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ITERATIVE APPROXIMATIONS FOR A FAMILY OF MULTIVALUED MAPPINGS IN BANACH SPACES

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Abstract. In this paper we consider the convergence of iterative processes for a family of multivalued nonexpansive mappings. Under somewhat different conditions the sequences of Noor, Mann and Ishikawa iterates converge to the common fixed point of the family of multivalued nonexpansive mappings.

1. Introduction

Let X be a Banach space and K a nonempty subset of X. Let 2^X denote the family of all subsets of X, $CB(X)$ the family of all nonempty closed bounded subsets of X and $C(X)$ the family of nonempty compact subsets of X. A multivalued mapping $T : K \to 2^X$ is said to be nonexpansive (resp, contractive) if

$$H(Tx,Ty) \leqslant \|x-y\|, \ \ x,y \in K,$$

$$(\text{resp. } H(Tx,Ty) \leqslant k\|x-y\|, \text{ for some } k \in (0,1)).$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B) := \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \ \sup_{y\in B}\inf_{x\in A}\|x-y\|\}, \ \ A,B \in CB(X).$$

A point x is called a fixed point of T if $x \in Tx$. Since Banach's Contraction Mapping Principle was extended nicely to multivalued mappings by Nadler in 1969 (see [8]), many authors have studied the fixed point theory for multivalued mappings (e.g. see[1, 4, 5, 6, 16, 21,]). For a single-valued nonexpansive mapping T, Mann [7] and Ishikawa [3] respectively introduced new iteration procedure for approximating its fixed point in a Banach space as follows:

$$x_{n+1} = (1-t_n)Tx_n + t_nx_n \tag{1}$$

and

$$x_{n+1} = (1-t_n)Ty_n + t_nx_n, y_n = (1-s_n)Tx_n + s_nx_n, \tag{2}$$

where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0,1]$. Obviously, Mann iteration is a special case of Ishikawa iteration. Subsequently, Mann iteration and Ishikawa iteration have extensively been studied for constructions of fixed points of nonlinear mappings and

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of solutions of nonlinear operator equations involving monotone, accretive and pseudocontractive operators. It is a very natural question whether the strongly convergent results of $\{x_n\}$ defined by (1) or (2) for a single-valued nonexpansive mapping T can be extended to the multivalued case.

In this paper we consider the following iteration for a family of multivalued nonexpansive mapping $\{T_n\}$. Let K be a nonempty closed convex subset of Banach space X and $T_n : K \to CB(K)$ be a family of multivalued nonexpansive mappings. For a given $x_1 \in K$ and $s_1 \in T_1x_1$ let

$$z_1 = (1-a_1)x_1 + a_1 s_1.$$

There exists $t_1 \in T_1 z_1$ such that $\|t_1 - s_1\| \leqslant H(T_1z_1, T_1x_1)$. Let

$$y_1 = (1-b_1-c_1)x_1 + b_1t_1 + c_1s_1.$$

There exists $r_1 \in T_1y_1$ such that $\|r_1 - t_1\| \leqslant H(T_1y_1, T_1z_1)$ and $\|r_1 - s_1\| \leqslant H(T_1y_1, T_1x_1)$. Let

$$x_2 = (1-\alpha_1-\beta_1-\gamma_1)x_1 + \alpha_1 r_1 + \beta_1 t_1 + \gamma_1 s_1.$$

There exists $s_2 \in T_2x_2$ such that $\|s_2 - r_1\| \leqslant H(T_2x_2, T_1y_1)$, $\|s_2 - t_1\| \leqslant H(T_2x_2, T_1z_1)$ and $\|s_2 - s_1\| \leqslant H(T_2x_2, T_1x_1)$. Inductively, we can get the sequence $\{x_n\}$ as follows:

