^* T\).

Theorem 1 and some of its consequences will be recaptured from Theorem 4 for the modules of the commuting gramian normal and bounded operators on Loynes spaces. An identity, similar to the identity in Theorem 4, for the modules of the commuting gramian normal and bounded operators on Loynes spaces will be obtained in Theorem 6. Furthermore, using the inequality of N.G. de Bruijn and Theorem 7, we can improve, relying on Proposition 2, Proposition 3 and Consequence 2, the inequalities from Theorem 1 and 2 from [2] for complex numbers. In addition, in Remark 4, similar results are presented for \(C^*\)-algebras.

### 2. The main results

In this section we assume \(\mathcal{H}\) is a Loynes \(Z\)-space. Then, we denote by \(\mathcal{B}^*(\mathcal{H})\) the space of linear and gramian bounded operators which admit gramian adjoint, see [6], [7], if \(\mathcal{H}\) is a Loynes \(Z\)-space. We denote by \(\mathcal{B}(\mathcal{H})\) the space of linear bounded operators if \(\mathcal{H}\) is a Hilbert space.

**Lemma 3.** For any operators \(A, B \in \mathcal{B}^*_h(\mathcal{H})\) and \(a, b > 0\), we have

\[
\frac{A^2}{a} + \frac{B^2}{b} \geq \frac{(A + B)^2}{a + b}.
\]

**Proof.** The obvious inequality

\[
\left(\sqrt{\frac{b}{a}}A - \sqrt{\frac{a}{b}}B\right)^2 \geq 0
\]

leads to

\[
A^2 + B^2 + \frac{b}{a}A^2 + \frac{a}{b}B^2 \geq A^2 + B^2 + AB + BA. \quad \Box
\]

**Lemma 4.** If \(n \in \mathbb{N}, n \geq 2, N_1, N_2, \ldots, N_n \in \mathcal{B}^*(\mathcal{H})\) are commuting gramian normal operators, then

\[
|\sum_{k=1}^{n} N_k|^2 = \sum_{k=1}^{n} |N_k|^2 + \frac{1}{4} \sum_{i,j=1}^{n} [(N_i + N_i^*)(N_j + N_j^*) - (N_i - N_i^*)(N_j - N_j^*)],
\]

where \(|N| = (N^*N)^{\frac{1}{2}}\).
We start this section by giving a generalization of the identity (4) from Theorem 3, (see [10]) for a particular class of operators on Løynes spaces. In the proof we will use the same techniques as in [10]. This theorem can also be proven using induction.

**Theorem 4.** If \( n \in \mathbb{N}, n \geq 2, N_1, N_2, \ldots, N_n \in \mathcal{B}^*(\mathcal{H}) \) are commuting gramian normal operators and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( \sum_{k=1}^{n} a_k \neq 0 \), then

\[
\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + \ldots + N_n|^2}{a_1 + a_2 + \ldots + a_n} = \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j},
\]

(1)

where \( |N| = (N^*N)^{1/2} \) is known as modulus of \( N \).

**Proof.** If the operators \( N_i, i \in \{1, \ldots, n\} \) are gramian normal then from [4], Proposition 1.2.2, we have the following decompositions for \( N_k, k \in \{1, \ldots, n\}, N_k = A_k + iB_k \) with \( A_k, B_k \) being commuting gramian self-adjoint operators.

Since \( N_k, k = 1, \ldots, n \), are mutually commuting gramian normal operators, operators \( A_k, B_k, k = 1, \ldots, n \), also commute (it follows from their forms and Fuglede’s theorem applied to \( N_i \) and \( N_j \)).

