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MAPPING PROPERTIES OF SOME CLASSES OF ANALYTIC FUNCTIONS UNDER AN INTEGRAL OPERATOR

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Abstract. In this paper, we consider certain subclasses of analytic functions with bounded boundary and bounded radius rotation related with Robertson functions and study the mapping properties of these classes under certain integral operator introduced by Serap Bulut recently.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. A function $f(z) \in A$ is said to be spiral-like if there exists a real number λ $\left(|\lambda| < \frac{\pi}{2}\right)$ such that

$$\operatorname{Re} e^{i\lambda} \frac{zf'(z)}{f(z)} > 0 \ (z \in E).$$

The class of all spiral-like functions was introduced by L. Spacek [21] in 1933 and we denote it by S_λ^*. Later in 1969, Robertson [20] considered the class C_λ of analytic functions in E for which $zf'(z) \in S_\lambda^*$.

Let $P_k^\lambda(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} e^{i\lambda} p(z) - \rho \cos\lambda}{1-\rho} \right| d\theta \leqslant k\pi \cos\lambda,\ z = re^{i\theta}, \tag{1.2}$$

where $k \geqslant 2$, $0 \leqslant \rho < 1$, λ is real with $|\lambda| < \frac{\pi}{2}$. For $\lambda = 0$, this class was introduced in [18] and for $\beta = 0$, see [19]. For $k = 2$, $\lambda = 0$ and $\rho = 0$, the class $P_k^\alpha(\rho)$ reduces to the class P of functions $p(z)$ analytic in E with $p(0) = 1$ and whose real part is positive.

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We define the following classes

$$R_k^\lambda(\rho) = \left\{ f(z) : f(z) \in A \text{ and } \frac{zf'(z)}{f(z)} \in P_k^\lambda(\rho),\ 0 \leqslant \rho < 1 \right\},$$

$$V_k^\lambda(\rho) = \left\{ f(z) : f(z) \in A \text{ and } \frac{(zf'(z))'}{f'(z)} \in P_k^\lambda(\rho),\ 0 \leqslant \rho < 1 \right\}.$$

These classes were introduced and studied in some details in [12]. For $\lambda = 0$ and $\rho = 0$, we obtain the well known classes R_k and V_k of analytic functions with bounded radius and bounded boundary rotations studied by Tammi [22] and Paatero [17] respectively. For details see [13, 14, 16]. Also it can easily be seen that $R_2^\lambda(0) = S_\lambda^*$ and $V_2^\lambda(0) = C_\lambda$.

For $f(z) \in A$, Al-Oboudi [2] introduced the following operator

$$D^0 f(z) = f(z), \tag{1.3}$$

$$D^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z),\ \delta \geqslant 0, \tag{1.4}$$

$$\vdots$$

$$D^n f(z) = D_\delta\left(D^{n-1} f(z)\right),\ (n \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1.5}$$

If $f(z)$ is given by (1.1), then from (1.4) and (1.5), we see that

$$D^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n\ a_j\, z^j,\ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{1.6}$$

with $D^n f(0) = 0$.

Let us consider the integral operator $I_n(f_1(z), \ldots f_m(z)) : A^m \to A$ defined as

$$\begin{aligned} F_n(z) &= I_n(f_1(z), \ldots f_m(z)) \\ &= \int_0^z \left(\frac{D^n f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{D^n f_m(t)}{t} \right)^{\alpha_m} dt,\ (z \in E), \end{aligned} \tag{1.7}$$

where $f_l(z) \in A^l$, $\alpha_l > 0$ for $1 \leqslant l \leqslant m$ and $n, m \in \mathbb{N}_0$ while the operator D^n is the Al-Oboudi differential operator.

The operator, given by (1.7), was introduced and studied by Serap Bulut [8]. For $n = 0$, we have the integral operator discussed by Breaz and Breaz [3] and Breaz et al. [5], for more details see [4, 6, 7, 9, 10, 15].

In the present paper, we establish a relation between the classes of bounded boundary and bounded radius rotation related with Robertson functions. We also discuss some propeties of the above integral operator $F_n(z)$ for these classes.

2. Preliminary Results

In order to derive our main result, we need the following lemmas.