$$\begin{aligned} z_n &= (1-a_n)x_n + a_n s_n \\ y_n &= (1-b_n-c_n)x_n + b_n t_n + c_n s_n \\ x_{n+1} &= (1-\alpha_n-\beta_n-\gamma_n)x_n + \alpha_n r_n + \beta_n t_n + \gamma_n s_n, \end{aligned} \tag{3}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{b_n+c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n+\beta_n+\gamma_n\}$ are appropriate sequence in $[0,1]$, furthermore $s_n \in T_nx_n, t_n \in T_nz_n, r_n \in T_ny_n$ such that $\|t_n - s_n\| \leqslant H(T_nz_n, T_nx_n)$, $\|r_n - t_n\| \leqslant H(T_ny_n, T_nz_n)$, $\|r_n - s_n\| \leqslant H(T_ny_n, T_nx_n)$, $\|s_{n+1} - r_n\| \leqslant H(T_{n+1}x_{n+1}, T_ny_n)$, $\|s_{n+1} - t_n\| \leqslant H(T_{n+1}x_{n+1}, T_nz_n)$ and $\|s_{n+1} - s_n\| \leqslant H(T_{n+1}x_{n+1}, T_nx_n)$. The iterative scheme (3) is called Noor multivalued iterative scheme. If $a_n = c_n = \beta_n = \gamma_n \equiv 0$ or let $a_n = b_n = c_n = \beta_n = \gamma_n \equiv 0$, we get the algorithms in [22]

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n r_n, \qquad y_n = (1-b_n)x_n + b_n t_n \ \ \forall n \in N, \tag{4}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n r_n, \tag{5}$$

We call the iteration (4) and (5) is Ishikawa iteration and Mann iteration for a family of multivalued nonexpansive mappings. In fact let $\gamma_n \equiv 0$ or $c_n = \beta_n = \gamma_n \equiv 0$ or $b_n = c_n = \alpha_n = \gamma_n \equiv 0$, we also have the other three algorithms.

DEFINITION 1.1. A family of multivalued mappings $T_n : K \to CB(K)$ is said to satisfy Condition (A') if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(x) > 0$ for $x \in (0,\infty)$ such that

$$d(x, T_nx) \geqslant f(d(x, F(\cap_n T_n)) \quad \text{for all } x \in K,.$$

where $F(\cap_n T_n) \neq \emptyset$ is the common fixed point set of the family of multivalued mappings $\{T_n\}$. From now on, $F(\cap_n T_n)$ stands for the common fixed point set of the family of multivalued mappings $\{T_n\}$.

2. Preliminaries

A Banach space X is said to be satisfy Opial's condition [14] if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x (n \to \infty)$ implies the following inequality:

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known that Hilbert spaces and l_p $(1 < p < \infty)$ have the Opial's condition. The following Lemmas will be useful in this paper.

LEMMA 2.1. *Let $\{b_n\}$, $\{\alpha_n\}$ be two real sequences such that*

(i) $b_n, \alpha_n \in [0,1]$;

(ii) $\lim_{n\to\infty} b_n \to 0$;

(iii) $\sum_{n=1}^{\infty} b_n \alpha_n = \infty$.

Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum_{n=1}^{\infty} b_n \alpha_n (1-b_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.

LEMMA 2.2. (see [20]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad n \geqslant 0$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$$

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

LEMMA 2.3. (see [10]) *Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequence in uniformly convex Banach space X. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\limsup_n \|x_n\| \leqslant d, \limsup_n \|y_n\| \leqslant d, \limsup_n \|z_n\| \leqslant d$, and $\lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = d$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.*

LEMMA 2.4. (see [9]) *Let X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leqslant r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\begin{aligned}\|\lambda x + \mu y + \xi z + \vartheta \omega\|^2 &\leqslant \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \vartheta \|\omega\|^2 \\ &\quad - \frac{1}{3}\vartheta(\lambda g(\|x-\omega\|) + \mu g(\|y-\omega\|) + \xi g(\|z-\omega\|))\end{aligned}$$

for all $x, y, z, \omega \in B_r$ and $\lambda, \mu, \xi, \vartheta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

3. Main results

THEOREM 3.1. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Suppose that $T_n : K \to CB(K)$ be a family of multivalued nonexpansive mappings and $F(\cap_n T_n) \neq \emptyset$ satisfying $T_n(p) = \{p\}$ for any fixed point $p \in F(\cap_n T_n)$ and $\{x_n\}$ be the sequence of Ishikawa iterates defined by (4). Assume that*

(i) $\alpha_n, b_n \in [0,1]$;

(ii) $\lim_{n\to\infty} b_n \to 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n b_n = \infty$.

