SHARP BOUNDS FOR SEIFFERT MEANS
IN TERMS OF LEHMER MEANS

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Abstract. In this paper, we establish two sharp inequalities as follows: \( P(a, b) > L_{-\frac{1}{6}}(a, b) \) and \( T(a, b) < L_{\frac{1}{3}}(a, b) \) for all \( a, b > 0 \) with \( a \neq b \). Here, \( L_r(a, b) \), \( P(a, b) \) and \( T(a, b) \) are the Lehmer, first and second Seiffert means of \( a \) and \( b \), respectively.

1. Introduction

For \( r \in \mathbb{R} \) and \( a, b > 0 \), the Lehmer mean \( L_r(a, b) \) was introduced by Lehmer [1] as follows:

\[
L_r(a, b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r} \tag{1.1}
\]

It is well known that \( L_r(a, b) \) is increasing with respect to \( r \in \mathbb{R} \) for fixed \( a \) and \( b \). Many means are the special cases of Lehmer mean, for example,

\[
A(a, b) = \frac{a + b}{2} = L_0(a, b) \quad \text{is the arithmetic mean,}
\]

\[
G(a, b) = \sqrt{ab} = L_{-\frac{1}{2}}(a, b) \quad \text{is the geometric mean,}
\]

\[
H(a, b) = \frac{2ab}{a+b} = L_{-1}(a, b) \quad \text{is the harmonic mean.}
\]

Investigation of the inequalities between Lehmer and other means has attracted the attention of a considerable number of mathematicians [2–5].

The first and second Seiffert means \( P(a, b) \) [6] and \( T(a, b) \) [7] of two positive numbers \( a \) and \( b \) are defined by

\[
P(a, b) = \begin{cases} 
\frac{a - b}{4 \arctan(\sqrt{a^{-\frac{1}{2}}} - \pi)}, & a \neq b, \\
\frac{a}{a+b}, & a = b
\end{cases} \tag{1.2}
\]

and

\[
T(a, b) = \begin{cases} 
\frac{a - b}{2 \arctan(\frac{a-b}{a+b})}, & a \neq b, \\
\frac{a}{a+b}, & a = b
\end{cases} \tag{1.3}
\]
respectively.

Recently, both means \( P \) and \( T \) have been the subject of intensive research. In particular, many remarkable inequalities for \( P \) and \( T \) can be found in the literature [7–11]. The first Seiffert mean \( P(a,b) \) can be rewritten as (see [10, Eq.(2.4)])

\[
P(a,b) = \begin{cases} \frac{a-b}{2\arcsin\left(\frac{a-b}{a+b}\right)}, & a \neq b, \\ a, & a = b. \end{cases}
\] (1.4)

The power mean of order \( p \) of the positive real numbers \( a \) and \( b \) is defined by

\[
M_p(a,b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}
\]

The main properties of the power mean \( M_p \) are given in [12]. In particular, \( M_p(a,b) \) is continuous and strictly increasing with respect to \( p \in \mathbb{R} \) for fixed \( a \) and \( b \) with \( a \neq b \).

Let

\[
I(a,b) = \begin{cases} \frac{1}{e} \left(\frac{b^p}{a^p}\right)^{\frac{1}{p-1}}, & b \neq a, \\ a, & b = a \end{cases}
\]

and

\[
L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a \end{cases}
\]

be the identric and logarithmic means of two positive numbers \( a \) and \( b \), respectively. Then it is well known that

\[
\min\{a,b\} < H(a,b) = L_{-1}(a,b) = M_{-1}(a,b) < G(a,b)
= L_{-\frac{1}{2}}(a,b) = M_{0}(a,b) < L(a,b) < I(a,b) < A(a,b)
= L_{0}(a,b) = M_{1}(a,b) < \max\{a,b\} \tag{1.5}
\]

for all \( a,b > 0 \) with \( a \neq b \).

In [6], Seiffert proved that

\[
L(a,b) < P(a,b) < I(a,b) \tag{1.6}
\]

for all \( a,b > 0 \) with \( a \neq b \).

Alzer [4] established that

\[
I(a,b) > L_{-\frac{1}{6}}(a,b)
\]

for all \( a,b > 0 \) with \( a \neq b \).

Seiffert [7] obtained the power mean bounds for the second Seiffert mean \( T \) as follows:

\[
M_1(a,b) < T(a,b) < M_2(a,b) \tag{1.7}
\]

for all \( a,b > 0 \) with \( a \neq b \).
The following bounds for the first Seiffert mean $P$ in terms of power means are proved by H"ast"o [8]:

$$M_{\log_2} (a, b) < P(a, b) < M_{\frac{2}{3}} (a, b)$$

for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to present the optimal upper and lower Lehmer mean bounds for the first and second Seiffert means.

