Journal of Mathematical Inequalities
Volume 4, Number 4 (2010), 505–516

ON BERGSTROM INEQUALITY FOR COMMUTING GRAMIAN NORMAL OPERATORS

LOREDANA CIURDARIU

(*Communicated by A. Čižmešija*)

Abstract. We hereby present a variant of Bergstrom inequality for gramian normal operators on Loynes spaces and Hilbert spaces, as well as several consequences.

1. Introduction

The aim of this paper is to present some equalities and inequalities for operators on Loynes spaces. In fact, in Theorem 5 (ii), Bergstrom′s classical inequality for commuting gramian normal and bounded operators is given. Moreover, in Consequence 2 the two inequalities presented in [2] are improved. Similar results are given in [8, 9].

In order to fulfill this goal, we first recall some equalities for complex numbers from [2] and [10].

LEMMA 1. *If* $z_1, z_2, \ldots, z_n$, $(n \geqslant 2)$ *is a sequence of complex numbers, then*

$$\sum_{1 \leqslant i < j \leqslant n} |z_i + z_j|^2 = (n-2)\sum_{k=1}^{n} |z_k|^2 + |\sum_{k=1}^{n} z_k|^2.$$

LEMMA 2. *If* $z_1, z_2, \ldots, z_n$, $(n \geqslant 2)$ *is a sequence of complex numbers, then*

$$\sum_{1 \leqslant i < j \leqslant n} |z_i - z_j|^2 = n\sum_{k=1}^{n} |z_k|^2 - |\sum_{k=1}^{n} z_k|^2.$$

As a generalization of the well-known identity,

$$\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} - \frac{|z_1+z_2|^2}{a_1+a_2} = \frac{|a_1 z_2 - a_2 z_1|^2}{a_1 a_2 (a_1+a_2)}, \tag{$*$}$$

when a_1, a_2 are real numbers with $a_1,\ a_2 \neq 0,\ a_1 + a_2 \neq 0$, in [10], the following was proven.

Mathematics subject classification (2010): Primary 47A45; Secondary 42B10.
Keywords and phrases: Pseudo-Hilbert spaces (Loynes spaces), seminorms, Bergstrom inequality.

Paper JMI-04-45

THEOREM 1. *If* $n \in \mathbb{N}$, $n \geqslant 2$, $z_1, z_2, ..., z_n \in \mathbb{C}$ *and* $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ *with* $\sum_{k=1}^{n} a_k \neq 0$, *then we have*

$$\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} + ... + \frac{|z_n|^2}{a_n} - \frac{|z_1 + z_2 + ... + z_n|^2}{a_1 + a_2 + ... + a_n}$$
$$= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leqslant i < j \leqslant n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j}.$$

If we replace z_2 by $-z_2$ in (*) we also obtain

$$\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} - \frac{|z_1 - z_2|^2}{a_1 + a_2} = \frac{|a_1 z_2 + a_2 z_1|^2}{a_1 a_2 (a_1 + a_2)},$$

where a_1, a_2 are real numbers with $a_1,\ a_2 \neq 0,\ a_1 + a_2 \neq 0$.

Note that the identity from Theorem 1 reduces to the identity from Lemma 2 for $a_i = 1,\ \forall i \in \{1, 2, ..., n\}$.

Also, we need a refinement of the classical Cauchy-Buniakowsky-Schwarz inequality, which was proven in [1] by N. G. de Bruijin.

THEOREM 2. *If* $a_1,\ a_2, ..., a_n$ *is a sequence of real numbers and* $z_1,\ z_2, ..., z_n$ *is a sequence of complex numbers, then*

$$|\sum_{k=1}^{n} a_k z_k|^2 \leqslant \frac{1}{2} \sum_{k=1}^{n} a_k^2 [\sum_{k=1}^{n} |z_k|^2 + |\sum_{k=1}^{n} z_k^2|].$$

The equality holds when there is a complex number α *such that* $\alpha^2 \sum_{k=1}^{n} z_k^2 \geqslant 0$ *and* $a_k = Re(\alpha z_k)$ *for all* k.

In what follows, we mention some basic definitions such as the definition of the admissible spaces, or of Loynes spaces (called pseudo-Hilbert spaces in the papers of S.A. Chobanyan and A. Weron, see [3], [12] and [13]) in order to state and prove the results in the next section, and also several properties for operators in these spaces.