Then, as $n \to \infty$, the sequence $\{x_n\}$ strongly converges to a common fixed point of T_n.

Proof. Take $p \in F(\cap_n T_n)$, noting that $T_n(p) = \{p\}$ for any fixed point $p \in F(\cap_n T_n)$, then $\|u_n - p\| = d(u_n, T_n p)$. Using Lemma 2.4, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|r_n-p\|^2-\frac{1}{3}\alpha_n(1-\alpha_n)g(\|x_n-r_n\|)\\
&\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n(H(T_ny_n,T_np))^2\\
&\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|y_n-p\|^2\\
&\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n[(1-b_n)\|x_n-p\|^2+b_n\|t_n-p\|^2\\
&\quad -\frac{1}{3}b_n(1-b_n)g(\|x_n-t_n\|)]\\
&\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n[(1-b_n)\|x_n-p\|^2+b_n(H(T_nx_n,T_np))^2\\
&\quad -\frac{1}{3}b_n(1-b_n)g(\|x_n-t_n\|)]\\
&\leqslant \|x_n-p\|^2-\frac{1}{3}\alpha_nb_n(1-b_n)g(\|x_n-t_n\|).
\end{aligned}$$

Therefore,

$$\|x_{n+1}-p\|^2 \leqslant \|x_n-p\|^2,\;\; \frac{1}{3}\alpha_nb_n(1-b_n)g(\|x_n-t_n\|) \leqslant \|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Then $\{\|x_n-p\|\}$ is a decreasing sequence and further $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p \in F(\cap_n T_n)$. It follows from that we have

$$\frac{1}{3}\sum_{n=0}^{\infty}\alpha_nb_n(1-b_n)g(\|x_n-t_n\|) \leqslant \|x_1-p\|^2.$$

From Lemma 2.1, there exists a subsequence $\{x_{n_k}-t_{n_k}\}$ of $\{x_n-t_n\}$ such that $g(\|x_{n_k}-t_{n_k}\|) \to 0$ as $k \to \infty$, therefore we get $\|x_{n_k}-t_{n_k}\| \to 0$, by the continuity and strictly increasing nature of g. By the compactness of K, we may assume that $x_{n_k} \to q$ for some $q \in K$. Thus for any $n \in \mathbb{N}$,

$$\begin{aligned}
d(q,T_nq) &\leqslant \|q-x_{n_k}\|+d(x_{n_k},T_nx_{n_k})+H(T_nx_{n_k},T_nq)\\
&\leqslant \|q-x_{n_k}\|+\|x_{n_k}-t_{n_k}\|+\|x_{n_k}-q\| \to 0.
\end{aligned}$$

Hence q is a common fixed point of $\{T_n\}$. Now we can take q in place of p, we get that $\{\|x_n - q\|\}$ is a decreasing sequence, Since $\|x_{n_k} - q\| \to 0$ as $k \to \infty$, it follows that $\{\|x_n - q\|\} \to 0$, so the desired conclusion follows.

THEOREM 3.2. *Let X be a Banach space which satifies Opial's condition and K be a nonempty weakly compact convex subset of X. T_n, $\{x_n\}$ and the condition be the same as Theorem 3.1, furthermore*

(i) $b_n, \alpha_n \in [0,1]$;

(ii) $\lim_{n\to\infty} b_n \to 0$;

(iii) $\sum_{n=1}^{\infty} b_n \alpha_n = \infty$.

Then, as $n \to \infty$, the sequence $\{x_n\}$ weakly converges to a common fixed point of T_n.

