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SPHERICAL CAP DISCREPANCY AND INEQUALITIES ON THE SPHERE

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(*Communicated by A. Čižmešija*)

Abstract. The aim of this work is to present a method of covering the unit sphere by means of spherical caps of fixed radius. The method based on a set of rotations provides an explicit formula for the number of spherical caps that cover the whole unit sphere and the exact positioning of their centers.

1. Introduction

Covering the unit sphere with n spherical caps of smallest possible radius is still a challenging unsolved problem. This led to the investigation of the general question: how many spherical caps of radius h do we need to cover the unit sphere? We are not looking for the optimal solution because it is beyond reach for the time being, rather we are looking for an explicit formula for the number of spherical caps needed and an exact positioning of the centers of the spherical caps that cover the whole unit sphere without giving any preferences to any region on the sphere.

The centers of the spherical caps will be placed on the unit sphere using a method described in [1]. This method uses Hecke operators on $L^2(S^2)$, the Hilbert space of square integrable functions on the unit sphere, to generate very evenly distributed sequences of three-dimensional rotations. Moreover, bounds on the discrepancy and the mean square discrepancy for spherical caps were obtained in [1]. We have previously used this method in [2] to compress functions on the unit sphere and it did perform uniformly well independently of the location of the support of the function on the sphere, and for functions supported in a small subset of S^2.

In this paper, we will refine a main result in [1] that estimates the discrepancy of spherical caps generated of special sequences of three-dimensional rotations and use it to produce an explicit formula for the number of spherical caps that cover the whole unit sphere and the exact positioning of their centers.

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2. Ramanujan set

The group $G = SO(3)$ of proper rotations preserving the dot product, acts on $L^2(S^2)$ as follows:

$$\rho(\gamma)f(\sigma) = f(\gamma^{-1}\sigma) \qquad (\gamma \in G,\ \sigma \in S^2,\ f \in L^2(S^2)). \tag{2.1}$$

Each operator $\rho(\gamma)$ defined in (2.1) is unitary. For any $f,g \in L^2(S^2)$ the function $G \ni \gamma \to (\rho(\gamma)f,g) \in \mathbb{C}$ is continuous. Moreover, $\rho(\gamma_1)\rho(\gamma_2) = \rho(\gamma_1\gamma_2)$ for any $\gamma_1,\gamma_2 \in G$. In other words, $(\rho, L^2(S^2))$ is a unitary representation of the group G, see [3].

Let $S \subseteq SO(3)$ be a finite symmetric set. In other words, the number of elements of S, denoted by $|S| = 2N$, is even and $\gamma \in S$ if and only if $\gamma^{-1} \in S$. Let $(T_S f)(x) = \sum_{\gamma \in S} f(\gamma x)$, where $f \in L^2(S^2)$. Furthermore, let $u(x) = 1$, $x \in S^2$, denote the unit function and $\mathscr{H}_0 = \mathbb{C}u$. The orthogonal projection $P_{\mathscr{H}_0} : L^2(S^2) \to \mathscr{H}_0$ is given by:

$$P_{\mathscr{H}_0} f = \left(\frac{1}{4\pi}\int_{S^2} f(x)\,dx\right) u \qquad (f \in L^2(S^2)). \tag{2.2}$$

THEOREM 2.1. [1] *For any finite symmetric set* $S \subseteq SO(3)$

$$\left\| \frac{1}{|S|} T_S - P_{\mathscr{H}_0} \right\| \geqslant 2\frac{\sqrt{|S|-1}}{|S|}. \tag{2.3}$$

A set where the equality holds is called a **Ramanujan set**. Let p be a prime, equal to 1 modulo 4. Then there exists in [1] an explicitly described Ramanujan set, S_p, with $|S_p| = p+1$.