2. Main Results

**Theorem 2.1.** Inequality $L_{-\frac{1}{6}} (a, b) < P(a, b) < L_0 (a, b)$ holds for all $a, b > 0$ with $a \neq b$, and $L_{-\frac{1}{6}} (a, b)$ and $L_0 (a, b)$ are the best possible lower and upper Lehmer mean bounds for the first Seiffert mean $P(a, b)$.

*Proof.* From (1.1) and (1.4) we clearly see that both $L_r (a, b)$ and $P(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b = 1$. Let $t = \sqrt[6]{a} > 1$. Then (1.1) and (1.2) give

$$P(a, b) - L_{-\frac{1}{6}} (a, b) = -\frac{t(t^5 + 1)}{(4 \arctan t^3 - \pi)(t + 1)}\{4 \arctan t^3 - \frac{(t + 1)(t^6 - 1)}{t(t^5 + 1)} - \pi\}. \tag{2.1}$$

Let

$$f(t) = 4 \arctan t^3 - \frac{(t + 1)(t^6 - 1)}{t(t^5 + 1)} - \pi, \tag{2.2}$$

then simple computations yield that

$$\lim_{t \to 1} f(t) = 0 \tag{2.3}$$

and

$$f'(t) = -\frac{(t - 1)^4(t + 1)^2(t^2 + t + 1)}{t^2(t^5 + 1)^2(t^6 + 1)} f_1(t), \tag{2.4}$$

where

$$f_1(t) = t^{10} + t^9 + 3t^8 + 4t^7 - 5t^6 + 3t^5 - 5t^4 + 4t^3 + 3t^2 + t + 1 = t^6(t^4 + t^3 + 3t^2 - 5) + t^4(4t^3 + 3t - 5) + 4t^3 + 3t^2 + t + 1 \tag{2.5}$$

for all $t > 0$.

Therefore, $P(a, b) > L_{-\frac{1}{6}} (a, b)$ follows from (2.1)–(2.5).

On the other hand, $P(a, b) < L_0 (a, b)$ follows from (1.5) and (1.6).

Next, we prove that $L_{-\frac{1}{6}} (a, b)$ and $L_0 (a, b)$ are the best possible lower and upper Lehmer mean bounds for $P(a, b)$.

For any $\varepsilon > 0$ and $x > 0$, from (1.1) and (1.2) one has

$$L_{-\frac{1}{6} + \varepsilon} (1, 1 + x) - P(1, 1 + x) = \frac{g_1(x)}{(4 \arctan \sqrt{x + 1} - \pi)[1 + (x + 1)^{-\frac{1}{6} + \varepsilon}]} \tag{2.6}$$
and
\[ \lim_{x \to +\infty} \frac{P(1,x)}{L_\varepsilon(1,x)} = \lim_{x \to +\infty} \frac{x - 1}{\pi(x^{1-\varepsilon} + 1)} = +\infty, \quad (2.7) \]

where \( g_1(x) = [1 + (x + 1)^{\frac{\varepsilon}{2} + \varepsilon}](4 \arctan \sqrt{x + 1} - \pi) - x[1 + (x + 1)^{-\frac{1}{2} + \varepsilon}] \).

Let \( x \to 0 \), making use of the Taylor expansion we get
\[ g_1(x) = \left[ 2 + \left( \frac{5}{6} + \varepsilon \right)x + \left( \frac{5}{6} + \varepsilon \right) \left( \frac{\varepsilon}{2} - \frac{1}{12} \right) x^2 + o(x^2) \right] \left[ x - \frac{1}{2} x^2 + \frac{7}{24} x^3 + o(x^3) \right] - x \left[ 2 + \left( \varepsilon - \frac{1}{6} \right)x + \left( \varepsilon - \frac{1}{6} \right) \left( \frac{\varepsilon}{2} - \frac{7}{12} \right) x^2 + o(x^2) \right] = \frac{1}{2} \varepsilon x^3 + o(x^3). \quad (2.8) \]

Equations (2.6) and (2.8) imply that for any \( \varepsilon > 0 \) there exists \( \delta_1 = \delta_1(\varepsilon) > 0 \), such that \( L_{-\frac{1}{2} + \varepsilon}(1,1+x) > P(1,1+x) \) for \( x \in (0, \delta_1) \).