Proof. From Theorem 3.1, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \rightharpoonup p \in F(\cap_n T_n) \text{ as } n_k \to \infty.$$

(Here $\rightharpoonup$ denotes the weak convergence.) Suppose that x_n is not weakly convergent to $p \in F(\cap_n T_n)$, then there exist a subsequence $\{x_{n_i}\} \subset \{x_n\}$ $(i \neq k)$, such that $\{x_{n_i}\} \rightharpoonup q \in F(\cap_n T_n)$, and $p \neq q$. Since X satisfies Opial's condition, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| \\ &< \lim_{k\to\infty} \|x_{n_k} - q\| = \lim_{i\to\infty} \|x_{n_i} - q\| \\ &< \lim_{i\to\infty} \|x_{n_k} - p\| = \lim_{n\to\infty} \|x_n - p\|. \end{aligned}$$

This is a contraction, so the desired conclusion follows.

THEOREM 3.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Suppose that T_n, $\{x_n\}$ be the same as in Theorem 3.1 and $b_n, \alpha_n \in [a,b] \subset (0,1)$. Then, as $n \to \infty$, the sequence $\{x_n\}$ strongly converges to a common fixed point of T_n.*

Proof. Using a similar proof of Theorem 3.1, we obtain that $\lim_{n\to\infty} \|x_n - p\|^2$ exists for $p \in F(\cap_n T_n)$ and

$$\frac{1}{3}\alpha_n b_n (1 - b_n) \varphi(\|x_n - t_n\|) \leqslant \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Then we have

$$\frac{1}{3}a^2 (1 - b) \varphi(\|x_n - t_n\|) \leqslant \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Therefore $\lim_{n\to\infty} \varphi(\|x_n - t_n\|) = 0$ and $\lim_{n\to\infty} \|x_n - t_n\| = 0$. Since $t_n \in T_n x_n$, then $d(x_n, T_n x_n) \leqslant \|x_n - t_n\|$. Therefore, $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$, and condition A' implies $\lim_{n\to\infty} d(x_n, F(\cap_n T_n)) = 0$. Since $\lim_{n\to\infty} d(x_n, F(\cap_n T_n)) = 0$, given $\varepsilon > 0$, there exist

$N_\varepsilon > 0$ and $z_\varepsilon \in F(\cap_n T_n)$ such that $\|x_n - z_\varepsilon\| < \varepsilon$ for all $n \geqslant N_\varepsilon$. Take $\varepsilon_k = \frac{1}{2^k}$ for $k \in \mathbb{N}$ then corresponding to each ε_k there is an $N_k > 0$ and a $z_k \in F(\cap_n T_n)$ such that

$$\|x_n - z_k\| \leqslant \frac{\varepsilon_k}{4} \quad \text{for all } n \geqslant N_k.$$

When $N_{k+1} \geqslant N_k$ for all $k \in \mathbb{N}$,

$$\|z_k - z_{k+1}\| = \|z_k - x_{N_{k+1}} + x_{N_{k+1}} - z_{k+1}\| < \frac{\varepsilon_k}{4} + \frac{\varepsilon_{k+1}}{4} = \frac{3\varepsilon_{k+1}}{4}.$$

Let $S(z,r) = \{x \in X : \|x - z\| \leqslant r\}$. For $x \in S(z_{k+1}, \varepsilon_{k+1})$ we have

$$\|z_k - x\| = \|z_k - z_{k+1} + z_{k+1} - x\| < \frac{3\varepsilon_{k+1}}{4} + \varepsilon_{k+1} < 2\varepsilon_{k+1} = \varepsilon_k.$$

This implies $S(z_{k+1}, \varepsilon_{k+1}) \subset S(z_k, \varepsilon_k)$ for $k \in \mathbb{N}$. By the Cantor intersection theorem, there exist a single point p such that

$$\cap_{k=1}^{\infty} S(z_k, \varepsilon_k) = \{p\},$$

then $\|z_k - p\| \leqslant \varepsilon_k \to 0$ as $k \to \infty$ which assure $\lim_{n\to\infty} \|x_n - p\| = 0$ since $\lim_{k\to\infty} N_k = \infty$ implies $n \to \infty$. For any $n \in \mathbb{N}$ and $x \in T_n p$, noting $T_n z_k = \{z_k\}$, we get

$$\begin{aligned} \|p - x\| &\leqslant \|p - z_k\| + d(x, T_n z_k) \leqslant \|p - z_k\| + H(T_n p, T_n z_k) \\ &\leqslant 2\|p - z_k\| \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Then p is a common fixed point of T_n and $\{x_n\}$ strongly converges to p.