DEFINITION 1. ([4, 7]) A locally convex space Z is called admissible in the Loynes sense if the following conditions are satisfied:

(A.1) Z is complete;

(A.2) there is a closed convex cone in Z, denoted Z_+, that defines an order relation on Z (that is $z_1 \leqslant z_2$ if $z_2 - z_1 \in Z_+$);

(A.3) there is an involution in Z, $Z \ni z \to z^* \in Z$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), so that $z \in Z_+$ implies $z^* = z$;

(A.4) the topology of Z is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);

(A.5) any monotonously decreasing sequence in Z_+ is convergent.

OBSERVATION 1. ([4]) A set $C \in Z$ is called *solid* if $0 \leqslant z' \leqslant z''$ and $z'' \in C$ implies $z' \in C$.

DEFINITION 2. ([4, 7]) Let Z be an admissible space in the Loynes sense. A linear topological space $\mathscr{H}$ is called pre-Loynes Z–space if it satisfies the following properties:

(L1) $\mathscr{H}$ is endowed with a Z–valued *inner product* (gramian), i.e. there exists an application $\mathscr{H} \times \mathscr{H} \ni (h,k) \to [h,k] \in Z$ having the following properties:

(G$_1$) $[h,h] \geqslant 0$; $[h,h] = 0$ implies $h = 0$;

(G$_2$) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;

(G$_3$) $[\lambda h, k] = \lambda [h,k]$;

(G$_4$) $[h,k]^* = [k,h]$;

for all $h, k, h_1, h_2 \in \mathscr{H}$ and $\lambda \in \mathbb{C}$.

(L2) The topology of $\mathscr{H}$ is the weakest locally convex topology on $\mathscr{H}$ for which the application $\mathscr{H} \ni h \to [h,h] \in Z$ is continuous.

Moreover, if $\mathscr{H}$ is a complete space with this topology, then $\mathscr{H}$ is called Loynes Z*–space*.

An operator $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is called gramian bounded (see [4]) if there exists a constant $\mu > 0$ so that in the sense of the order in Z

$$[Th, Th]_{\mathscr{K}} \leqslant \mu [h,h]_{\mathscr{H}}, \quad h \in \mathscr{H}, \tag{$**$}$$

holds.

We denote the class of such operators by $\mathscr{B}(\mathscr{H}, \mathscr{K})$, and $\mathscr{B}^*(\mathscr{H}, \mathscr{K}) = \mathscr{B}(\mathscr{H}, \mathscr{K}) \cap \mathscr{L}^*(\mathscr{H}, \mathscr{K})$. Considering $\mathscr{C}(\mathscr{H}, \mathscr{K})$ the class of linear and continuous operators from $\mathscr{H}$ to $\mathscr{K}$ we define $\mathscr{C}^*(\mathscr{H}, \mathscr{K}) := \mathscr{L}^*(\mathscr{H}, \mathscr{K}) \cap \mathscr{C}(\mathscr{H}, \mathscr{K})$. If $\mathscr{H} = \mathscr{K}$, then $\mathscr{C}^*(\mathscr{H}, \mathscr{H})$ is $\mathscr{C}^*(\mathscr{H})$.

We also denote the introduced norm by

$$\|T\| = \inf\{\sqrt{\mu},\ \mu > 0 \text{ and satisfies } (**)\}.$$

COROLLARY 1. ([4, 6, 7]) *The space $\mathscr{B}^*(\mathscr{H}, \mathscr{K})$ is a Banach space, and its involution from $\mathscr{B}^*(\mathscr{H}, \mathscr{K})$ to $\mathscr{B}^*(\mathscr{K}, \mathscr{H})$ satisfies*

$$\|T^*T\| = \|T\|^2, \quad T \in \mathscr{B}^*(\mathscr{H}, \mathscr{K}).$$

In particular, $\mathscr{B}^*(\mathscr{H})$ is a C^*–algebra.

Let $\mathscr{H}$ be a Loynes Z- space, where Z is an admissible space. Next, we recall the well-known results which will be used below in proving Theorem 4.