Thus \( |N_k|^2 = |A_k + iB_k|^2 = (A_k + iB_k)^*(A_k + iB_k) = (A_k - iB_k)(A_k + iB_k) = A_k^2 + B_k^2, k \in \{1, \ldots, n\}, \) and

\[
|N_1 + N_2 + \ldots + N_n|^2 = |(A_1 + A_2 + \ldots + A_n) + i(B_1 + B_2 + \ldots + B_n)|^2
= (A_1 + A_2 + \ldots + A_n)^2 + (B_1 + B_2 + \ldots + B_n)^2.
\]

Moreover,

\[
|a_k N_j - a_j N_k|^2 = |a_k (A_j + iB_j) - a_j (A_k + iB_k)|^2 = (a_k A_j - a_j A_k)^2 + (a_k B_j - a_j B_k)^2,
\]

\( j, k \in \{1, \ldots, n\}. \)

The expression on the left-hand side of (1) is equivalent to

\[
\frac{A_1^2 + B_1^2}{a_1} + \frac{A_2^2 + B_2^2}{a_2} + \ldots + \frac{A_n^2 + B_n^2}{a_n} - \frac{(A_1 + A_2 + \ldots + A_n)^2 + (B_1 + B_2 + \ldots + B_n)^2}{a_1 + a_2 + \ldots + a_n}
\]

\[
= \frac{1}{a_1 a_2 \ldots a_n (a_1 + a_2 + \ldots + a_n)} \left( (a_2^2 a_3 a_4 \ldots a_n + a_2 a_3^2 a_4 \ldots a_n + \ldots + a_2 a_3 \ldots a_n - 1 a_n^2) (A_1^2 + B_1^2) + (a_3^2 a_4 a_5 \ldots a_n + a_1 a_3^2 a_4 \ldots a_n + \ldots + a_1 a_3 \ldots a_n - 1 a_n^2) (A_2^2 + B_2^2) + \ldots + (a_2 a_3 a_4 \ldots a_n - 1 + a_1 a_2 a_3 \ldots a_n - 1 + \ldots + a_1 a_2 \ldots a_n - 2 a_{n-1}^2) (A_n^2 + B_n^2) - 2 a_1 a_2 \ldots a_n ((A_1 A_2 + A_1 A_3 + \ldots + A_1 A_n + \ldots + A_n - 1 A_n) + (B_1 B_2 + B_1 B_3 + \ldots + B_1 B_n + \ldots + B_{n-1} B_n)) \right)
\]
\[ = \frac{1}{a_1a_2\ldots a_n(a_1 + a_2 + \ldots + a_n)} \left( a_3a_4\ldots a_n((a_1A_2 - a_2A_1)^2 + (a_1B_2 - a_2B_1)^2) \right. \]

\[ + a_2a_4a_5\ldots a_n((a_1A_3 - a_3A_1)^2 + (a_1B_3 - a_3B_1)^2) + \ldots \]

\[ + a_1a_2\ldots a_{n-2}(a_{n-1}A_n - a_nA_{n-1})^2 \]

\[ + (a_{n-1}B_n - a_nB_{n-1})^2). \]

Thus, we obtain

\[
\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + \ldots + N_n|^2}{a_1 + a_2 + \ldots + a_n} \\
= \frac{A_1^2 + B_1^2}{a_1} + \frac{A_2^2 + B_2^2}{a_2} + \ldots + \frac{A_n^2 + B_n^2}{a_n} - \frac{(A_1 + A_2 + \ldots + A_n)^2 + (B_1 + B_2 + \ldots + B_n)^2}{a_1 + a_2 + \ldots + a_n} \\
= \frac{1}{a_1a_2\ldots a_n(a_1 + a_2 + \ldots + a_n)}(a_3a_4\ldots a_n|a_1N_2 - a_2N_1|^2 + a_2a_4a_5\ldots a_n|a_1N_3 - a_3N_1|^2 \\
+ \ldots + a_1a_2\ldots a_{n-2}|a_{n-1}N_n - a_nN_{n-1}|^2). \]

**Theorem 5.**

(i) If \( n \in \mathbb{N}, n \geq 2, N_1, N_2, \ldots, N_n \in \mathcal{B}^+(\mathcal{H}) \) are commuting gramian normal operators and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( \sum_{k=1}^{n} a_k \neq 0 \), then

\[
\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_n|^2}{a_n} \geq \frac{1}{\sum_{k=1}^{n} a_k} \sum_{1 \leq i < j \leq n} |a_iN_j - a_jN_i|^2. \tag{2}
\]

(ii) Under the above-stated conditions, if \( a_1, a_2, \ldots, a_n \in (0, \infty) \) we also have,

\[
\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_n|^2}{a_n} \geq \frac{|N_1 + N_2 + \ldots + N_n|^2}{a_1 + a_2 + \ldots + a_n},
\]

with equality if and only if \( a_iN_j = a_jN_i \), for any \( i, j \in \{1, 2, \ldots, n\} \).