REMARK 3.4. The above results holds for Mann iteration (5).

THEOREM 3.5. *Let K be a nonempty compact convex subset of a Banach space X. Suppose that $T_n : K \to CB(K)$ be a family of multivalued mappings satisfying $H(T_i x, T_j y) \leqslant \|x - y\|$ for any $i, j \in \mathbb{N}$. Let $\{x_n\}$ be the sequence of Mann iterates defined by (5). Assume that $F(\cap_n T_n) \neq \emptyset$ and $\forall n \in \mathbb{N}$, $T_n(p) = \{p\}$ for any fixed point $p \in F(\cap_n T_n)$. Assume that*

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1.$$

Then, as $n \to \infty$, the sequence $\{x_n\}$ strongly converges to a common fixed point of T_n.

Proof. Take $p \in F(\cap_n T_n)$, noting that $T_n p = \{p\}$ and $\|r_n - p\| = d(r_n, T_n p)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant (1 - \alpha_n)\|x_n - p\| + \alpha_n \|r_n - p\| \\ &\leqslant (1 - \alpha_n)\|x_n - p\| + \alpha_n (H(T_n x_n, T_n p)) \\ &\leqslant \|x_n - p\|. \end{aligned}$$

Then $\{\|x_n - p\|\}$ is a decreasing sequence and $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(\cap_n T_n)$. It follows from the definition of Mann iteration (5) that

$$\|r_{n+1} - r_n\| \leqslant H(T_{n+1}x_{n+1}, T_n x_n) \leqslant \|x_{n+1} - x_n\|.$$

Therefore we get $\limsup_{n\to\infty}(\|r_{n+1} - r_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. By Lemma 2.2, we obtain $\lim_{n\to\infty}\|r_n - x_n\| = 0$. Since $r_n \in T_n x_n$, then $d(x_n, T_n x_n) \leqslant \|x_n - r_n\|$, which assure that $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$. The remainder of the proof is the same as Theorem 3.1.

THEOREM 3.6. *Let K be a nonempty closed convex subset of Banach space X. Suppose that $T_n : K \to CB(K)$ be a family of multivalued nonexpansive mappings satisfying Condition A' and for any $i, j \in \mathbb{N}$ $H(T_i x, T_j y) \leqslant \|x - y\|$. T_n, $\{x_n\}$ and condition be the same as Theorem 3.5. Assume that*

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1.$$

Then as $n \to \infty$, the sequence $\{x_n\}$ strongly converges to common fixed point of T_n.

Proof. It follows from the proof of Theorem 3.5 that $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(\cap_n T_n)$ and $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$. Since T_n satisfying the Condition A', then we have $\lim_{n\to\infty} d(x_n, F(\cap_n T_n)) = 0$. The remainder of the proof is the same as Theorem 3.3.

THEOREM 3.7. *Let X be a Banach space satisfying Opial's condition and K be a nonempty weakly compact convex subset of X. Suppose that $T_n : K \to C(K)$ be a family of multivalued nonexpansive mappings that satisfies for any $i, j \in \mathbb{N}$ $H(T_i x, T_j y) \leqslant \|x - y\|$. T_n, $\{x_n\}$ and condition be the same as Theorem 3.5. Assume that*

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1.$$

Then as $n \to \infty$, the sequence $\{x_n\}$ weakly converges to a common fixed point of T_n.