LEMMA 2.1. [11]. *Let* $u = u_1 + iu_2$, $v = v_1 + iv_2$ *and* $\Psi(u,v)$ *be a complex valued function satisfying the conditions:*

(i). $\Psi(u,v)$ *is continuous in a domain* $D \subset \mathbb{C}^2$,

(ii). $(1,0) \in D$ *and* $\operatorname{Re}\Psi(1,0) > 0$,

(iii). $\operatorname{Re}\Psi(iu_2, v_1) \leqslant 0$, *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leqslant -\frac{1}{2}\left(1 + u_2^2\right)$.

If $h(z) = 1 + c_1 z + \cdots$ *is a function analytic in* E *such that* $(h(z), zh'(z)) \in D$ *and* $\operatorname{Re}\Psi(h(z), zh'(z)) > 0$ *for* $z \in E$, *then* $\operatorname{Re} h(z) > 0$ *in* E.

LEMMA 2.2. *Let* $f(z) \in V_k^\lambda(\rho)$, $0 \leqslant \rho < 1$ *and* λ *is real with* $|\lambda| < \frac{\pi}{2}$. *Then* $f(z) \in R_k^\lambda(\beta)$, *where* β *is one of the root of*

$$2\beta^3 + (1-2\rho)\beta^2 + \left(3\sec^2\lambda - 4\right)\beta - (1+2\rho)\tan^2\lambda = 0. \tag{2.1}$$

Proof. Let us suppose

$$\begin{aligned} e^{i\lambda}\frac{zf'(z)}{f(z)} &= \cos\lambda\left[(1-\beta)p(z)+\beta\right] + i\sin\lambda \\ &= \cos\lambda\left[(1-\beta)\left\{\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right\} + \beta\right] + i\sin\lambda \end{aligned} \tag{2.2}$$

where $p(z)$ is analytic in E with $p(0) = 1$. Then

$$\frac{(zf'(z))'}{f'(z)} = \frac{zf'(z)}{f(z)} + \frac{(1-\beta)\cos\lambda\ zp'(z)}{\cos\lambda\left[(1-\beta)p(z)+\beta\right] + i\sin\lambda},$$

or, equivalently

$$\begin{aligned} &\frac{1}{\cos\lambda}\left[e^{i\lambda}\frac{(zf'(z))'}{f'(z)} - i\sin\lambda\right] \\ &\qquad = \beta + (1-\beta)\left[p(z) + \frac{(1+i\tan\lambda)\ zp'(z)}{\left[(1-\beta)p(z)+\beta\right] + i\tan\lambda}\right], \end{aligned}$$

that is,

$$\begin{aligned} &\frac{1}{(1-\rho)\cos\lambda}\left[e^{i\lambda}\frac{(zf'(z))'}{f'(z)} - i\sin\lambda - \rho\cos\lambda\right] \\ &\qquad = \frac{\beta-\rho}{(1-\rho)} + \frac{(1-\beta)}{(1-\rho)}\left[p(z) + \frac{(1+i\tan\lambda)\ zp'(z)}{\left[(1-\beta)p(z)+\beta\right] + i\tan\lambda}\right]. \end{aligned} \tag{2.3}$$

Since $f(z) \in V_k^\lambda(\rho)$, it implies that

$$\frac{\beta-\rho}{(1-\rho)}+\frac{(1-\beta)}{(1-\rho)}\left[p(z)+\frac{(1+i\tan\lambda)\ zp'(z)}{[(1-\beta)p(z)+\beta]+i\tan\lambda}\right] \in P_k. \tag{2.4}$$

Now we define a function

$$\Phi_{a,b}(z)=\frac{1}{1+b}\frac{z}{(1-z)^a}+\frac{b}{1+b}\frac{z}{(1-z)^{a+1}}$$

with $a=\frac{1+i\tan\lambda}{(1-\beta)}$ and $b=\frac{\beta+i\tan\lambda}{(1-\beta)}$.