PROPOSITION 1. [4] *For an operator $N \in \mathscr{C}^*(\mathscr{H})$ the following assertions are equivalent:*

(i) *N is gramian normal;*

(ii) $[Nh,Nh] = [N^*h,N^*h]$, $h \in \mathscr{H}$;

(iii) *There exist two commuting gramian self-adjoint operators $A,B \in \mathscr{L}_h^*(\mathscr{H})$, so that $N = A + iB$.*

THEOREM 3. [4] *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Loynes Z–spaces and $N_1 \in \mathscr{B}^*(\mathscr{H}_1)$, $N_2 \in \mathscr{B}^*(\mathscr{H}_2)$ two gramian normal operators on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. If there exists $T \in \mathscr{B}^*(\mathscr{H}_1,\mathscr{H}_2)$ which inverts N_1 and N_2, that is $TN_1 = N_2T$, then T also inverts the adjoints N_1^* and N_2^*, i.e. $TN_1^* = N_2^*T$.*

Theorem 1 and some of its consequences will be recaptured from Theorem 4 for the modules of the commuting gramian normal and bounded operators on Loynes spaces. An identity, similar to the identity in Theorem 4, for the modules of the commuting gramian normal and bounded operators on Loynes spaces will be obtained in Theorem 6. Furthermore, using the inequality of N.G. de Bruijn and Theorem 7, we can improve, relying on Proposition 2, Proposition 3 and Consequence 2, the inequalities from Theorem 1 and 2 from [2] for complex numbers. In addition, in Remark 4, similar results are presented for C^* - algebras.

2. The main results

In this section we assume $\mathscr{H}$ is a Loynes Z- space. Then, we denote by $\mathscr{B}^*(\mathscr{H})$ the space of linear and gramian bounded operators which admit gramian adjoint, see [6], [7], if $\mathscr{H}$ is a Loynes Z- space. We denote by $\mathscr{B}(\mathscr{H})$ the space of linear bounded operators if $\mathscr{H}$ is a Hilbert space.

LEMMA 3. *For any operators $A,B \in \mathscr{B}_h^*(\mathscr{H})$ and a, $b > 0$, we have*

$$\frac{A^2}{a} + \frac{B^2}{b} \geqslant \frac{(A+B)^2}{a+b}.$$

Proof. The obvious inequality

$$\left(\sqrt{\frac{b}{a}}A - \sqrt{\frac{a}{b}}B\right)^2 \geqslant 0$$

leads to

$$A^2 + B^2 + \frac{b}{a}A^2 + \frac{a}{b}B^2 \geqslant A^2 + B^2 + AB + BA. \quad \square$$

LEMMA 4. *If $n \in \mathbb{N}$, $n \geqslant 2$, $N_1,N_2,\ldots,N_n \in \mathscr{B}^*(\mathscr{H})$ are commuting gramian normal operators, then*

$$|\sum_{k=1}^n N_k|^2 = \sum_{k=1}^n |N_k|^2 + \frac{1}{4}\sum_{i,j=1}^n [(N_i + N_i^*)(N_j + N_j^*) - (N_i - N_i^*)(N_j - N_j^*)],$$

where $|N| = (N^*N)^{\frac{1}{2}}$.

We start this section by giving a generalization of the identity (4) from Theorem 3, (see [10]) for a particular class of operators on Loynes spaces. In the proof we will use the same techniques as in [10]. This theorem can also be proven using induction.

THEOREM 4. *If* $n \in \mathbb{N}$, $n \geqslant 2$, $N_1, N_2, \ldots, N_n \in \mathscr{B}^*(\mathscr{H})$ *are commuting gramian normal operators and* $a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\}$ *with* $\sum_{k=1}^{n} a_k \neq 0$, *then*

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + \ldots + N_n|^2}{a_1 + a_2 + \ldots + a_n}$$

$$= \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leqslant i < j \leqslant n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j}, \tag{1}$$

where $|N| = (N^* N)^{\frac{1}{2}}$ *is known as modulus of* N.

Proof. If the operators N_i, $i \in \{1, \ldots, n\}$ are gramian normal then from [4], Proposition 1.2.2, we have the following decompositions for N_k, $k \in \{1, \ldots, n\}$, $N_k = A_k + iB_k$ with A_k, B_k being commuting gramian self-adjoint operators.

Since N_k, $k = 1, \ldots, n$, are mutually commuting gramian normal operators, operators A_k, B_k, $k = 1, \ldots, n$, also commute (it follows from their forms and Fuglede$'$s theorem applied to N_i and N_k).

Thus $|N_k|^2 = |A_k + iB_k|^2 = (A_k + iB_k)^*(A_k + iB_k) = (A_k - iB_k)(A_k + iB_k) = A_k^2 + B_k^2$, $k \in \{1, \ldots, n\}$, and

$$\begin{aligned} |N_1 + N_2 + \ldots + N_n|^2 &= |(A_1 + A_2 + \ldots + A_n) + i(B_1 + B_2 + \ldots + B_n)|^2 \\ &= (A_1 + A_2 + \ldots + A_n)^2 + (B_1 + B_2 + \ldots + B_n)^2. \end{aligned}$$

Moreover,

$$|a_k N_j - a_j N_k|^2 = |a_k(A_j + iB_j) - a_j(A_k + iB_k)|^2 = (a_k A_j - a_j A_k)^2 + (a_k B_j - a_j B_k)^2,$$

$j, k \in \{1, \ldots, n\}$.