Proof. From the proof of Theorem 3.5 that $\lim\limits_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(\cap_n T_n)$. Since K is weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ for some $x^* \in K$. Suppose there exists $n \in \mathbb{N}$ and x^* does not belong to $T_n x^*$. By the compactness of $T_n x^*$, for any given x_{n_k}, there is $p_k \in T_n x^*$ such that $\|x_{n_k} - p_k\| = d(x_{n_k}, T_n x^*)$ and $p_k \to p \in T_n x^*$ then $x^* \neq p$. Since X satisfies Opial's condition, then we get

$$\begin{aligned}
\limsup_{k\to\infty}\|x_{n_k} - p\| &\leqslant \limsup_{k\to\infty}[\|x_{n_k} - p_k\| + \|p_k - p\|] = \limsup_{k\to\infty}\|x_{n_k} - p_k\| \\
&\leqslant \limsup_{k\to\infty}[d(x_{n_k}, T_n x_{n_k}) + H(T_n x_{n_k}, T_n x^*)] \\
&\leqslant \limsup_{k\to\infty}\|x_{n_k} - x^*\| < \limsup_{k\to\infty}\|x_{n_k} - p\|.
\end{aligned}$$

This is a contraction. Hence $x^* \in T_n x^*$ for any $n \in \mathbb{N}$. The remainder of the proof is the same as Theorem 3.2.

LEMMA 3.8. *Let X be a real Banach space and K be a nonempty convex subset of X. Let $T_n : K \to CB(K)$ be a multivalued nonexpansive mapping for which $F(\cap_n T_n) \neq \emptyset$ and $T_n(p) = \{p\}$ for any fixed point $p \in F(\cap_n T_n)$. Let $\{x_n\}$ be a sequence in K defined by (3), then we have the following conclusions:*

$$\lim_n \|x_n - p\| \text{ exists for any } p \in F(\cap_n T_n)$$

Proof. . Let $p \in F(\cap_n T_n)$, from iterative scheme (3), note that $T_n(p) = \{p\}$ for any fixed point $p \in F(\cap_n T_n)$, we have

$$\begin{aligned}\|z_n - p\| &\leqslant (1-a_n)\|x_n - p\| + a_n\|s_n - p\| \\ &= (1-a_n)\|x_n - p\| + a_n d(s_n, T_n p) \\ &\leqslant (1-a_n)\|x_n - p\| + a_n H(T_n x_n, T_n p) \\ &\leqslant \|x_n - p\|,\end{aligned}$$

similarly $\|y_n - p\| \leqslant \|x_n - p\|$, and so we have

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| + \alpha_n\|r_n - p\| \\ &\quad +\beta_n\|t_n - p\| + \gamma_n\|s_n - p\| \\ &\leqslant (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| + \alpha_n d(T_n y_n, T_n p) \\ &\quad +\beta_n H(T_n z_n, T_n p) + \gamma_n H(T_n x_n, T_n p) \\ &\leqslant (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\ &\quad +\beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\ &\leqslant \|x_n - p\|.\end{aligned}$$

Then $\{\|x_n - p\|\}$ is a decreasing sequence and hence $\lim_n \|x_n - p\|$ exists for any $p \in F(\cap_n T_n)$.

LEMMA 3.9. *Let X be a uniformly convex Banach space and K be a nonempty convex subset of X. Let $T_n : K \to CB(K)$ be a multivalued nonexpansive mapping for which $F(\cap_n T_n) \neq \emptyset$ and $T_n(p) = \{p\}$ for any fixed point $p \in F(\cap_n T_n)$. Let $\{x_n\}$ be a sequence in K defined by (3). If the coefficient satisfy one of the following control conditions:*

(i) $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and* $\limsup_n a_n < 1$;

(ii) $0 < \liminf_n \gamma_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$;

(iii) $0 < \liminf_n(\alpha_n b_n + \beta_n)$ *and* $0 < \liminf_n a_n \leqslant \limsup_n a_n < 1$,

then we have $\lim_n d(x_n, T_n x_n) = 0$.

Proof. By Lemma 3.8, it is well known that $\lim_n \|x_n - p\|$ exists for any $p \in F(\cap_n T_n)$, then it follows that $\{s_n - p\}, \{t_n - p\}, and \{r_n - p\}$ are all bounded. We may assume that these sequences belong to B_r where $r>0$. Note that $T_n(p)=\{p\}$ for any fixed point $p\in F(\cap_n T_n)$. By Lemma 2.4, we get

