

## A NOTE ON THE HERMITE–HADAMARD INEQUALITY

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*Abstract.* In this note we give a new generalization of the well-known Hermite-Hadamard inequality

### 1. Introduction And Main Results

The research of beautiful inequalities which have symmetry is very interesting and important to Analysis and PDE. A well-known example is the famous Hermite-Hadamard inequality which was first published in [1]. It gives us an estimate of the mean value of a convex function which works great in Analysis and PDE. In this note, we give a new generalization of the Hermite-Hadamard inequality, and believe that our inequality would also have some use in Analysis and PDE.

Throughout this note, we denote by  $I$  the closed interval  $[a, b]$ . A real-valued function  $f$  is said to be convex on  $I$  if  $\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y)$  for all  $x, y \in I, 0 \leq \lambda \leq 1$ , and a function  $f$  that is continuous on  $I$  and twice differentiable on  $(a, b)$  is convex on  $I$  if and only if  $f''(x) \geq 0$  for all  $x \in (a, b)$ . The classical Hermite-Hadamard inequality is:

**THEOREM 1.1.** (Hermite-Hadamard Inequality) *If  $f : I \rightarrow \mathbb{R}$  is a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

An account on the history of this inequality can be found in [2]. Surveys on various generalizations and developments can be found in [3] and [4]. In this note, we will prove that for arbitrary non-negative real-valued integrable function  $\Phi : I \rightarrow \mathbb{R}$ , there exist real numbers  $l, L$  such that:

$$f\left(\frac{1}{b-a} \int_a^b \Phi(x) dx\right) \leq l \leq \frac{1}{b-a} \int_a^b f \circ \Phi(x) dx \leq L \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}. \quad (2)$$

This is a generalization of the following result in [5]:

$$f\left(\frac{a+b}{2}\right) \leq l \leq \frac{1}{b-a} \int_a^b f(x) dx \leq L \leq \frac{f(a)+f(b)}{2}. \quad (3)$$

In fact, we prove the following theorem:

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**THEOREM 1.2.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function. If the non-negative real-valued integrable function  $\Phi : I \rightarrow \mathbb{R}$  such that  $f \circ \Phi(x)$  is also convex, then for  $n \in \mathbb{N}$ ,  $\lambda_0 = 0, \lambda_{n+1} = 1$  and arbitrary  $0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq 1$ , we have*

$$\begin{aligned} f\left(\frac{1}{b-a} \int_a^b \Phi(x) dx\right) &\leq l(\lambda_1, \dots, \lambda_n) \leq \frac{1}{b-a} \int_a^b f \circ \Phi(x) dx \\ &\leq L(\lambda_1, \dots, \lambda_n) \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} l(\lambda_1, \dots, \lambda_n) &= \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) f\left(\frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \int_{(1-\lambda_k)a + \lambda_k b}^{(1-\lambda_{k+1})a + \lambda_{k+1}b} \Phi(x) dx\right), \\ L(\lambda_1, \dots, \lambda_n) &= \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \frac{f \circ \Phi((1-\lambda_k)a + \lambda_k b) + f \circ \Phi((1-\lambda_{k+1})a + \lambda_{k+1}b)}{2}. \end{aligned}$$

**COROLLARY 1.3.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function. If the non-negative real-valued integrable function  $\Phi : I \rightarrow \mathbb{R}$  such that  $f \circ \Phi(x)$  is also convex, then for  $n \in \mathbb{N}$ ,  $\lambda_0 = 0, \lambda_{n+1} = 1$  and arbitrary  $0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq 1$ , we have*

$$\begin{aligned} f\left(\frac{1}{b-a} \int_a^b \Phi(x) dx\right) &\leq \sup_{0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq 1} l(\lambda_1, \dots, \lambda_n) \leq \frac{1}{b-a} \int_a^b f \circ \Phi(x) dx \\ &\leq \sup_{0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq 1} L(\lambda_1, \dots, \lambda_n) \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}, \end{aligned} \quad (5)$$

where  $l(\lambda_1, \dots, \lambda_n)$  and  $L(\lambda_1, \dots, \lambda_n)$  are defined in Theorem 1.2.

Applying Theorem 1.2 for  $\Phi(x) = x$  and  $n = 1$ , we get the result proven by A. El Farissi in [5].

**COROLLARY 1.4.** *Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a convex function on  $I$ . Then for an arbitrary  $\lambda \in [0, 1]$ , we have*

$$f\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq L(\lambda) \leq \frac{f(a) + f(b)}{2},$$

where

$$l(\lambda) = \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right),$$

$$L(\lambda) = \frac{1}{2} (f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)).$$

### 2. Lemma And Proof Of The Theorem

In order to prove Theorem 1.2, we shall need the following Lemma:

LEMMA 2.1. (Jensen’s Inequality) *If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a convex function, then for an arbitrary non-negative real-valued integrable function  $\Phi : I \rightarrow \mathbb{R}$ , we have*

$$f\left(\frac{1}{b-a} \int_a^b \Phi(x) dx\right) \leq \frac{1}{b-a} \int_a^b f \circ \Phi(x) dx. \tag{6}$$

*Proof.* See [6].  $\square$

With the help of Lemma 2.1, we now turn to prove Theorem 1.2.

