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# A NOTE ON THE HERMITE–HADAMARD INEQUALITY

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*Abstract.* In this note we give a new generalization of the well-known Hermite-Hadamard inequality

## 1. Introduction And Main Results

The research of beautiful inequalities which have symmetry is very interesting and important to Analysis and PDE. A well-known example is the famous Hermite-Hadamard inequality which was first published in [1]. It gives us an estimate of the mean value of a convex function which works great in Analysis and PDE. In this note, we give a new generalization of the Hermite-Hadamard inequality, and believe that our inequality would also have some use in Analysis and PDE.

Throughout this note, we denote by $I$ the closed interval $[a,b]$. A real-valued function $f$ is said to be convex on $I$ if $\lambda f(x)+(1-\lambda)f(y) \geqslant f(\lambda x+(1-\lambda)y)$ for all $x,y \in I, 0 \leqslant \lambda \leqslant 1$, and a function $f$ that is continuous on $I$ and twice differentiable on (a, b) is convex on $I$ if and only if $f''(x) \geqslant 0$ for all $x \in (a,b)$. The classical Hermite-Hadamard inequality is:

THEOREM 1.1. (Hermite-Hadamard Inequality) *If $f: I \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

An account on the history of this inequality can be found in [2]. Surveys on various generalizations and developments can be found in [3] and [4]. In this note, we will prove that for arbitrary non-negative real-valued integrable function $\Phi: I \to \mathbb{R}$, there exist real numbers $l$, $L$ such that:

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) \leqslant l \leqslant \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \leqslant L \leqslant \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \tag{2}$$

This is a generalization of the following result in [5]:

$$f\left(\frac{a+b}{2}\right) \leqslant l \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L \leqslant \frac{f(a)+f(b)}{2}. \tag{3}$$

In fact, we prove the following theorem:

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THEOREM 1.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function. If the non-negative real-valued integrable function $\Phi: I\to\mathbb{R}$ such that $f\circ\Phi(x)$ is also convex, then for $n\in\mathbb{N}$, $\lambda_0=0,\lambda_{n+1}=1$ and arbitrary $0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n\leqslant 1$, we have*

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) &\leqslant l(\lambda_1,\cdots,\lambda_n)\leqslant \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \\ &\leqslant L(\lambda_1,\cdots,\lambda_n)\leqslant \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}, \end{aligned} \tag{4}$$

*where*

$$l(\lambda_1,\cdots,\lambda_n)=\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\,f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right),$$

$$L(\lambda_1,\cdots,\lambda_n)=\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}.$$

COROLLARY 1.3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function. If the non-negative real-valued integrable function $\Phi: I\to\mathbb{R}$ such that $f\circ\Phi(x)$ is also convex, then for $n\in\mathbb{N}$, $\lambda_0=0,\lambda_{n+1}=1$ and arbitrary $0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n\leqslant 1$, we have*

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) &\leqslant \sup_{0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n\leqslant 1} l(\lambda_1,\cdots,\lambda_n)\leqslant \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \\ &\leqslant \sup_{0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n\leqslant 1} L(\lambda_1,\cdots,\lambda_n)\leqslant \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}, \end{aligned} \tag{5}$$

*where $l(\lambda_1,\cdots,\lambda_n)$ and $L(\lambda_1,\cdots,\lambda_n)$ are defined in Theorem 1.2.*

Applying Theorem 1.2 for $\Phi(x)=x$ and $n=1$, we get the result proven by A. El Farissi in [5].

COROLLARY 1.4. *Assume that $f:\mathbb{R}\to\mathbb{R}$ is a convex function on I. Then for an arbitrary $\lambda\in[0,1]$, we have*

$$f\left(\frac{a+b}{2}\right)\leqslant l(\lambda)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant L(\lambda)\leqslant\frac{f(a)+f(b)}{2},$$

*where*

$$l(\lambda)=\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right),$$

$$L(\lambda)=\frac{1}{2}\left(f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)\right).$$

## 2. Lemma And Proof Of The Theorem

In order to prove Theorem 1.2, we shall need the following Lemma:

LEMMA 2.1. (Jensen's Inequality) *If $f:\mathbb{R}\to\mathbb{R}$ is a convex function, then for an arbitrary non-negative real-valued integrable function $\Phi: I\to\mathbb{R}$, we have*

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) \leqslant \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx. \tag{6}$$

*Proof.* See [6]. □

With the help of Lemma 2.1, we now turn to prove Theorem 1.2.