Let S_p be a Ramanujan set, $S_p = \{\gamma_1, ..., \gamma_{\frac{p+1}{2}}, \gamma_1^{-1}, ..., \gamma_{\frac{p+1}{2}}^{-1}\}$, and let $S_p^M \subseteq SO(3)$ denote the set of reduced words of length at most $M = 1,2,3,...$ in S_p (by reduced we mean all the obvious cancelations such as $\gamma\gamma^{-1}$ have been carried out). It is straightforward to verify by induction that

$$|S_p^M| = \frac{p^{M+1} + p^M - p - 1}{p-1}. \tag{2.4}$$

In [1], we have the following theorem:

THEOREM 2.2.

$$\left\| \frac{1}{|S_p^M|} T_{S_p^M} - P_{\mathscr{H}_0} \right\| \leqslant \text{const}\,\frac{\log(|S_p^M|)}{\sqrt{|S_p^M|}}. \tag{2.5}$$

3. Main theorem

We will use in this paper the Ramanujan set of rotations only in the case when $p = 5$. For $p=5$, the construction can be described quite concisely. $S_5 = \{A,B,C,A^{-1},B^{-1}, C^{-1}\}$, where A, B, C are rotations about the X, Y, and Z axes, each through an angle of $\arccos(\frac{-3}{5})$.

$$S_5^M = \{A,B,C,A^{-1},B^{-1},C^{-1},AA,AB,AC,AB^{-1},AC^{-1},...\}. \tag{3.1}$$

Using (2.4), S_5^M contains $\frac{3}{2}(5^M-1)$ elements of rotations.

Let $A \subseteq S^2$ be a spherical cap with center $y \in S^2$ and radius h. The area $|A| = 2\pi h$. Denote by χ_A the characteristic function of A. In order for the set $\{\gamma A\}_{\gamma \in S_5^M}$ to cover the whole unit sphere, one has to make sure that for every $x \in S^2$, there exists at least one spherical cap, say γA, where $\gamma \in S_5^M$, such that $x \in \gamma A$.

THEOREM 3.1. *Let $C_M = 5^{\frac{M}{2}}(M+1+\frac{M}{\sqrt{5}})$, and let $k = 16\frac{(4+\sqrt{\pi})}{\pi}$. Then, for every cap $A \subseteq S^2$ and for all $x \in S^2$ we have*

$$\left| |A| - \frac{1}{|S_5^M|} \sum_{\gamma \in S_5^M} \chi_{\gamma A}(x) \right| \leqslant \frac{3}{4^{\frac{1}{3}}}(4\pi)^{\frac{1}{3}} \left[\frac{C_M}{|S_5^M|} k \right]^{\frac{2}{3}}. \tag{3.2}$$

4. Main lemma

The Legendre polynomials $P_n(x)$ are defined as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n. \tag{4.1}$$

Also $P_n(1) = 1$ and

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{m,n}. \tag{4.2}$$

To prove Theorem 3.1, we will use the following lemma.

LEMMA 4.1. *For $0 \leqslant \theta \leqslant \pi$, we have the following inequality*

$$|P_{n-1}(\cos(\theta)) - P_{n+1}(\cos(\theta))| \leqslant \frac{8}{\sqrt{\pi}} \sqrt{\frac{|\sin(\theta)|}{n}}. \tag{4.3}$$

Proof. Recall a result of Stieltjes ([4], Theorem 7.33, page 165)

$$\sqrt{\sin(\theta)}\,|P_n(\cos(\theta)| \leqslant \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} \quad (0 \leqslant \theta \leqslant \pi). \tag{4.4}$$

Bernstein's theorem ([4], Theorem 1.22.1, page 5) states that for any trigonometric polynomial $g(\theta)$ of degree n, we have

$$\left|g'(\theta)\right| \leqslant n \, . \max_{0\leqslant\theta_1\leqslant 2\pi} |g(\theta_1)| \quad (0 \leqslant \theta \leqslant 2\pi). \tag{4.5}$$

Let $0 < \theta_0 \leqslant \frac{\pi}{2}$. We see from (4.4) that

$$|P_n(\cos(\theta)| \leqslant \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{\sin(\theta)}} \leqslant \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{\sin(\theta_0)}} \quad (\theta_0 \leqslant \theta \leqslant \frac{\pi}{2}). \tag{4.6}$$