(iii) In (i), if we take \( a_1 = a_2 = \ldots = a_n = 1 \), the inequality becomes

\[
|N_1|^2 + |N_2|^2 + \ldots + |N_n|^2 \geq \frac{1}{n} \sum_{1 \leq i < j \leq n} |N_j - N_i|^2. \tag{3}
\]

**Proof.** It follows immediately from Theorem 4. The proof of equality in (ii) is contained in the proof of Theorem 4. \( \square \)

In the proof of the theorem below we need the following identity:

\[
|N_1 - N_2|^2 + |N_1 + N_2|^2 = 2|N_1|^2 + 2|N_2|^2, \tag{5}
\]

where \( |N_1| = (N_1^*N_1)^{\frac{1}{2}} \) is the modulus of \( N_1 \) and \( N_1, N_2 \in \mathcal{B}^+(\mathcal{H}) \) are arbitrary operators. This identity is easily verified by calculation.
Theorem 6. If $n \in \mathbb{N}$, $n \geq 2$, $N_1, N_2, \ldots, N_n \in \mathcal{B}^+(\mathcal{H})$ are gramian normal operators which commute as pairs and $a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^n a_k \neq 0$, then

$$\left| N_1 + N_2 + \ldots + N_n \right|^2 \left( a_1 + a_2 + \ldots + a_n \right) + \sum_{k=1}^n \left| N_k \right|^2 - \frac{2}{a_1 + a_2 + \ldots + a_n} \sum_{k=1}^n \left| N_k \right|^2$$

$$= \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n} \left| a_i N_j + a_j N_i \right|^2. \quad (6)$$

**Proof.** We will use identity (1) from Theorem 4 and identity (5).

Let

$$S = \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n} \left| a_i N_j + a_j N_i \right|^2.$$ 

Now, adding $S$ and the right member of identity (1), we will have

$$\frac{1}{\sum_{k=1}^n a_k} \left( \sum_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j} + \sum_{1 \leq i < j \leq n} \frac{|a_i N_j + a_j N_i|^2}{a_i a_j} \right)$$

$$= \frac{2}{\sum_{k=1}^n a_k} \sum_{j=2}^n \frac{|N_j|^2}{a_j} \sum_{i=1}^{j-1} \frac{|a_i|^2}{a_i} + \sum_{j=2}^n \frac{|a_j|^2}{a_j} \sum_{i=1}^{j-1} \frac{|N_i|^2}{a_i}$$

$$= \frac{2}{\sum_{k=1}^n a_k} \left( \sum_{j=2}^n \frac{|N_j|^2}{a_j} (a_1 + a_2 + \ldots + a_{j-1}) + \sum_{j=2}^n a_j \left( \frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_{j-1}|^2}{a_{j-1}} \right) \right)$$

$$= \frac{2}{\sum_{k=1}^n a_k} \left( \frac{|N_2|^2}{a_2} (a_1 + a_2 + \ldots + a_{n-1}) + \frac{|N_3|^2}{a_3} (a_1 + a_2 + \ldots + a_{n-1}) + \ldots + \frac{|N_n|^2}{a_n} (a_1 + a_2 + \ldots + a_{n-1}) \right)$$

$$+ a_2 \frac{|N_1|^2}{a_1} + a_3 \left( \frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} \right) + \ldots + a_n \left( \frac{|N_1|^2}{a_1} + \ldots + \frac{|N_{n-1}|^2}{a_{n-1}} \right)$$

$$= \frac{2}{\sum_{k=1}^n a_k} \left( \sum_{k=1}^n a_k \cdot \sum_{k=1}^n \frac{|N_k|^2}{a_k} - \sum_{k=1}^n |N_k|^2 \right).$$

