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SUMS OF REAL PARTS OF EIGENVALUES OF PERTURBED MATRICES

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Abstract. Let A be $\tilde{A}$ be $n\times n$ matrices, whose eigenvalues are λ_k and $\tilde{\lambda}_k$, respectively. Assuming that A is Hermitian, we prove the inequality

$$[\sum_{k=1}^{n} |\text{ Re } \tilde{\lambda}_k - \lambda_k|^p]^{1/p} \leqslant N_p(E_R) + \tilde{b}_p N_p(E_I) \ \ (2 \leqslant p < \infty)$$

where $N_p(A)$ is the Schatten-von Neumann norm of A, $E = \tilde{A} - A$, $E_R = (E+E^*)/2$, $E_I = (E-E^*)/2i$, and $\tilde{b}_p \leqslant pe^{1/3}$. That inequality is generalized then to the Schatten-von Neumann operators.

1. Introduction

Let A and $\tilde{A}$ be linear operators (matrices) in the complex Euclidean n-dimensional space $\mathbb{C}^n, n < \infty$, whose eigenvalues counted with their multiplicities are λ_k and $\tilde{\lambda}_k$ $(k = 1,...,n)$, respectively. By $N_p(A)$ $(1 \leqslant p < \infty)$ we denote the Schatten-von Neumann norm of A:

$$N_p^p(A) := \text{trace } [(A^*A)^{p/2}],$$

cf. [1, 4]; the asterisk means the adjoint operator. In particular, $N_2(.)$ is the Hilbert-Schmidt (Frobeinus) norm, cf. [1, 4]. Furthermore, $A_R = (A+A^*)/2$, $A_I = (A-A^*)/2i$ and $E = \tilde{A} - A$.

Introduce the quantity

$$m_p(A,\tilde{A}) := \min_{\pi} \sum_{k=1}^{n} |\lambda_{\pi(k)} - \tilde{\lambda}_k|^p \ (p \geqslant 1)$$

where π ranges over all permutations of the integers $1,2,..,n$. It plays an essential role in the perturbation theory of matrices, cf. [8, 11]. One of the famous results on $m_2(A,\tilde{A})$ is the Hoffman-Wiellandt theorem proved in [6] (see also [11, p. 189] and [8, p. 126]) which asserts that for all normal matrices A and $\tilde{A}$, the inequality $m_2(A,\tilde{A}) \leqslant N_2(A-\tilde{A})$ is valid.

In [9] L. Mirsky has proved that for all Hermitian matrices A and $\tilde{A}$ we have

$$m_p(A,\tilde{A}) \leqslant N_p(A-\tilde{A}) \ \ (p \geqslant 1) \tag{1.1}$$

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(see also [11, p. 194] and [8, p. 126]).

In 1975 W. Kahan [7] (see also [11, Theorem IV.5.2, p. 213]) derived the following result: let A be a Hermitian operator and $\tilde{A}$ an arbitrary one in $\mathbb{C}^n$, and

$$\lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_n \text{ and } \mathrm{Re}\,\tilde{\lambda}_1 \leqslant \mathrm{Re}\,\tilde{\lambda}_2 \leqslant ... \leqslant \mathrm{Re}\,\tilde{\lambda}_n. \tag{1.2}$$

Then

$$[\sum_{k=1}^{n} (\mathrm{Re}\,\tilde{\lambda}_k - \lambda_k)^2]^{1/2} \leqslant N_2(E_R) + [N_2^2(E_I) - \sum_{k=1}^{n} (\mathrm{Im}\,\lambda_k)^2]^{1/2} \leqslant \sqrt{2} N_2(E). \tag{1.3}$$

Here $E_R = (E + E^*)/2$, $E_I = (E - E^*)/2i$.

The Kahan theorem generalizes the Mirsky result in the case $p = 2$. Inequality (1.3) can be easily generalized to the Hilbert-Schmidt operators. In the present paper we establish an analogous result for a $p \in (2, \infty)$. The results obtained below enable us to derive estimates for the sums of the eigenvalues of perturbed Schatten-von Neumann operators.