Let $\cos(\theta) = \cos(\theta_0)\cos(u)$. Then by (4.6) for all real u

$$|P_n(\cos(\theta_0)\cos(u))| \leqslant \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{\sin(\theta_0)}}. \tag{4.7}$$

Hence, (4.5) and (4.7) imply

$$\left|\cos(\theta_0)\sin(u)P_n'(\cos(\theta_0)\cos(u))\right| \leqslant \sqrt{\frac{2}{\pi}}\frac{\sqrt{n}}{\sqrt{\sin(\theta_0)}}. \tag{4.8}$$

Equivalently, for $\theta_0 \leqslant \theta \leqslant \frac{\pi}{2}$,

$$\sqrt{\cos^2(\theta_0) - \cos^2(\theta)}. \left|P_n'(\cos(\theta))\right| \leqslant \sqrt{\frac{2}{\pi}}\frac{\sqrt{n}}{\sqrt{\sin(\theta_0)}}. \tag{4.9}$$

Given $0 < \gamma \leqslant \frac{\pi}{2}$ set $\theta_0 = \frac{\gamma}{2}$, $\theta = \gamma$. Then (4.9) implies

$$\left|P_n'(\cos(\gamma))\right| \leqslant \sqrt{\frac{2}{\pi}}\frac{\sqrt{n}}{\sqrt{\cos^2(\frac{\gamma}{2}) - \cos^2(\gamma)}}\frac{1}{\sqrt{\sin(\frac{\gamma}{2})}}. \tag{4.10}$$

But

$$\cos^2\left(\frac{\gamma}{2}\right) - \cos^2(\gamma) = \sin^2(\gamma) - \sin^2\left(\frac{\gamma}{2}\right) = \sin^2\left(\frac{\gamma}{2}\right)\left(4\cos^2\left(\frac{\gamma}{2}\right) - 1\right) \geqslant \sin^2\left(\frac{\gamma}{2}\right).$$

Hence, by (4.10),

$$\left|P_n'(\cos(\gamma))\right| \leqslant \sqrt{\frac{2}{\pi}}\frac{\sqrt{n}}{(\sin(\frac{\gamma}{2}))^{\frac{3}{2}}}. \tag{4.11}$$

From ([4], (1), page 360) we have

$$P_{n-1}(x) - P_{n+1}(x) = \frac{2n+1}{n(n+1)}(1-x^2)P_n'(x). \tag{4.12}$$

By combining (4.11) and (4.12) we get

$$\begin{aligned} |P_{n-1}(\cos(\gamma)) - P_{n+1}(\cos(\gamma))| &\leqslant \frac{2n+1}{n(n+1)}\sin^2(\gamma)\sqrt{\frac{2}{\pi}}\frac{\sqrt{n}}{(\sin(\frac{\gamma}{2}))^{\frac{3}{2}}} \\ &= \frac{2n+1}{n+1}\frac{1}{\sqrt{n}}\sqrt{\frac{2}{\pi}}\frac{\sin^2(\gamma)}{(\sin(\frac{\gamma}{2}))^{\frac{3}{2}}}. \end{aligned} \tag{4.13}$$

But

$$\sin(\gamma) = 2\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right).$$

So

$$\frac{1}{\sin(\frac{\gamma}{2})} = \frac{2\cos(\frac{\gamma}{2})}{\sin(\gamma)} \leqslant \frac{2}{\sin(\gamma)}.$$

Therefore

$$|P_{n-1}(\cos(\gamma)) - P_{n+1}(\cos(\gamma))| \leqslant 4\frac{2n+1}{n+1}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{\pi}}\sqrt{\sin(\gamma)}. \tag{4.14}$$

Consequently, for $0 \leqslant \gamma \leqslant \frac{\pi}{2}$,

$$|P_{n-1}(\cos(\gamma)) - P_{n+1}(\cos(\gamma))| \leqslant \frac{8}{\sqrt{\pi}}\sqrt{\frac{|\sin(\gamma)|}{n}}. \tag{4.15}$$

Hence, the lemma is proven. □

A proof of Lemma 4.1 can be found in the book of Szegö (see [5], Theorem 7.33.3, page 172.) However, the proof does not provide the constant explicitly as we did.