Thus, we have

$$S = \sum_{k=1}^n \frac{|N_k|^2}{a_k} - \frac{2}{\sum_{k=1}^n a_k} \sum_{k=1}^n |N_k|^2 + \frac{\sum_{k=1}^n |N_k|^2}{\sum_{k=1}^n a_k}. \quad \Box$$

As a particular case of the previous theorem, we can also obtain the following identity for complex numbers:
**Consequence 1.** If \( n \in \mathbb{N}, \ n \geq 2, \ z_1, z_2, \ldots, z_n \in \mathbb{C} \) are complex numbers and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( \sum_{k=1}^{n} a_k \neq 0 \), then
\[
\frac{|z_1 + z_2 + \ldots + z_n|^2}{a_1 + a_2 + \ldots + a_n} + \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - \frac{2}{a_1 + a_2 + \ldots + a_n} \sum_{k=1}^{n} |z_k|^2 \]
\[= \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n} |a_i z_j + a_j z_i|^2.
\]

**Remark 1.** If \( n \in \mathbb{N}, \ n \geq 2, \ N_1, N_2, \ldots, N_n \in \mathcal{B}^*(\mathcal{H}) \) are gramian normal operators which commute as pairs and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( \sum_{k=1}^{n} a_k > 0 \), then
\[
\frac{|N_1 + N_2 + \ldots + N_n|^2}{a_1 + a_2 + \ldots + a_n} + \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} \geq \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n} |a_i N_j + a_j N_i|^2.
\]

The inequality of N. G. de Bruijn can be written in the following form:

**Theorem 7.** If \( a_1, a_2, \ldots, a_n \) is a sequence of real numbers and \( N_1, N_2, \ldots, N_n \) is a sequence of commuting gramian normal operators in \( \mathcal{B}^*(\mathcal{H}) \) so that \( \sum_{k=1}^{n} a_k N_k \) is hermitian, then
\[
|\sum_{k=1}^{n} a_k N_k|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 \left[ \sum_{k=1}^{n} |N_k|^2 + |\sum_{k=1}^{n} N_k^2| \right],
\]
where \( |N| \) is the modulus of the operator \( N \).

**Proof.** The proof will be as in [5]. Using the fact that \( \sum_{k=1}^{n} a_k N_k \) is hermitian and the definition of the modulus of an operator, we obtain
\[
|\sum_{k=1}^{n} a_k N_k|^2 = \left( \sum_{k=1}^{n} a_k N_k \right)^2,
\]
where \( N_k = A_k + iB_k, \ k = 1, \ldots, n. \)

Again, from \( \sum_{k=1}^{n} a_k N_k \) being hermitian, it results that \( \sum_{k=1}^{n} a_k N_k = \sum_{k=1}^{n} a_k N_k^* \) or \( \sum_{k=1}^{n} a_k B_k = 0. \)

Thus
\[
|\sum_{k=1}^{n} a_k N_k|^2 = |\sum_{k=1}^{n} a_k A_k + i \sum_{k=1}^{n} a_k B_k|^2 = \left( \sum_{k=1}^{n} a_k A_k \right)^2 + \left( \sum_{k=1}^{n} a_k B_k \right)^2
\]
\[= \left( \sum_{k=1}^{n} a_k A_k \right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} A_k^2,
\]
the last inequality being the Cauchy-Schwarz inequality for gramian self-adjoint operators which commute as pairs. A proof for this last inequality can be done by induction,
using the fact that for commuting gramian self-adjoint operators $A_k, A_l$ and $a_k, a_l$ real numbers, where $k, l \in \{1, \ldots, n\}$, we have $(a_k A_l - a_l A_k)^2 \geq 0$. Now taking into account that for all $k \in \{1, \ldots, n\}, 2A_k^2 = |N_k|^2 + C_k$, where $C_k = A_k^2 - B_k^2$, this will give $N_k^2 = C_k + 2iA_kB_k$, and then

$$|\sum_{k=1}^{n} a_k N_k|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 [\sum_{k=1}^{n} |N_k|^2 + \sum_{k=1}^{n} C_k].$$