2. The main result

Let c_m $(m = 1, 2, ...)$ be a sequence of positive numbers defined by by the recursive relation

$$c_1 = 1,\ c_m = c_{m-1} + \sqrt{c_{m-1}^2 + 1}\ \ (m = 2, 3, ...).$$

To formulate our main result, for a $p \in [2^m, 2^{m+1}]$ $(m = 1, 2, ...)$, put

$$b_p = c_m^t c_{m+1}^{1-t} \text{ with } t = 2 - 2^{-m} p.$$

As it is proved in [3, Corollary 1.3],

$$b_p \leqslant \frac{p e^{1/3}}{2} \leqslant p\ (p \geqslant 2).$$

Again assume that (1.2) holds. Now we are in a position to formulate the main result of the paper.

THEOREM 2.1. *Let A be a Hermitian operator and $\tilde{A}$ an arbitrary one in $\mathbb{C}^n$. Then for any $p \in [2, \infty)$,*

$$[\sum_{k=1}^{n} |\mathrm{Re}\,\tilde{\lambda}_k - \lambda_k|^p]^{1/p} \leqslant N_p(E_R) + 2 b_p N_p(E_I). \tag{2.1}$$

Proof. As it is well known, according to the Schur theorem, cf. [11], we can write

$$\tilde{A} = Q \tilde{T} Q^{-1}$$

where $\tilde{T}$ is an upper triangular matrix. Since $\tilde{T}$ and $\tilde{A}$ are similar, they have the same eigenvalues, and without loss of generality we can assume that $\tilde{A}$ is already upper triangular, i.e.

$$\tilde{A} = \tilde{D} + \tilde{V} \ \ (\sigma(\tilde{A}) = \sigma(\tilde{D})) \tag{2.2}$$

where $\tilde{D}$ is the diagonal matrix and $\tilde{V}$ is the strictly upper triangular matrix. Here and below $\sigma(A)$ denotes the spectrum of A. We have $\tilde{A} = \tilde{D}_R + i\tilde{D}_I + \tilde{V}$ and thus, the real and imaginary part of A are

$$\tilde{A}_R = A + E_R = \tilde{D}_R + \tilde{V}_R \text{ and } \tilde{A}_I = E_I = \tilde{D}_I + \tilde{V}_I,$$

respectively. Since A and $\tilde{D}_R$ are Hermitian, by (1.1) we obtain

$$[\sum_{k=1}^{n} |\mathrm{Re}\,\tilde{\lambda}_k - \lambda_k|^p]^{1/p} \leqslant N_p(A - \tilde{D}_R) = N_p(A - A_R + \tilde{V}_R) =$$

$$N_p(E_R + \tilde{V}_R) \ \ (1 \leqslant p < \infty).$$

Thus

$$[\sum_{k=1}^{n} |\mathrm{Re}\,\tilde{\lambda}_k - \lambda_k|^p]^{1/p} \leqslant N_p(E_R) + N_p(\tilde{V}_R) \ \ (1 \leqslant p < \infty). \tag{2.3}$$

Making use Lemma 2.2 from [3], we get the inequality

$$N_p(\tilde{V}_R) \leqslant b_p N_p(\tilde{V}_I) \ \ (2 \leqslant p < \infty) \tag{2.4}$$

(see also [5, Section 3.6] and [2]). In addition, by (2.2) $\tilde{V}_I = \tilde{A}_I - \tilde{D}_I$ and therefore

$$N_p(\tilde{V}_I) \leqslant N_p(\tilde{A}_I) + N_p(\tilde{D}_I) \ \ (1 \leqslant p < \infty).$$

Thanks to the Weyl inequalities [4],

$$N_p(\tilde{D}_I) \leqslant N_p(\tilde{A}_I) \text{ and } N_p(\tilde{D}_R) \leqslant N_p(\tilde{A}_R) \ \ (1 \leqslant p < \infty).$$