The expression on the left-hand side of (1) is equivalent to

$$\begin{aligned} &\frac{A_1^2 + B_1^2}{a_1} + \frac{A_2^2 + B_2^2}{a_2} + \ldots + \frac{A_n^2 + B_n^2}{a_n} - \frac{(A_1 + A_2 + \ldots + A_n)^2 + (B_1 + B_2 + \ldots + B_n)^2}{a_1 + a_2 + \ldots + a_n} \\ &= \frac{1}{a_1 a_2 \ldots a_n (a_1 + a_2 + \ldots + a_n)} ((a_2^2 a_3 a_4 \ldots a_n + a_2 a_3^3 a_4 \ldots a_n + \ldots + a_2 a_3 \ldots a_{n-1} a_n^2)(A_1^2 + B_1^2) \\ &\quad + (a_1^2 a_3 a_4 \ldots a_n + a_1 a_3^2 a_4 \ldots a_n + \ldots + a_1 a_3 \ldots a_{n-1} a_n^2)(A_2^2 + B_2^2) + \ldots \\ &\quad + (a_1^2 a_2 a_3 \ldots a_{n-1} + a_1 a_2^2 a_3 \ldots a_{n-1} + \ldots + a_1 a_2 \ldots a_{n-2} a_{n-1}^2)(A_n^2 + B_n^2) \\ &\quad - 2 a_1 a_2 \ldots a_n ((A_1 A_2 + A_1 A_3 + \ldots + A_1 A_n + \ldots + A_{n-1} A_n) \\ &\quad + (B_1 B_2 + B_1 B_3 + \ldots + B_1 B_n + \ldots + B_{n-1} B_n))) \end{aligned}$$

$$= \frac{1}{a_1a_2...a_n(a_1+a_2+...+a_n)} \cdot (a_3a_4...a_n((a_1A_2-a_2A_1)^2+(a_1B_2-a_2B_1)^2))$$
$$+a_2a_4a_5...a_n((a_1A_3-a_3A_1)^2+(a_1B_3-a_3B_1)^2)+...$$
$$+a_1a_2...a_{n-2}((a_{n-1}A_n-a_nA_{n-1})^2$$
$$+(a_{n-1}B_n-a_nB_{n-1})^2)).$$

Thus, we obtain

$$\frac{|N_1|^2}{a_1}+\frac{|N_2|^2}{a_2}+...+\frac{|N_n|^2}{a_n}-\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n}$$
$$=\frac{A_1^2+B_1^2}{a_1}+\frac{A_2^2+B_2^2}{a_2}+...+\frac{A_n^2+B_n^2}{a_n}-\frac{(A_1+A_2+...+A_n)^2+(B_1+B_2+...+B_n)^2}{a_1+a_2+...+a_n}$$
$$=\frac{1}{a_1a_2...a_n(a_1+a_2+...+a_n)}(a_3a_4...a_n|a_1N_2-a_2N_1|^2+a_2a_4a_5...a_n|a_1N_3-a_3N_1|^2$$
$$+...+a_1a_2...a_{n-2}|a_{n-1}N_n-a_nN_{n-1}|^2). \quad \square$$

THEOREM 5. *(i) If $n\in\mathbb{N}$, $n\geqslant 2$, $N_1,N_2,...,N_n\in\mathscr{B}^*(\mathscr{H})$ are commuting gramian normal operators and $a_1,a_2,...,a_n\in\mathbb{R}\setminus\{0\}$ with $\sum_{k=1}^n a_k\neq 0$, then*

$$\frac{|N_1|^2}{a_1}+\frac{|N_2|^2}{a_2}+...+\frac{|N_n|^2}{a_n}\geqslant\frac{1}{\sum_{k=1}^n a_k}\sum_{1\leqslant i<j\leqslant n}\frac{|a_iN_j-a_jN_i|^2}{a_ia_j}. \tag{2}$$

(ii) Under the above-stated conditions, if $a_1,\ a_2,\ ....,a_n\in(0,\infty)$ we also have,

$$\frac{|N_1|^2}{a_1}+\frac{|N_2|^2}{a_2}+...+\frac{|N_n|^2}{a_n}\geqslant\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n},$$

with equality if and only if $a_iN_j=a_jN_i$, for any $i,j\in\{1,2,...,n\}$.