Because

$$\sum_{k=1}^{n} C_k \leq |\sum_{k=1}^{n} N_k^2|,$$

we have the desired inequality. $\Box$

Furthermore, we can state the following result for particular operators on Loynes spaces:

**PROPOSITION 2.** Let $N_1, N_2, \ldots, N_n, (n \geq 2)$ be a sequence of gramian normal operators in $B^n(\mathcal{H})$ commuting as pairs so that $\sum_{k=1}^{n} a_k N_k$ is hermitian and $a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^{n} a_k > 0$. Then

$$\sum_{1 \leq i < j \leq n} \frac{|a_i N_j + a_j N_i|^2}{a_ia_j} \leq \frac{n-4}{2} \sum_{k=1}^{n} |N_k|^2 + (\sum_{k=1}^{n} a_k) \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} + \frac{n}{2} |\sum_{k=1}^{n} N_k^2|.$$

**Proof.** If we replace $a_k, k \in \{1, 2, \ldots, n\}$ by $\frac{1}{\sqrt{a_1 + a_2 + \ldots + a_n}}$ in the inequality from Theorem 7, then we have

$$\left|\sum_{k=1}^{n} \frac{N_k}{\sqrt{a_1 + a_2 + \ldots + a_n}}\right|^2 = \frac{\left|\sum_{k=1}^{n} N_k\right|^2}{\sum_{k=1}^{n} a_k} \leq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{a_1 + a_2 + \ldots + a_n} \left[\sum_{k=1}^{n} |N_k|^2 + \sum_{k=1}^{n} N_k^2\right]$$

or

$$\frac{\left|\sum_{k=1}^{n} N_k\right|^2}{\sum_{k=1}^{n} a_k} \leq \frac{n}{2} \sum_{k=1}^{n} |N_k|^2 + \sum_{k=1}^{n} N_k^2|.$$

Now using Theorem 6, we find

$$\frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j + a_j N_i|^2}{a_ia_j} \leq \frac{n-4}{2} \sum_{k=1}^{n} |N_k|^2 + \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{2}{\sum_{k=1}^{n} a_k} \cdot \sum_{k=1}^{n} |N_k|^2$$

i.e.

$$\sum_{1 \leq i < j \leq n} \frac{|a_i N_j + a_j N_i|^2}{a_ia_j} \leq \frac{n-4}{2} \sum_{k=1}^{n} |N_k|^2 + \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} + \frac{n}{2} |\sum_{k=1}^{n} N_k^2|. \quad \Box$$
REMARK 2. If we take \( a_k = 1, \ k \in \{1, 2, \ldots, n\} \) in the above-stated inequality, then we get
\[
\sum_{1 \leq i < j \leq n} |N_j - N_i|^2 \leq \frac{3n-4}{2} \sum_{k=1}^{n} |N_k|^2 + \frac{n}{2} \sum_{k=1}^{n} N_k^2.
\]

PROPOSITION 3. Let \( N_1, N_2, \ldots, N_n, \ (n \geq 2) \) be a sequence of gramian normal operators in \( \mathcal{B}(\mathcal{H}) \) commuting as pairs, so that \( \sum_{k=1}^{n} a_k N_k \) is hermitian and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( \sum_{k=1}^{n} a_k > 0 \). Then
\[
\sum_{1 \leq i < j \leq n} \frac{|a_iN_j - a_jN_i|^2}{a_ia_j} \geq \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{n}{2} \sum_{k=1}^{n} |N_k|^2 - \frac{n}{2} \sum_{k=1}^{n} N_k^2.
\]