*Proof of Theorem 1.2.* It follows from the hypothesis that  $f(x)$  and  $f \circ \Phi(x)$  are both convex functions, therefore by applying Jensen’s inequality for  $f(x)$  and the right-hand side of the Hermite-Hadamard inequality for  $f \circ \Phi(x)$  we get

$$f\left(\frac{1}{b-a} \int_a^b \Phi(x) dx\right) \leq \frac{1}{b-a} \int_a^b f \circ \Phi(x) dx \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}. \tag{7}$$

By assumption  $\lambda_0 = 0$ , so

$$[a, (1 - \lambda_1)a + \lambda_1b] = [(1 - \lambda_0)a + \lambda_0b, (1 - \lambda_1)a + \lambda_1b].$$

Then applying (7) to  $[(1 - \lambda_k)a + \lambda_kb, (1 - \lambda_{k+1})a + \lambda_{k+1}b]$ , for  $k = 0, 1, \dots, n$  we get

$$\begin{aligned} f\left(\frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} \Phi(x) dx\right) \\ \leq \frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f \circ \Phi(x) dx \\ \leq \frac{f \circ \Phi((1 - \lambda_k)a + \lambda_kb) + f \circ \Phi((1 - \lambda_{k+1})a + \lambda_{k+1}b)}{2}. \end{aligned} \tag{8}$$

Multiplying each term in (8) by corresponding  $(\lambda_{k+1} - \lambda_k)$ , and adding the resulting inequalities, we get

$$\begin{aligned} \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) f\left(\frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} \Phi(x) dx\right) \\ \leq \frac{1}{b-a} \sum_{k=0}^n \int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f \circ \Phi(x) dx \\ \leq \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \frac{f \circ \Phi((1 - \lambda_k)a + \lambda_kb) + f \circ \Phi((1 - \lambda_{k+1})a + \lambda_{k+1}b)}{2}, \end{aligned}$$

that is

$$l(\lambda_1, \dots, \lambda_n) \leq \frac{1}{b-a} \int_a^b f \circ \Phi(x) dx \leq L(\lambda_1, \dots, \lambda_n),$$

where  $l(\lambda_1, \dots, \lambda_n)$  and  $L(\lambda_1, \dots, \lambda_n)$  are defined as in Theorem 1.2.

To prove the remaining two inequalities:

$$f\left(\frac{1}{b-a} \int_a^b \Phi(x) dx\right) \leq l(\lambda_1, \dots, \lambda_n) \leq L(\lambda_1, \dots, \lambda_n) \leq \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}, \quad (9)$$

we use the fact  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \circ \Phi(x)$  are both convex functions and observe that

$$\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) = 1$$

$$\begin{aligned} & f\left(\frac{1}{b-a} \int_a^b \Phi(x) dx\right) \\ &= f\left(\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \int_{(1-\lambda_k)a + \lambda_k b}^{(1-\lambda_{k+1})a + \lambda_{k+1}b} \Phi(x) dx\right) \\ &\leq \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) f\left(\frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \int_{(1-\lambda_k)a + \lambda_k b}^{(1-\lambda_{k+1})a + \lambda_{k+1}b} \Phi(x) dx\right) \\ &\leq \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \frac{f \circ \Phi((1-\lambda_k)a + \lambda_k b) + f \circ \Phi((1-\lambda_{k+1})a + \lambda_{k+1}b)}{2} \\ &\leq \frac{1}{2} \sum_{k=0}^n (((1-\lambda_k) - (1-\lambda_{k+1}))((1-\lambda_k) + (1-\lambda_{k+1}))) f \circ \Phi(a) \\ &\quad + \frac{1}{2} \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) (\lambda_{k+1} + \lambda_k) f \circ \Phi(b) \\ &= \frac{1}{2} \sum_{k=0}^n \left( ((1-\lambda_k)^2 - (1-\lambda_{k+1})^2) f \circ \Phi(a) + (\lambda_{k+1}^2 - \lambda_k^2) f \circ \Phi(b) \right) \\ &= \frac{f \circ \Phi(a) + f \circ \Phi(b)}{2}. \quad \square \end{aligned}$$

EXAMPLE. Let  $a_0 = a_{n+1} = 0, a_1, \dots, a_n \geq 0$  and suppose that there is at least one  $a_i$  such that  $a_i \neq 0$ . For  $\lambda_k = \frac{\sum_{i=0}^k a_i}{\sum_{i=0}^n a_i}$  from Theorem 1.2 we get

$$l(\lambda_1, \dots, \lambda_n) = \frac{1}{\sum_{k=0}^n a_k} \sum_{k=0}^n a_{k+1} f\left(\frac{\sum_{k=0}^n a_k}{a_{k+1}(b-a)} \int_{c_k}^{c_{k+1}} \Phi(x) dx\right), \quad (10)$$

$$L(\lambda_1, \dots, \lambda_n) = \frac{1}{\sum_{k=0}^n a_k} \sum_{k=0}^n a_{k+1} \frac{f \circ \Phi(c_k) + f \circ \Phi(c_{k+1})}{2}, \quad (11)$$

where  $c_k = (1 - \lambda_k)a + \lambda_k b$ .

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