5. Proof of main theorem

Proof of Theorem 3.1. Here we follow Theorem 2.5 in [1] making it more precise at various points.

Let A_1, A_2 be two spherical caps about y with radii $h-2\varepsilon$, $h+2\varepsilon$, respectively. Therefore, $|A_1| = 2\pi(h-2\varepsilon)$, and $|A_2| = 2\pi(h+2\varepsilon)$. We have

$$||A_\nu| - |A|| = 2\pi\,|(h\pm 2\varepsilon) - h| = 4\pi\varepsilon, \quad (\nu = 1,\ 2). \tag{5.1}$$

For $\varepsilon > 0$ let $k_\varepsilon(z,\xi)$ be the point-pair invariant

$$k_\varepsilon(z,\xi) = \begin{cases} \dfrac{1}{2\pi(1-\cos(\varepsilon))} & \text{if } d(z,\xi) < \varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Define $k * f(z)$ by

$$k * f(z) = \int_{S^2} k(z,\xi) f(\xi)\, d\omega(\xi). \tag{5.3}$$

We get

$$\begin{aligned} \left|\frac{1}{|S_5^M|}\sum_{\gamma\in S_5^M}(\chi_{A_\nu} * k_\varepsilon)(\gamma x) - |A|\right| &\leqslant \left|\frac{1}{|S_5^M|}\sum_{\gamma\in S_5^M}(\chi_{A_\nu} * k_\varepsilon)(\gamma x) - |A_\nu|\right| + ||A_\nu| - |A|| \\ &= \left|\frac{1}{|S_5^M|}\sum_{\gamma\in S_5^M}(\chi_{A_\nu} * k_\varepsilon)(\gamma x) - |A_\nu|\right| + 4\pi\varepsilon. \end{aligned} \tag{5.4}$$

Let

$$I_\nu = \left| \sum_{\gamma \in S_5^M} (\chi_{A_\nu} * k_\varepsilon)(\gamma x) - \left|S_5^M\right| |A_\nu| \right|. \tag{5.5}$$

Then

$$\left| \frac{1}{\left|S_5^M\right|} \sum_{\gamma \in S_5^M} (\chi_{A_\nu} * k_\varepsilon)(\gamma x) - |A| \right| \leqslant \frac{1}{\left|S_5^M\right|} I_\nu + 4\pi\varepsilon. \tag{5.6}$$

Now, we need to estimate I_ν. By (2.9) in [1], we have

$$I_\nu = \left| \sum_{m=1}^{\infty} \hat{k}_\varepsilon(m) \hat{k}_{A_\nu}(m) \sum_{|j| \leqslant m} \varphi_{j,m}(y) \sum_{\gamma \in S_5^M} \varphi_{j,m}(\gamma x) \right| \tag{5.7}$$

where

$$\hat{k}_{A_\nu}(m) = 2\pi \int_{\cos\rho_\nu}^{1} P_m(x)dx \qquad (\rho_1 = h + 2\varepsilon,\ \rho_2 = h - 2\varepsilon), \tag{5.8}$$

and

$$\hat{k}_\varepsilon(m) = \frac{1}{1 - \cos\varepsilon} \int_{\cos\varepsilon}^{1} P_m(x)dx. \tag{5.9}$$

Furthermore, the $\varphi_{j,m}$ are simultaneous eigenvectors for the averaging operator T_S, and for the operator defined by k_ε.