(iii) In (i), if we take $a_1=a_2=...=a_n=1$, the inequality becomes

$$|N_1|^2+|N_2|^2+...+|N_n|^2\geqslant\frac{1}{n}\sum_{1\leqslant i<j\leqslant n}|N_j-N_i|^2. \tag{3}$$

Proof. It follows immediately from Theorem 4. The proof of equality in (ii) is contained in the proof of Theorem 4. $\square$

In the proof of the theorem below we need the following identity:

$$|N_1-N_2|^2+|N_1+N_2|^2=2|N_1|^2+2|N_2|^2, \tag{5}$$

where $|N_1|=(N_1^*N_1)^{\frac{1}{2}}$ is the modulus of N_1 and $N_1,\ N_2\in\mathscr{B}^*(\mathscr{H})$ are arbitrary operators. This identity is easily verified by calculation.

THEOREM 6. *If $n \in \mathbb{N}$, $n \geqslant 2$, $N_1, N_2, ..., N_n \in \mathscr{B}^*(\mathscr{H})$ are gramian normal operators which commute as pairs and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^n a_k \neq 0$, then*

$$\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n}+\sum_{k=1}^n \frac{|N_k|^2}{a_k}-\frac{2}{a_1+a_2+...+a_n}\sum_{k=1}^n |N_k|^2$$

$$=\frac{1}{a_1+a_2+...+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{|a_iN_j+a_jN_i|^2}{a_ia_j}. \tag{6}$$

Proof. We will use identity (1) from Theorem 4 and identity (5).
Let

$$S=\frac{1}{a_1+a_2+...+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{|a_iN_j+a_jN_i|^2}{a_ia_j}.$$

Now, adding S and the right member of identity (1), we will have

$$\begin{aligned}
&\frac{1}{\sum_{k=1}^n a_k}\left(\sum_{1\leqslant i<j\leqslant n}\frac{|a_iN_j-a_jN_i|^2}{a_ia_j}+\sum_{1\leqslant i<j\leqslant n}\frac{|a_iN_j+a_jN_i|^2}{a_ia_j}\right)\\
&=\frac{2}{\sum_{k=1}^n a_k}\sum_{1\leqslant i<j\leqslant n}\frac{|a_iN_j|^2+|a_jN_i|^2}{a_ia_j}\\
&=\frac{2}{\sum_{k=1}^n a_k}\left(\sum_{j=2}^n\frac{|N_j|^2}{a_j}\sum_{i=1}^{j-1}\frac{|a_i|^2}{a_i}+\sum_{j=2}^n\frac{|a_j|^2}{a_j}\sum_{i=1}^{j-1}\frac{|N_i|^2}{a_i}\right)\\
&=\frac{2}{\sum_{k=1}^n a_k}\left(\sum_{j=2}^n\frac{|N_j|^2}{a_j}\sum_{i=1}^{j-1}a_i+\sum_{j=2}^n a_j\sum_{i=1}^{j-1}\frac{|N_i|^2}{a_i}\right)\\
&=\frac{2}{\sum_{k=1}^n a_k}\left(\sum_{j=2}^n\frac{|N_j|^2}{a_j}(a_1+a_2+...+a_{j-1})+\sum_{j=2}^n a_j\left(\frac{|N_1|^2}{a_1}+\frac{|N_2|^2}{a_2}+...+\frac{|N_{j-1}|^2}{a_{j-1}}\right)\right)\\
&=\frac{2}{\sum_{k=1}^n a_k}\left(\frac{|N_2|^2}{a_2}a_1+\frac{|N_3|^2}{a_3}(a_1+a_2)+...+\frac{|N_n|^2}{a_n}(a_1+a_2+...+a_{n-1})\right.\\
&\quad\left.+a_2\frac{|N_1|^2}{a_1}+a_3\left(\frac{|N_1|^2}{a_1}+\frac{|N_2|^2}{a_2}\right)+...+a_n\left(\frac{|N_1|^2}{a_1}+...+\frac{|N_{n-1}|^2}{a_{n-1}}\right)\right)\\
&=\frac{2}{\sum_{k=1}^n a_k}\left(\sum_{k=1}^n a_k\cdot\sum_{k=1}^n\frac{|N_k|^2}{a_k}-\sum_{k=1}^n|N_k|^2\right).
\end{aligned}$$