$$\begin{aligned}
\|z_n-p\|^2 &\leqslant (1-a_n)\|x_n-p\|^2+a_n\|s_n-p\|^2\\
&= (1-a_n)\|x_n-p\|^2+a_n d(s_n,T_np)^2\\
&\leqslant (1-a_n)\|x_n-p\|^2+a_n H(T_nx_n,T_np)^2\\
&\leqslant \|x_n-p\|^2,\\
\|y_n-p\|^2 &\leqslant (1-b_n-c_n)\|x_n-p\|^2+b_n\|t_n-p\|^2+c_n\|s_n-p\|^2\\
&\quad -\frac{1}{3}(1-b_n-c_n)(b_ng(\|t_n-x_n\|)+c_ng(\|s_n-x_n\|)\\
&\leqslant (1-b_n-c_n)\|x_n-p\|^2+b_nH(T_nz_n,T_np)^2+c_nH(T_nx_n,T_np)^2\\
&\quad -\frac{1}{3}(1-b_n-c_n)b_ng(\|t_n-x_n\|)\\
&\leqslant \|x_n-p\|^2-\frac{1}{3}(1-b_n-c_n)b_ng(\|t_n-x_n\|),
\end{aligned}$$

therefore we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leqslant (1-\alpha_n-\beta_n-\gamma_n)\|x_n-p\|^2+\alpha_n\|r_n-p\|^2+\beta_n\|t_n-p\|^2\\
&\quad +\gamma_n\|s_n-p\|^2-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)[\alpha_ng(\|x_n-r_n\|)\\
&\quad +\beta_ng(\|x_n-t_n\|)+\gamma_ng(\|x_n-s_n\|)]\\
&\leqslant (1-\alpha_n-\beta_n-\gamma_n)\|x_n-p\|^2+\alpha_nH(T_ny_n,T_np)^2+\beta_nH(T_nz_n,T_np)^2\\
&\quad +\gamma_nH(T_nx_n,T_np)^2-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)[\alpha_ng(\|x_n-r_n\|)\\
&\quad +\beta_ng(\|x_n-t_n\|)+\gamma_ng(\|x_n-s_n\|)]\\
&\leqslant \|x_n-p\|^2-\frac{\alpha_n}{3}(1-b_n-c_n)b_ng(\|t_n-x_n\|)-\frac{1}{3}(1-\alpha_n-\beta_n-\\
&\quad \gamma_n)[\alpha_ng(\|x_n-r_n\|)+\beta_ng(\|x_n-t_n\|)+\gamma_ng(\|x_n-s_n\|)].
\end{aligned}$$

Then

$$(1-\alpha_n-\beta_n-\gamma_n)\alpha_ng(\|x_n-r_n\|)\leqslant 3(\|x_n-p\|^2-\|x_{n+1}-p\|^2), \tag{6}$$

$$(1-\alpha_n-\beta_n-\gamma_n)\beta_ng(\|x_n-t_n\|)\leqslant 3(\|x_n-p\|^2-\|x_{n+1}-p\|^2), \tag{7}$$

$$(1-\alpha_n-\beta_n-\gamma_n)\gamma_ng(\|x_n-s_n\|)\leqslant 3(\|x_n-p\|^2-\|x_{n+1}-p\|^2), \tag{8}$$

and

$$\alpha_n(1-b_n-c_n)b_ng(\|t_n-x_n\|)\leqslant 3(\|x_n-p\|^2-\|x_{n+1}-p\|^2). \tag{9}$$

Since $\lim_n\|x_n-p\|$ exists for any $p\in F(\cap_nT_n)$, it follows from (6) that $\lim_n(1-\alpha_n-\beta_n-\gamma_n)\alpha_ng(\|x_n-r_n\|)=0$. From g is continuous strictly increasing with $g(0)=0$ and $0<\liminf_n\alpha_n\leqslant\limsup_n(\alpha_n+\beta_n+\gamma_n)<1$, then

$$\lim_n\|x_n-r_n\|=0. \tag{10}$$

Using a similarly method together with inequalities (7) and $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, then

$$\lim_n \|x_n - t_n\| = 0. \tag{11}$$

Similarly, from (8) and $0 < \liminf_n \gamma_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, we have $\lim_n \|x_n - s_n\| = 0$, since $s_n \in T_n x_n$, then $0 \leqslant \lim_n d(x_n, T_n x_n) \leqslant \lim_n \|x_n - s_n\| = 0$, thus we get (ii).