Using (1.25) and (1.26) in [1], we have

$$\sum_{\gamma \in S_5^M} \varphi_{j,m}(\gamma x) = 5^{\frac{M}{2}} \left(\frac{\sin((M+1)\theta)}{\sin(\theta)} + \frac{\sin(M\theta)}{\sqrt{5}\ \sin(\theta)} \right) \varphi_{j,m}(x). \tag{5.10}$$

Thus we have

$$\begin{aligned}
\left| \sum_{\gamma \in S_5^M} \varphi_{j,m}(\gamma x) \right| &\leqslant 5^{\frac{M}{2}} \left(\frac{|\sin((M+1)\theta)|}{|\sin(\theta)|} + \frac{|\sin(M\theta)|}{\sqrt{5}|\sin(\theta)|} \right) \left|\varphi_{j,m}(x)\right| \\
&= 5^{\frac{M}{2}} \left(\frac{\left| e^{i(M+1)\theta} - e^{-i(M+1)\theta} \right|}{|e^{i\theta} - e^{-i\theta}|} + \frac{|e^{iM\theta} - e^{-iM\theta}|}{\sqrt{5}|e^{i\theta} - e^{-i\theta}|} \right) \left|\varphi_{j,m}(x)\right| \\
&= 5^{\frac{M}{2}} \left(\frac{\left| 1 - e^{-i2(M+1)\theta} \right|}{|1 - e^{-i2\theta}|} + \frac{|1 - e^{-i2M\theta}|}{\sqrt{5}|1 - e^{-i2\theta}|} \right) \left|\varphi_{j,m}(x)\right| \\
&= 5^{\frac{M}{2}} \left(\left| \sum_{k=0}^{M} e^{-i2\theta k} \right| + \frac{1}{\sqrt{5}} \left| \sum_{k=0}^{M-1} e^{-i2\theta k} \right| \right) \left|\varphi_{j,m}(x)\right| \\
&\leqslant 5^{\frac{M}{2}} \left(\sum_{k=0}^{M} \left|e^{-i2\theta k}\right| + \frac{1}{\sqrt{5}} \sum_{k=0}^{M-1} \left|e^{-i2\theta k}\right| \right) \left|\varphi_{j,m}(x)\right| \\
&= 5^{\frac{M}{2}} \left(M + 1 + \frac{M}{\sqrt{5}} \right) \left|\varphi_{j,m}(x)\right|.
\end{aligned} \tag{5.11}$$

Therefore, we have the following inequality

$$\left|\sum_{\gamma\in S_5^M}\varphi_{j,m}(\gamma x)\right|\leqslant C_M\left|\varphi_{j,m}(x)\right|. \tag{5.12}$$

Hence,

$$I_\nu\leqslant C_M\sum_{m=1}^{\infty}\left|\hat{k}_\varepsilon(m)\hat{k}_{A_\nu}(m)\right|\sum_{|j|\leqslant m}\left|\varphi_{j,m}(x)\varphi_{j,m}(y)\right|. \tag{5.13}$$

By the Cauchy-Schwartz inequality, we obtain

$$\sum_{|j|\leqslant m}\left|\varphi_{j,m}(x)\varphi_{j,m}(y)\right|\leqslant\left(\sum_{|j|\leqslant m}\left|\varphi_{j,m}(x)\right|^2\right)^{\frac{1}{2}}\left(\sum_{|j|\leqslant m}\left|\varphi_{j,m}(y)\right|^2\right)^{\frac{1}{2}}. \tag{5.14}$$

Furthermore, for all z in S^2 we have

$$\sum_{|j|\leqslant m}\left|\varphi_{j,m}(z)\right|^2=\frac{2m+1}{2\pi}. \tag{5.15}$$

Consequently,

$$I_\nu\leqslant C_M\sum_{m=1}^{\infty}\left|\hat{k}_\varepsilon(m)\hat{k}_{A_\nu}(m)\right|\frac{2m+1}{2\pi}. \tag{5.16}$$