Thus, we have

$$S=\sum_{k=1}^n\frac{|N_k|^2}{a_k}-\frac{2}{\sum_{k=1}^n a_k}\sum_{k=1}^n|N_k|^2+\frac{|\sum_{k=1}^n N_k|^2}{\sum_{k=1}^n a_k}. \quad \square$$

As a particular case of the previous theorem, we can also obtain the following identity for complex numbers:

CONSEQUENCE 1. *If* $n \in \mathbb{N}$, $n \geqslant 2$, $z_1, z_2, ..., z_n \in \mathbb{C}$ *are complex numbers and* $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ *with* $\sum_{k=1}^n a_k \neq 0$, *then*

$$\frac{|z_1+z_2+...+z_n|^2}{a_1+a_2+...+a_n} + \sum_{k=1}^n \frac{|z_k|^2}{a_k} - \frac{2}{a_1+a_2+...+a_n}\sum_{k=1}^n |z_k|^2$$
$$= \frac{1}{a_1+a_2+...+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{|a_i z_j + a_j z_i|^2}{a_i a_j}.$$

REMARK 1. If $n \in \mathbb{N}$, $n \geqslant 2$, N_1, $N_2, ..., N_n \in \mathscr{B}^*(\mathscr{H})$ are gramian normal operators which commute as pairs and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^n a_k > 0$, then

$$\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n} + \sum_{k=1}^n \frac{|N_k|^2}{a_k} \geqslant \frac{1}{a_1+a_2+...+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{|a_i N_j + a_j N_i|^2}{a_i a_j}. \qquad (7)$$

The inequality of N. G. de Bruijn can be written in the following form:

THEOREM 7. *If* a_1, $a_2, ..., a_n$ *is a sequence of real numbers and* N_1, $N_2, ..., N_n$ *is a sequence of commuting gramian normal operators in* $\mathscr{B}^*(\mathscr{H})$ *so that* $\sum_{k=1}^n a_k N_k$ *is hermitian, then*

$$|\sum_{k=1}^n a_k N_k|^2 \leqslant \frac{1}{2}\sum_{k=1}^n a_k^2 [\sum_{k=1}^n |N_k|^2 + |\sum_{k=1}^n N_k^2|],$$

where $|N|$ *is the modulus of the operator* N.

Proof. The proof will be as in [5]. Using the fact that $\sum_{k=1}^n a_k N_k$ is hermitian and the definition of the modulus of an operator, we obtain

$$|\sum_{k=1}^n a_k N_k|^2 = (\sum_{k=1}^n a_k N_k)^2,$$

where $N_k = A_k + iB_k$, $k = 1...,, n$.

Again, from $\sum_{k=1}^n a_k N_k$ being hermitian, it results that $\sum_{k=1}^n a_k N_k = \sum_{k=1}^n a_k N_k^*$ or

$$\sum_{k=1}^n a_k B_k = 0.$$

Thus

$$|\sum_{k=1}^n a_k N_k|^2 = |\sum_{k=1}^n a_k A_k + i\sum_{k=1}^n a_k B_k|^2 = (\sum_{k=1}^n a_k A_k)^2 + (\sum_{k=1}^n a_k B_k)^2$$
$$= (\sum_{k=1}^n a_k A_k)^2 \leqslant \sum_{k=1}^n a_k^2 \sum_{k=1}^n A_k^2,$$

the last inequality being the Cauchy-Schwarz inequality for gramian self-adjoint operators which commute as pairs. A proof for this last inequality can be done by induction, using the fact that for commuting gramian self-adjoint operators A_k, A_l and a_k, a_l real numbers, where $k,l \in \{1,...,n\}$, we have $(a_k A_l - a_l A_k)^2 \geqslant 0$. Now taking into account that for all $k \in \{1,...,n\}$, $2A_k^2 = |N_k|^2 + C_k$, where $C_k = A_k^2 - B_k^2$, this will give $N_k^2 = C_k + 2iA_kB_k$, and then

$$|\sum_{k=1}^{n} a_k N_k|^2 \leqslant \frac{1}{2}\sum_{k=1}^{n} a_k^2 [\sum_{k=1}^{n} |N_k|^2 + \sum_{k=1}^{n} C_k].$$

Because

$$\sum_{k=1}^{n} C_k \leqslant |\sum_{k=1}^{n} N_k^2|,$$

we have the desired inequality. $\square$

Furthermore, we can state the following result for particular operators on Loynes spaces:

PROPOSITION 2. *Let* $N_1, N_2, ..., N_n$, $(n \geqslant 2)$ *be a sequence of gramian normal operators in* $\mathscr{B}^*(\mathscr{H})$ *commuting as pairs so that* $\sum_{k=1}^{n} a_k N_k$ *is hermitian and* $a_1, a_2, ...,$ $a_n \in \mathbb{R} \setminus \{0\}$ *with* $\sum_{k=1}^{n} a_k > 0$. *Then*

$$\sum_{1 \leqslant i<j \leqslant n} \frac{|a_i N_j + a_j N_i|^2}{a_i a_j} \leqslant \frac{n-4}{2}\sum_{k=1}^{n} |N_k|^2 + (\sum_{k=1}^{n} a_k) \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} + \frac{n}{2}|\sum_{k=1}^{n} N_k^2|.$$

Proof. If we replace a_k, $k \in \{1, 2, ..., n\}$ by $\frac{1}{\sqrt{a_1+a_2+...+a_n}}$ in the inequality from Theorem 7, then we have

$$|\sum_{k=1}^{n} \frac{N_k}{\sqrt{a_1+a_2+...+a_n}}|^2 = \frac{|\sum_{k=1}^{n} N_k|^2}{\sum_{k=1}^{n} a_k} \leqslant \frac{1}{2}\sum_{k=1}^{n} \frac{1}{a_1+a_2+...+a_n}[\sum_{k=1}^{n} |N_k|^2 + |\sum_{k=1}^{n} N_k^2|]$$

or

$$\frac{|\sum_{k=1}^{n} N_k|^2}{\sum_{k=1}^{n} a_k} \leqslant \frac{n}{2\sum_{k=1}^{n} a_k}[\sum_{k=1}^{n} |N_k|^2 + |\sum_{k=1}^{n} N_k^2|].$$

Now using Theorem 6, we find

$$\frac{1}{a_1+a_2+...+a_n}\sum_{1 \leqslant i<j \leqslant n} \frac{|a_i N_j + a_j N_i|^2}{a_i a_j}$$

$$\leqslant \frac{n}{2\sum_{k=1}^{n} a_k}[\sum_{k=1}^{n} |N_k|^2 + |\sum_{k=1}^{n} N_k^2|] + \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{2}{\sum_{k=1}^{n} a_k} \cdot \sum_{k=1}^{n} |N_k|^2$$

i.e.

$$\sum_{1 \leqslant i<j \leqslant n} \frac{|a_i N_j + a_j N_i|^2}{a_i a_j} \leqslant \frac{n}{2}[\sum_{k=1}^{n} |N_k|^2 + |\sum_{k=1}^{n} N_k^2|] + \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - 2\sum_{k=1}^{n} |N_k|^2$$

$$= \frac{n-4}{2}\sum_{k=1}^{n} |N_k|^2 + \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} + \frac{n}{2}|\sum_{k=1}^{n} N_k^2|. \quad \square$$

REMARK 2. If we take $a_k = 1$, $k \in \{1, 2, ..., n\}$ in the above-stated inequality, then we get

$$\sum_{1 \leqslant i < j \leqslant n} |N_j + N_i|^2 \leqslant \frac{3n-4}{2} \sum_{k=1}^{n} |N_k|^2 + \frac{n}{2} \Big| \sum_{k=1}^{n} N_k^2 \Big|.$$

PROPOSITION 3. *Let $N_1, N_2, ..., N_n$, $(n \geqslant 2)$ be a sequence of gramian normal operators in $\mathscr{B}^*(\mathscr{H})$ commuting as pairs, so that $\sum_{k=1}^{n} a_k N_k$ is hermitian and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^{n} a_k > 0$. Then*

$$\sum_{1 \leqslant i < j \leqslant n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j} \geqslant \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{n}{2} \sum_{k=1}^{n} |N_k|^2 - \frac{n}{2} \Big| \sum_{k=1}^{n} N_k^2 \Big|.$$

Proof. Adding the identities from Theorem 4 and 6, we find

$$\frac{1}{\sum_{k=1}^{n} a_k} \sum_{1 \leqslant i < j \leqslant n} \frac{|a_i N_j - a_j N_i|^2 + |a_i N_j + a_j N_i|^2}{a_i a_j} = 2 \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{2}{\sum_{k=1}^{n} a_k} \cdot \sum_{k=1}^{n} |N_k|^2$$

or

$$\sum_{1 \leqslant i < j \leqslant n} \left(\frac{|a_i N_j - a_j N_i|^2}{a_i a_j} + \frac{|a_i N_j + a_j N_i|^2}{a_i a_j} \right) = 2 \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - 2 \sum_{k=1}^{n} |N_k|^2.$$