Thus,

$$N_p(\tilde{V}_I) \leqslant 2N_p(\tilde{A}_I) \ \ (1 \leqslant p < \infty). \tag{2.5}$$

Now (2.4) implies the inequality

$$N_p(\tilde{V}_R) \leqslant 2b_p N_p(\tilde{A}_I) \ \ (2 \leqslant p < \infty).$$

So by (2.3) we get the desired inequality

$$[\sum_{k=1}^{n} |\mathrm{Re}\,\tilde{\lambda}_k - \lambda_k|^p]^{1/p} \leqslant N_p(E_R) + N_p(\tilde{V}_R) \leqslant N_p(E_R) + 2b_p N_p(E_I). \qquad \square$$

The proved theorem is sharp in the following sense: if $\tilde{A}$ is Hermitian, then $N_p(E_I) = 0$ and inequality (2.1) becomes the Mirsky result (1.1).

COROLLARY 2.2. *Let a matrix $\tilde{A} = (a_{jk})_{j,k=1}^n$ have the real diagonal entries. Let W be the off-diagonal part of $\tilde{A}$: $W = \tilde{A} - \text{diag}\,(a_{11}, ..., a_{nn})$. Then for any $p \in [2, \infty)$,*

$$[\sum_{k=1}^n |\text{Re}\,\tilde{\lambda}_k - a_{kk}|^p]^{1/p} \leqslant N_p(W_R) + 2b_p N_p(W_I)$$

and therefore,

$$[\sum_{k=1}^n |\text{Re}\,\tilde{\lambda}_k|^p]^{1/p} \geqslant [\sum_{k=1}^n |a_{kk}|^p]^{1/p} - N_p(W_R) - 2b_p N_p(W_I). \tag{2.6}$$

Indeed, this result is due to the previous theorem with $A = diag\,[a_{jj}]$.

Certainly, inequality (2.6) has a sense only if its right-hand side is positive.

The latter corollary complements the Weyl inequality

$$\sum_{k=1}^n |\text{Re}\,\tilde{\lambda}_k|^p \leqslant N_p^p(\tilde{A}_R) \;\; (p \geqslant 1).$$

Furthermore, for a $p \geqslant 1$, let S_p be the Schatten-von Neumann ideal of compact operators A in a separable Hilbert space with the finite norm $N_p(A)$ [4, 1]. Since any operator from S_p can be considered as a limit in N_p of finite rank operators [1], Theorem 2.1 implies

COROLLARY 2.3. *Let $A \in S_p$ $(2 \leqslant p < \infty)$ be a Hermitian operator and $\tilde{A} \in S_p$ an arbitrary one. Then*

$$[\sum_{k=1}^\infty |\text{Re}\,\tilde{\lambda}_k - \lambda_k|^p]^{1/p} \leqslant N_p(E_R) + 2b_p N_p(E_I).$$

3. The case $p = 1$ and perturbations of determinants

The case $1 \leqslant p < 2$ should be considered separately from the case $p \geqslant 2$, since the relations between $N_p(\tilde{V}_R)$ and $N_p(\tilde{V}_I)$ similar to inequality (2.3) are unknown if $p = 1$, and we could not use the arguments of the proof of Theorem 2.1.

Furthermore, by (2.2) one can write out

$$\sum_{k=1}^n |\text{Re}\,\tilde{\lambda}_k - \lambda_k| \leqslant N_1(E_R) + N_1(\tilde{V}_R).$$

But by the well-known Theorem 3.2.1 from [5],

$$N_1(V_R) \leqslant N_1(V_I) v_n \text{ where } v_n := \frac{4}{\pi} \sum_{k=1}^n \frac{1}{2k-1}. \tag{3.1}$$

Thus (2.4) and (3.1) yield the inequality

$$N_1(V_R) \leqslant N_1(V_I) v_n \leqslant 2N_1(A_I) v_n.$$

Taking into account that

$$\sum_{k=1}^{n} |\mathrm{Im}\,\tilde{\lambda}_k| \leqslant N_1(\tilde{A}_I) = N_1(E_I),$$

cf. [4, Section II.6], we obtain the following Theorem.