The next step in the proof is to bound $\hat{k}_{A_\nu}(m)$ and $\hat{k}_\varepsilon(m)$. Recall that

$$(2m+1)P_m(x)=P'_{m+1}(x)-P'_{m-1}(x). \tag{5.17}$$

Using the above equation, $\hat{k}_{A_\nu}(m)$ and $\hat{k}_\varepsilon(m)$, defined in (5.8) and (5.9), can be rewritten as

$$\hat{k}_{A_\nu}(m)=\frac{2\pi}{2m+1}\left[P_{m-1}(\cos\rho_\nu)-P_{m+1}(\cos\rho_\nu)\right] \tag{5.18}$$

and

$$\hat{k}_\varepsilon(m)=\frac{1}{(2m+1)(1-\cos\varepsilon)}\left[P_{m-1}(\cos\varepsilon)-P_{m+1}(\cos\varepsilon)\right]. \tag{5.19}$$

To bound $\hat{k}_{A_\nu}(m)$ and $\hat{k}_\varepsilon(m)$ we use lemma 4.1. Setting $t=\frac{8}{\sqrt{\pi}}$, we have the following inequalities

$$\left|\hat{k}_{A_\nu}(m)\right|\leqslant\frac{2\pi}{2m+1}t\sqrt{\frac{|\sin(\rho_\nu)|}{m}}\leqslant\frac{2\pi t}{(2m+1)\sqrt{m}}. \tag{5.20}$$

Similarly

$$\left|\hat{k}_\varepsilon(m)\right|\leqslant\frac{1}{(2m+1)(1-\cos(\varepsilon))}t\frac{\sqrt{\sin(\varepsilon)}}{\sqrt{m}}. \tag{5.21}$$

Also, based on (5.9), and using the fact that for $|x| \leqslant 1$ we have $|P_n(x)| \leqslant 1$ (see [6], Theorem 60) we have $\left|\hat{k}_\varepsilon(m)\right| \leqslant 1$ for all m, to conclude that

$$\begin{aligned}
I_\nu &\leqslant C_M \sum_{1\leqslant m\leqslant \frac{1}{\varepsilon}} \left|\hat{k}_\varepsilon(m)\hat{k}_{A_\nu}(m)\right| \frac{2m+1}{2\pi} + C_M \sum_{m>\frac{1}{\varepsilon}} \left|\hat{k}_\varepsilon(m)\hat{k}_{A_\nu}(m)\right| \frac{2m+1}{2\pi} \\
&\leqslant C_M \left[\sum_{1\leqslant m\leqslant \frac{1}{\varepsilon}} \frac{2\pi t}{(2m+1)\sqrt{m}} \frac{2m+1}{2\pi} + \sum_{m>\frac{1}{\varepsilon}} \frac{t\sqrt{\sin(\varepsilon)}}{(2m+1)(1-\cos(\varepsilon))\sqrt{m}} \frac{2\pi t}{(2m+1)\sqrt{m}} \frac{2m+1}{2\pi} \right] \\
&= C_M \left[\sum_{1\leqslant m\leqslant \frac{1}{\varepsilon}} \frac{t}{\sqrt{m}} + \sum_{m>\frac{1}{\varepsilon}} \frac{t^2\sqrt{\sin(\varepsilon)}}{(1-\cos(\varepsilon))} \frac{1}{(2m+1)m} \right] \\
&= C_M t \left[\sum_{1\leqslant m\leqslant \frac{1}{\varepsilon}} \frac{1}{\sqrt{m}} + t \frac{\sqrt{\sin(\varepsilon)}}{(1-\cos(\varepsilon))} \sum_{m>\frac{1}{\varepsilon}} \frac{1}{(2m+1)m} \right].
\end{aligned} \tag{5.22}$$

Notice that for the first part of the sum we have

$$\sum_{1\leqslant m\leqslant \frac{1}{\varepsilon}} \frac{1}{\sqrt{m}} \leqslant \int_0^{\frac{1}{\varepsilon}} \frac{1}{\sqrt{x}} dx = 2\frac{1}{\sqrt{\varepsilon}}. \tag{5.23}$$

Furthermore, for the second part we need the following lemma

LEMMA 5.1.

$$\max_{0\leqslant x\leqslant \pi} \frac{x^{\frac{3}{2}}\sqrt{\sin x}}{2(1-\cos x)} = 1. \tag{5.24}$$

Proof.