Proof. Adding the identities from Theorem 4 and 6, we find
\[
\frac{1}{\sum_{k=1}^{n} a_k} \sum_{1 \leq i < j \leq n} \frac{|a_iN_j - a_jN_i|^2 + |a_iN_j + a_jN_i|^2}{a_ia_j} = \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{|N_k|^2}{a_k}.
\]

or
\[
\sum_{1 \leq i < j \leq n} \left( \frac{|a_iN_j - a_jN_i|^2}{a_ia_j} + \frac{|a_iN_j + a_jN_i|^2}{a_ia_j} \right) = 2 \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - 2 \sum_{k=1}^{n} |N_k|^2.
\]

Now applying Proposition 2 we deduce that
\[
\sum_{1 \leq i < j \leq n} \frac{|a_iN_j - a_jN_i|^2}{a_ia_j} = 2 \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - 2 \sum_{k=1}^{n} |N_k|^2 - \sum_{1 \leq i < j \leq n} \frac{|a_iN_j + a_jN_i|^2}{a_ia_j}
\]
\[
\geq 2 \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - 2 \sum_{k=1}^{n} |N_k|^2 - \frac{n-4}{2} \sum_{k=1}^{n} |N_k|^2
\]
\[
- \frac{n}{2} \sum_{k=1}^{n} N_k^2,
\]

which leads to the desired inequality. \( \square \)

REMARK 3. If we take \( a_k = 1, \ k \in \{1, \ldots, n\} \) in the above-stated inequality, then we get
\[
\sum_{1 \leq i < j \leq n} |N_j - N_i|^2 \geq \frac{n}{2} \left( \sum_{k=1}^{n} |N_k|^2 - \sum_{k=1}^{n} N_k^2 \right).
\]

For complex numbers we can also state the following result as a special case of the Proposition 2 and 3.

CONSEQUENCE 2. Let \( z_1, z_2, \ldots, z_n, \ (n \geq 2) \) be a sequence of complex numbers and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( \sum_{k=1}^{n} a_k > 0 \). Then
\[
\sum_{1 \leq i < j \leq n} \frac{|a_iz_j + a_jz_i|^2}{a_ia_j} \leq \frac{n-4}{2} \sum_{k=1}^{n} |z_k|^2 + \left( \sum_{k=1}^{n} a_k \right) \cdot \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} + \frac{n}{2} \sum_{k=1}^{n} z_k^2.
\]
and
\[ \sum_{1 \leq i < j \leq n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j} \geq \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - \frac{n}{2} \sum_{k=1}^{n} |z_k|^2 - \frac{n}{2} \sum_{k=1}^{n} \frac{n}{2} |z_k|^2. \]

It is known that $\mathcal{B}^*(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ are particular $C^*$-algebras, see [7], [4], [11]. As in the case of Loynes spaces, we can also deduce by calculation in $C^*$-algebras the following equality:
\[ |b_1 - b_2|^2 + |b_1 + b_2|^2 = 2|b_1|^2 + 2|b_2|^2, \]

where $|b_1| = (b_1^* b_1)^{1/2}$ is the modulus of $b_1$ and $b_1, b_2 \in \mathcal{A}$ are arbitrary elements in a $C^*$-algebra $\mathcal{A}$. Then, the following result can be obtained, similarly as for the Loynes spaces and using analogous theorems (see [11]).

**Remark 4.** If $n \in \mathbb{N}, n \geq 2, b_1, b_2, ..., b_n \in \mathcal{A}$ are normal elements which commute as pairs and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^{n} a_k \neq 0$, then
\[ \frac{|b_1 + b_2 + ... + b_n|^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{|b_k|^2}{a_k} - \frac{2}{a_1 + a_2 + ... + a_n} \sum_{k=1}^{n} \frac{|b_k|^2}{a_k}, \]

Moreover, if $n \in \mathbb{N}, n \geq 2, b_1, b_2, ..., b_n \in \mathcal{A}$ are normal elements which commute as pairs and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^{n} a_k > 0$, then
\[ \frac{|b_1 + b_2 + ... + b_n|^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{|b_k|^2}{a_k} \geq \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i b_j + a_j b_i|^2}{a_i a_j}. \]

**References**


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