*Proof of Theorem 1.2.* It follows from the hypothesis that $f(x)$ and $f\circ\Phi(x)$ are both convex functions, therefore by applying Jensen's inequality for $f(x)$ and the right-hand side of the Hermite-Hadamard inequality for $f\circ\Phi(x)$ we get

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) \leqslant \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \leqslant \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \tag{7}$$

By assumption $\lambda_0 = 0$, so

$$[a,(1-\lambda_1)a+\lambda_1 b] = [(1-\lambda_0)a+\lambda_0 b,(1-\lambda_1)a+\lambda_1 b].$$

Then applying (7) to $[(1-\lambda_k)a+\lambda_k b,(1-\lambda_{k+1})a+\lambda_{k+1}b]$, for $k=0,1,\cdots,n$ we get

$$\begin{aligned} &f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right) \\ &\quad\leqslant \frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f\circ\Phi(x)\,dx \\ &\quad\leqslant \frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}. \end{aligned} \tag{8}$$

Multiplying each term in (8) by corresponding $(\lambda_{k+1}-\lambda_k)$, and adding the resulting inequalities, we get

$$\begin{aligned} &\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right) \\ &\quad\leqslant \frac{1}{b-a}\sum_{k=0}^{n}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f\circ\Phi(x)\,dx \\ &\quad\leqslant \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}, \end{aligned}$$

that is

$$l(\lambda_1,\cdots,\lambda_n) \leqslant \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \leqslant L(\lambda_1,\cdots,\lambda_n),$$

where $l(\lambda_1,\cdots,\lambda_n)$ and $L(\lambda_1,\cdots,\lambda_n)$ are defined as in Theorem 1.2.

To prove the remaining two inequalities:

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) \leqslant l(\lambda_1,\cdots,\lambda_n) \leqslant L(\lambda_1,\cdots,\lambda_n) \leqslant \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}, \quad (9)$$

we use the fact $f:\mathbb{R}\to\mathbb{R}$, $f\circ\Phi(x)$ are both convex functions and observe that $\sum\limits_{k=0}^{n}(\lambda_{k+1}-\lambda_k)=1$

$$\begin{aligned}
&f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right)\\
&\quad= f\left(\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right)\\
&\quad\leqslant \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\,f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right)\\
&\quad\leqslant \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}\\
&\quad\leqslant \frac{1}{2}\sum_{k=0}^{n}\left(((1-\lambda_k)-(1-\lambda_{k+1}))((1-\lambda_k)+(1-\lambda_{k+1}))\,f\circ\Phi(a)\right)\\
&\qquad+\frac{1}{2}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)(\lambda_{k+1}+\lambda_k)\,f\circ\Phi(b)\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\left(\left((1-\lambda_k)^2-(1-\lambda_{k+1})^2\right)f\circ\Phi(a)+\left(\lambda_{k+1}^2-\lambda_k^2\right)f\circ\Phi(b)\right)\\
&\quad=\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \qquad \square
\end{aligned}$$

EXAMPLE. Let $a_0=a_{n+1}=0, a_1,\cdots a_n\geqslant 0$ and suppose that there is at least one $a_i$ such that $a_i\neq 0$. For $\lambda_k=\frac{\sum_{i=0}^{k}a_i}{\sum_{i=0}^{n}a_i}$ from Theorem 1.2 we get

$$l(\lambda_1,\cdots,\lambda_n)=\frac{1}{\sum\limits_{k=0}^{n}a_k}\sum_{k=0}^{n}a_{k+1}f\left(\frac{\sum\limits_{k=0}^{n}a_k}{a_{k+1}(b-a)}\int_{c_k}^{c_{k+1}}\Phi(x)\,dx\right), \quad (10)$$

$$L(\lambda_1,\cdots,\lambda_n)=\frac{1}{\sum\limits_{k=0}^{n}a_k}\sum_{k=0}^{n}a_{k+1}\frac{f\circ\Phi(c_k)+f\circ\Phi(c_{k+1})}{2}, \quad (11)$$

where $c_k = (1-\lambda_k)\,a + \lambda_k b$.

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