Equation (2.7) implies that for any \( \varepsilon > 0 \) there exists \( X_1 = X_1(\varepsilon) > 1 \), such that \( P(1,x) > L_{-\varepsilon}(1,x) \) for \( x \in (X_1, \infty) \).

**THEOREM 2.2.** Inequality \( L_0(a,b) < T(a,b) < L_{\frac{1}{3}}(a,b) \) holds for all \( a,b > 0 \) with \( a \neq b \), and \( L_0(a,b) \) and \( L_{\frac{1}{3}}(a,b) \) are the best possible lower and upper Lehmer mean bounds for the second Seiffert mean \( T(a,b) \).

**Proof.** From (1.1) and (1.3) we clearly see that both \( L_r(a,b) \) and \( T(a,b) \) are symmetric and homogeneous of degree 1, without loss of generality, we assume that \( a > b = 1 \). Let \( t = \sqrt[3]{a} > 1 \). Then (1.1) and (1.3) give
\[ T(a,b) - L_{\frac{1}{3}}(a,b) = \frac{t^4 + 1}{2(t + 1) \arctan \frac{t^2 - 1}{t^3 + 1}} \left[ \frac{(t^3 - 1)(t + 1)}{t^4 + 1} - 2 \arctan \frac{t^3 - 1}{t^3 + 1} \right]. \quad (2.9) \]

Let
\[ g(t) = \frac{(t^3 - 1)(t + 1)}{t^4 + 1} - 2 \arctan \frac{t^3 - 1}{t^3 + 1}, \quad (2.10) \]
then simple computations yield that
\[ \lim_{t \to 1} g(t) = 0, \quad (2.11) \]
\[ g'(t) = -\frac{(t - 1)^2(t^2 + t + 1)}{(t^4 + 1)^2(t^6 + 1)}(t^6 + 3t^5 + 9t^4 + 12t^3 + 9t^2 + 3t + 1) < 0 \quad (2.12) \]
for \( t > 1 \).

Therefore, \( T(a,b) < L_{\frac{1}{3}}(a,b) \) follows from (2.9)–(2.12).

On the other hand, \( T(a,b) > L_0(a,b) \) follows from (1.5) and (1.7).

Next we prove that \( L_0(a,b) \) and \( L_{\frac{1}{3}}(a,b) \) are the best possible lower and upper Lehmer mean bounds for \( T(a,b) \).
For any $\varepsilon > 0$ and $x > 0$, from (1.1) and (1.3) one has

$$T(1,1+x) - L_{\frac{1}{3}-\varepsilon}(1,1+x) = \frac{g_2(x)}{2[1+(1+x)^{\frac{1}{3}-\varepsilon}]\arctan\frac{x}{2+x}}\tag{2.13}$$

and

$$\lim_{x\to\infty} \frac{L_\varepsilon(1,x)}{T(1,x)} = \lim_{x\to\infty} \frac{\pi(x^{\varepsilon+1}+1)}{2(x^\varepsilon+1)(x-1)} = \frac{\pi}{2} > 1, \tag{2.14}$$

where $g_2(x) = x[1+(1+x)^{\frac{1}{3}-\varepsilon}] - 2[1+(1+x)^{\frac{1}{3}-\varepsilon}]\arctan\frac{x}{2+x}$.

Let $x \to 0$, making use of the Taylor expansion we get

$$g_2(x) = x \left[ 2 + \left( \frac{1}{3} - \varepsilon \right) x - \frac{(1-3\varepsilon)(2+3\varepsilon)}{18} x^2 + o(x^2) \right]$$

$$- x \left[ 1 - \frac{1}{2} x + \frac{1}{6} x^2 + o(x^2) \right] \left[ 2 + \left( \frac{4}{3} - \varepsilon \right) x + \frac{(4-3\varepsilon)(1-3\varepsilon)}{18} x^2 + o(x^2) \right]$$

$$= \frac{1}{2} \varepsilon x^3 + o(x^3). \tag{2.15}$$

Equations (2.13) and (2.15) imply that for any $0 < \varepsilon < \frac{1}{3}$, there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that $T(1,1+x) > L_{\frac{1}{3}-\varepsilon}(1,1+x)$ for $x \in (0,\delta_2)$.

Equation (2.14) implies that for any $\varepsilon > 0$ there exists $X_2 = X_2(\varepsilon) > 1$, such that $L_\varepsilon(1,x) > T(1,x)$ for $x \in (X_2,\infty)$. \hfill \Box

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REFERENCES


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