$$\begin{aligned}
\frac{x^{\frac{3}{2}}\sqrt{\sin x}}{2(1-\cos x)} &= \frac{x^{\frac{3}{2}}\sqrt{2\sin\frac{x}{2}\cos\frac{x}{2}}}{2(2\sin^2\frac{x}{2})} \\
&= \frac{x^{\frac{3}{2}}\sqrt{\cos\frac{x}{2}}}{2\sqrt{2}(\sin\frac{x}{2})^{\frac{3}{2}}} \\
&= \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^{\frac{3}{2}} \sqrt{\cos\frac{x}{2}}
\end{aligned}$$

In [7], it was proved that the inequality

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \tag{5.25}$$

is valid for $x \in (0, \frac{\pi}{2}]$, and the exponent 3 is the best possible. Using (5.25), we have

$$0 \leqslant \left(\frac{x}{\sin x}\right)^3 \cos x < 1, \quad 0 < x \leqslant \frac{\pi}{2}. \tag{5.26}$$

Therefore,

$$\frac{x^{\frac{3}{2}}\sqrt{\sin x}}{2(1-\cos x)} < 1, \quad 0 < x \leqslant \pi.$$

Moreover,

$$\lim_{x\to 0}\frac{x^{\frac{3}{2}}\sqrt{\sin x}}{2(1-\cos x)} = \lim_{x\to 0}\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^{\frac{3}{2}}\sqrt{\cos\frac{x}{2}} = 1.$$

Consequently, the lemma follows. □

It follows then that from the previous lemma that

$$\frac{\sqrt{\sin(\varepsilon)}}{(1-\cos(\varepsilon))} \leqslant 2\frac{1}{\varepsilon^{\frac{3}{2}}}. \tag{5.27}$$

Moreover,

$$\sum_{m>\frac{1}{\varepsilon}}\frac{1}{(2m+1)m} \leqslant \frac{1}{2}\sum_{m>\frac{1}{\varepsilon}}\frac{1}{m^2} \leqslant \frac{1}{2}\int_{\frac{1}{\varepsilon}}^{\infty}\frac{1}{x^2}dx = \frac{\varepsilon}{2}. \tag{5.28}$$

Hence,

$$I_v \leqslant C_M t\left[2\frac{1}{\sqrt{\varepsilon}} + 2t\frac{1}{\varepsilon^{\frac{3}{2}}}\frac{\varepsilon}{2}\right] = C_M t(2+t)\frac{1}{\sqrt{\varepsilon}}. \tag{5.29}$$

By combining the previous formula with (5.6) we get

$$\left|\frac{1}{|S_5^M|}\sum_{\gamma\in S_5^M}(\chi_{A_v} * k_\varepsilon)(\gamma x) - |A|\right| \leqslant \frac{C_M}{|S_5^M|}t(2+t)\frac{1}{\sqrt{\varepsilon}} + 4\pi\varepsilon. \tag{5.30}$$

Set $k = t(2+t) = 16\frac{(4+\sqrt{\pi})}{\pi}$. It easy to verify that, for any $x_1,\ldots,x_n \in S^2$, we have

$$\sum_{r=1}^{n}(\chi_{A_1} * k_\varepsilon)(x_r) \leqslant \sum_{r=1}^{n}\chi_A(x_r) \leqslant \sum_{r=1}^{n}(\chi_{A_2} * k_\varepsilon)(x_r). \tag{5.31}$$

In conclusion,

$$\left||A| - \frac{1}{|S_5^M|}\sum_{\gamma\in S_5^M}\chi_A(\gamma x)\right| \leqslant \frac{C_M}{|S_5^M|}k\frac{1}{\sqrt{\varepsilon}} + 4\pi\varepsilon. \tag{5.32}$$

The previous inequality is valid for every ε, such that $0 < \varepsilon < 1$, so the following inequality still holds

$$\left||A| - \frac{1}{|S_5^M|}\sum_{\gamma\in S_5^M}\chi_A(\gamma x)\right| \leqslant \min_{0<\varepsilon<1}\left(\frac{C_M}{|S_5^M|}k\frac{1}{\sqrt{\varepsilon}} + 4\pi\varepsilon\right). \tag{5.33}$$