Now applying Proposition 2 we deduce that

$$\begin{aligned} \sum_{1 \leqslant i < j \leqslant n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j} &= 2 \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - 2 \sum_{k=1}^{n} |N_k|^2 - \sum_{1 \leqslant i < j \leqslant n} \frac{|a_i N_j + a_j N_i|^2}{a_i a_j} \\ &\geqslant 2 \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - 2 \sum_{k=1}^{n} |N_k|^2 - \frac{n-4}{2} \sum_{k=1}^{n} |N_k|^2 \\ &\quad - \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{n}{2} \Big| \sum_{k=1}^{n} N_k^2 \Big|, \end{aligned}$$

which leads to the desired inequality. $\square$

REMARK 3. If we take $a_k = 1$, $k \in \{1, ..., n\}$ in the above-stated inequality, then we get

$$\sum_{1 \leqslant i < j \leqslant n} |N_j - N_i|^2 \geqslant \frac{n}{2} \Big(\sum_{k=1}^{n} |N_k|^2 - \Big| \sum_{k=1}^{n} N_k^2 \Big| \Big).$$

For complex numbers we can also state the following result as a special case of the Proposition 2 and 3.

CONSEQUENCE 2. *Let $z_1, z_2, ..., z_n$, $(n \geqslant 2)$ be a sequence of complex numbers and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^{n} a_k > 0$. Then*

$$\sum_{1 \leqslant i < j \leqslant n} \frac{|a_i z_j + a_j z_i|^2}{a_i a_j} \leqslant \frac{n-4}{2} \sum_{k=1}^{n} |z_k|^2 + \Big(\sum_{k=1}^{n} a_k \Big) \cdot \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} + \frac{n}{2} \Big| \sum_{k=1}^{n} z_k^2 \Big|.$$

and

$$\sum_{1\leqslant i<j\leqslant n}\frac{|a_iz_j-a_jz_i|^2}{a_ia_j}\geqslant\sum_{k=1}^n a_k\cdot\sum_{k=1}^n\frac{|z_k|^2}{a_k}-\frac{n}{2}\sum_{k=1}^n|z_k|^2-\frac{n}{2}|\sum_{k=1}^n z_k^2|.$$

It is known that $\mathscr{B}^*(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$ are particular C^*-algebras, see [7], [4], [11]. As in the case of Loynes spaces, we can also deduce by calculation in C^*-algebras the following equality:

$$|b_1-b_2|^2+|b_1+b_2|^2=2|b_1|^2+2|b_2|^2,$$

where $|b_1|=(b_1^*b_1)^{\frac{1}{2}}$ is the modulus of b_1 and $b_1,\ b_2\in\mathscr{A}$ are arbitrary elements in a C^*-algebra $\mathscr{A}$. Then, the following result can be obtained, similarly as for the Loynes spaces and using analogous theorems (see [11]).

REMARK 4. If $n\in\mathbb{N},\ n\geqslant 2,\ b_1,\ b_2,\dots,b_n\in\mathscr{A}$ are normal elements which commute as pairs and $a_1,a_2,\dots,a_n\in\mathbb{R}\setminus\{0\}$ with $\sum_{k=1}^n a_k\neq 0$, then

$$\frac{|b_1+b_2+\dots+b_n|^2}{a_1+a_2+\dots+a_n}+\sum_{k=1}^n\frac{|b_k|^2}{a_k}-\frac{2}{a_1+a_2+\dots+a_n}\sum_{k=1}^n|b_k|^2$$
$$=\frac{1}{a_1+a_2+\dots+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{|a_ib_j+a_jb_i|^2}{a_ia_j}.$$

Moreover, if $n\in\mathbb{N},\ n\geqslant 2,\ b_1,\ b_2,\dots,b_n\in\mathscr{A}$ are normal elements which commute as pairs and $a_1,a_2,\dots,a_n\in\mathbb{R}\setminus\{0\}$ with $\sum_{k=1}^n a_k>0$, then

$$\frac{|b_1+b_2+\dots+b_n|^2}{a_1+a_2+\dots+a_n}+\sum_{k=1}^n\frac{|b_k|^2}{a_k}\geqslant\frac{1}{a_1+a_2+\dots+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{|a_ib_j+a_jb_i|^2}{a_ia_j}.$$

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(Received December 15, 2009)

Loredana Ciurdariu
Department of Mathematics
"Politehnica" University of Timisoara
P-ta. Victoriei, No.2
300006-Timisoara
Romania
e-mail: cloredana43@yahoo.com