THEOREM 3.1. *Let A be a Hermitian operator and $\tilde{A}$ an arbitrary one in $\mathbb{C}^n$. Then the inequalities*

$$\sum_{k=1}^{n} |\mathrm{Re}\,\tilde{\lambda}_k - \lambda_k| \leqslant N_1(E_R) + 2v_n N_1(E_I)$$

and

$$\sum_{k=1}^{n} |\tilde{\lambda}_k - \lambda_k| \leqslant \sum_{k=1}^{n} |\mathrm{Re}\,\tilde{\lambda}_k - \lambda_k| + \sum_{k=1}^{n} |\mathrm{Im}\,\tilde{\lambda}_k| \leqslant \eta_n(E)$$

are true, where

$$\eta_n(E) := N_1(E_R) + (1 + 2v_n)N_1(E_I).$$

Let us apply the latter theorem to determinants. To this end note that

$$\det A - \det \tilde{A} = \sum_{j=1}^{n} \prod_{k=1}^{j-1} \lambda_k \left(\lambda_j - \tilde{\lambda}_j\right) \prod_{k=j+1}^{n} \tilde{\lambda}_k.$$

Here we put

$$\prod_{k=1}^{0} \lambda_k = \prod_{k=n+1}^{n} \lambda_k = 1.$$

Hence,

$$|\det A - \det \tilde{A}| \leqslant \sum_{j=1}^{n} |\lambda_j - \tilde{\lambda}_j| \max_{1 \leqslant j \leqslant n} \left(\prod_{k=1}^{j-1} |\lambda_k| \prod_{k=j+1}^{n} |\tilde{\lambda}_k| \right). \tag{3.2}$$

According to the inequality for the arithmetic and geometric mean values,

$$\prod_{k=1}^{j-1} |\lambda_k| \prod_{k=j+1}^{n} |\tilde{\lambda}_k| \leqslant \left[\frac{1}{n-1} \left(\sum_{k=1}^{j-1} |\lambda_k| + \sum_{k=j+1}^{n} |\tilde{\lambda}_k| \right) \right]^{n-1}.$$

But thanks to Theorem 2.1,

$$\sum_{k=1}^{n} |\tilde{\lambda}_k| \leqslant \sum_{k=1}^{n} |\lambda_k| + \eta_n(E).$$

Thus

$$\prod_{k=1}^{j-1} |\lambda_k| \prod_{k=j+1}^{n} |\tilde{\lambda}_k| \leqslant \left[\frac{1}{n-1} \left(\sum_{k=1}^{n} |\lambda_k| + \eta_n(E) \right) \right]^{n-1}.$$

Making use Theorem 3.1 and (3.2), we arrive at the following result.

COROLLARY 3.2. *Let A be a Hermitian operator and $\tilde{A}$ an arbitrary one in $\mathbb{C}^n$. Then*

$$|\det A - \det \tilde{A}| \leqslant \eta_n(E) \left[\frac{1}{n-1} \left(\sum_{k=1}^{n} |\lambda_k| + \eta_n(E) \right) \right]^{n-1}.$$

Taking in this corollary $A = \operatorname{diag}(a_{11}, \ldots, a_{nn})$ we get

COROLLARY 3.3. *Let a matrix $\tilde{A} = (a_{jk})_{j,k=1}^{n}$ have the real diagonal entries. Then*

$$\left| \det \tilde{A} - \prod_{k=1}^{n} a_{kk} \right| \leqslant \eta_n(W) \left[\frac{1}{n-1} \left(\sum_{k=1}^{n} |a_{kk}| + \eta_n(W) \right) \right]^{n-1}.$$

Recall that W is the off-diagonal part of $\tilde{A}$. Besides,

$$\eta_n(W) = N_1(W_R) + (1 + 2\nu_n) N_1(W_I).$$

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