LEMMA 5.2.

$$\min_{0<\varepsilon<1}\left(\frac{C_M}{\left|S_5^M\right|}k\frac{1}{\sqrt{\varepsilon}}+4\pi\varepsilon\right)=\frac{3}{4^{\frac{1}{3}}}(4\pi)^{\frac{1}{3}}\left[\frac{C_M}{\left|S_5^M\right|}k\right]^{\frac{2}{3}}. \tag{5.34}$$

Proof. Let $J(\varepsilon)=k_1\varepsilon+\frac{k_2}{\sqrt{\varepsilon}}$, where $\varepsilon>0$. The minimum of $J(\varepsilon)$ occurs at $\varepsilon_0=(\frac{k_2}{2k_1})^{\frac{2}{3}}$, and $\min J(\varepsilon)=J(\varepsilon_0)=\frac{3}{4^{\frac{1}{3}}}k_1^{\frac{1}{3}}k_2^{\frac{2}{3}}$. In our case, $k_1=4\pi$ and $k_2=\frac{C_M}{\left|S_5^M\right|}k$. Therefore, the lemma follows. □

Combining (5.33) and Lemma 5.2, Theorem 3.1 is proven. □

6. Necessary condition for sphere covering

Further, in order to use Theorem 3.1 to guarantee the covering of the sphere we need the next lemma.

LEMMA 6.1.

$$\left||A|-\frac{1}{\left|S_5^M\right|}\sum_{\gamma\in S_5^M}\chi_{\gamma A}(x)\right|<|A|\Rightarrow\bigcup_{\gamma\in S_5^M}\gamma A=S^2. \tag{6.1}$$

Proof. As $\gamma A\subseteq S^2$, then $\bigcup_{\gamma\in S_5^M}\gamma A\subseteq S^2$. Now, let $x\in S^2$ we have obviously

$$\left||A|-\frac{1}{\left|S_5^M\right|}\sum_{\gamma\in S_5^M}\chi_{\gamma A}(x)\right|<|A|\Rightarrow\sum_{\gamma\in S_5^M}\chi_{\gamma A}(x)\neq 0. \tag{6.2}$$

So, we can see that there exists a $\gamma_0\in S_5^M$ such that $\chi_{\gamma_0 A}(x)=1$, which is equivalent to say that $x\in\gamma_0 A$. As $\gamma_0 A\subseteq\bigcup_{\gamma\in S_5^M}\gamma A$ the lemma follows. □

Based on this fact and on Theorem 3.1, we reach a covering if

$$\frac{3}{4^{\frac{1}{3}}}(4\pi)^{\frac{1}{3}}\left[\frac{5^{\frac{M}{2}}(M+1+\frac{M}{\sqrt{5}})}{\left|S_5^M\right|}16\left(\frac{4+\sqrt{\pi}}{\pi}\right)\right]^{\frac{2}{3}}<2\pi h. \tag{6.3}$$

As $\left|S_5^M\right|=\frac{3}{2}(5^M-1)$, then inequality (6.3) can be simplified as follows

$$\frac{5^{\frac{M}{2}}(M+1+\frac{M}{\sqrt{5}})}{5^M-1}<\frac{1}{16}\sqrt{\frac{2}{3}}\frac{\pi^2}{4+\sqrt{\pi}}h^{\frac{3}{2}}. \tag{6.4}$$

Thus we have the following proposition.

PROPOSITION 6.2. *If M satisfies the following inequality*

$$\frac{5^{\frac{M}{2}}(M+1+\frac{M}{\sqrt{5}})}{5^M-1} < \frac{1}{16}\sqrt{\frac{2}{3}}\frac{\pi^2}{4+\sqrt{\pi}}h^{\frac{3}{2}} \tag{6.5}$$

then the sphere covering is guaranteed

$$\bigcup_{\gamma\in S_5^M} \gamma A = S^2. \tag{6.6}$$

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