If $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n a_n < 1$, we will prove (i).

$$\begin{aligned}\|s_n - x_n\| &\leqslant \|s_n - t_n\| + \|t_n - x_n\| \leqslant H(T_n x_n, T_n z_n) + \|t_n - x_n\| \\ &\leqslant \|x_n - z_n\| + \|t_n - x_n\| \\ &\leqslant a_n\|x_n - s_n\| + \|t_n - x_n\|. \end{aligned} \tag{12}$$

Since $\limsup_n a_n < 1$, then

$$\liminf_n (1 - a_n) = 1 - \limsup_n a_n > 0.$$

This together with (11), (12), we obtain the result.
Finally, we will prove (iii). From iterative scheme (3) and Lemma 3.1, we have

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n\|r_n - p\| \\ &\quad + \beta_n\|t_n - p\| + \gamma_n\|s_n - p\| \\ &\leqslant (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\ &\quad + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\|. \\ &\leqslant (1 - \alpha_n - \beta_n)\|x_n - p\| + \alpha_n[(1 - b_n)\|x_n - p\| \\ &\quad + b_n\|z_n - p\|] + \beta_n\|z_n - p\|, \end{aligned}$$

which implies

$$\|x_{n+1} - p\| - \|x_n - p\| + (\alpha_n b_n + \beta_n)\|x_n - p\| \leqslant (\alpha_n b_n + \beta_n)\|z_n - p\|.$$

Notice that

$$0 < \liminf_n(\alpha_n b_n + \beta_n) \text{ and } \lim_n \|x_n - p\| \text{ exists.}$$

Hence we have

$$d = \lim_n \|x_n - p\| \leqslant \liminf_n \|z_n - p\| \leqslant \limsup \|z_n - p\| \leqslant d.$$

Thus we have

$$d = \lim_n \|z_n - p\| = \lim_n \|(1 - a_n)\|x_n - p\| + a_n\|s_n - p\|.$$

By Lemma 2.3 and $0 < \liminf_n a_n \leqslant \limsup_n a_n < 1$, we have $0 \leqslant \lim_n d(x_n, Tx_n) \leqslant \lim_n \|x_n - s_n\| = 0$.

THEOREM 3.10. *Let X, T_n and $\{x_n\}$ be the same as in Lemma 3.9, K be a nonempty compact convex subset of a Banach space X, then $\{x_n\}$ converges strongly to a common fixed point of T_n.*

Proof. By Lemma 3.9, we have $\lim_n d(x_n, T_n x_n) = 0$. Since K be a nonempty compact convex subset, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$ for some $q \in K$. Thus,

$$\begin{aligned} d(q, Tq) &\leqslant \|q - x_{n_k}\| + d(x_{n_k}, T_{n_k} x_{n_k}) + H(T_{n_k} x_{n_k}, T_{n_k} q) \\ &\leqslant 2\|q - x_{n_k}\| + d(x_{n_k}, T_{n_k} x_{n_k}) \to 0. \end{aligned}$$

Hence q is a fixed point of T_n. Now taking q in place of p, we get that $\lim_{n\to\infty} \|x_n - q\|$ exists. So the desired conclusion follows.

THEOREM 3.11. *Let X, K, T_n and $\{x_n\}$ be the same as in Lemma 3.9, if T_n satisfies Condition A' with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a common fixed point of T_n.*

Proof. By Lemma 3.9, we have $\lim_n d(x_n, T_n x_n) = 0$. Since T_n satisfies Condition A' with respect to $\{x_n\}$. Thus we get $d(x_n, F(\cap_n T_n)) = 0$. The remainder of the proof is similar as Theorem 2.4 in [19], we omit it.

THEOREM 3.12. *Let X, T_n and $\{x_n\}$ be the same as in Lemma 3.9 , K be a nonempty weakly compact convex subset of a Banach space X and X satisfies Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of T_n.*

Proof. The proof of the Theorem is similar as Theorem 2.5 in [19], we omit it.

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