Using (2.2) with the convolution techniques given by Noor [14, 16], we have

$$\begin{aligned}\left[p(z)+\frac{azp'(z)}{p(z)+b}\right] &= \cos\lambda\left[(1-\beta)\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z)+\frac{azp_1'(z)}{p_1(z)+b}\right)\right.\right.\\ &\left.\left.-\left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2(z)+\frac{azp_2'(z)}{p_2(z)+b}\right)\right\}+\beta\right]+i\sin\lambda. \end{aligned}\tag{2.5}$$

Thus, from (2.4) and (2.5), we have

$$\frac{\beta-\rho}{(1-\rho)}+\frac{(1-\beta)}{(1-\rho)}\left[p_i(z)+\frac{(1+i\tan\lambda)\ zp_i'(z)}{[(1-\beta)p_i(z)+\beta]+i\tan\lambda}\right] \in P. \tag{2.6}$$

We now form the functional $\Psi(u,v)$ by choosing $u=p_i(z),\ v=zp_i'(z)$ in (2.6) and note that the first two conditions of Lemma 2.1 are clearly satisfied. We check condition (iii) as follows.

$$\begin{aligned}\operatorname{Re}\Psi(iu,v_1) &= \frac{\beta-\rho}{(1-\rho)}+\frac{1-\beta}{(1-\rho)}\operatorname{Re}\left\{\frac{(1+i\tan\lambda)v_1}{[(1-\beta)iu_2+\beta]+i\tan\lambda}\right\}\\ &= \frac{\beta-\rho}{(1-\rho)}-\frac{1-\beta}{2(1-\rho)}\operatorname{Re}\left\{\frac{(1+i\tan\lambda)\left(1+u_2^2\right)}{\beta+i[(1-\beta)u_2+\tan\lambda]}\right\}\\ &= \frac{A+Bu_2+Cu_2^2-Du_2^3}{E},\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}A &= \left[2(\beta-\rho)\left(\beta^2+\tan^2\lambda\right)-(1-\beta)\left(\beta+\tan^2\lambda\right)\right]\cos^2\lambda,\\ B &= (5\beta-4\rho-1)(1-\beta)\sin\lambda\cos\lambda,\\ C &= (1-\beta)\left[2(\beta-\rho)(1-\beta)-\left(\beta+\tan^2\lambda\right)\right]\cos^2\lambda,\\ D &= (1-\beta)^2\sin\lambda\cos\lambda,\\ E &= 2(1-\rho)\left[\beta^2+((1-\beta)u_2+\tan\lambda)^2\right]\cos^2\lambda>0.\end{aligned}$$

The right hand side of (2.7) is negative if $A\leqslant 0$, $B\leqslant 0$, $C\leqslant 0$ and $D\geqslant 0$. From $A\leqslant 0$, we have β to be one of the root of

$$2\beta^3+(1-2\rho)\beta^2+\left(3\sec^2\lambda-4\right)\beta-(1+2\rho)\tan^2\lambda=0.$$

and from $B \leqslant 0$ and $C \leqslant 0$, it follows that $0 \leqslant \beta < 1$. Since all the conditions of Lemma 2.1 are satisfied, it follows that $p_i(z) \in P$ in E for $i = 1,2$ and consequently $p(z) \in P_k$ and hence $f(z) \in R_k^{\lambda}(\beta)$, where β is a root of the equation (2.1). □

3. Main Result

THEOREM 3.1. *Let $D^n f_l(z) \in R_k^{\lambda}(\rho)$ for $1 \leqslant l \leqslant m$ with $0 \leqslant \rho < 1$. Also let λ is real with $|\lambda| < \frac{\pi}{2}$, $\alpha_l > 0$, $1 \leqslant l \leqslant m$. If*

$$0 \leqslant (\rho - 1)\sum_{l=1}^{m} \alpha_l + 1 < 1,$$

then $F_n(z) \in V_k^{\lambda}(\eta)$ with

$$\eta = (\rho - 1)\sum_{l=1}^{m} \alpha_l + 1. \tag{3.1}$$

Proof. From (1.7), we have

$$\frac{F_n''(z)}{F_n'(z)} = \sum_{i=1}^{m} \alpha_l \left(\frac{(D^n f_l(z))'}{D^n f_l(z)} - \frac{1}{z} \right),$$

or, equivalently

$$e^{i\lambda}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) = \sum_{l=1}^{m} \alpha_l \left[\frac{z(D^n f_l(z))'}{D^n f_l(z)} - 1\right] e^{i\lambda} + e^{i\lambda}.$$

Subtracting and adding $\rho\cos\lambda \sum_{l=1}^{m} \alpha_l$ on the left hand side and then taking real part, we have

$$\begin{aligned} &\mathrm{Re}\left[e^{i\lambda}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) - \eta\cos\lambda\right] \\ &\qquad = \sum_{l=1}^{m} \alpha_l \mathrm{Re}\left[e^{i\lambda}\frac{z(D^n f_l(z))'}{D^n f_l(z)} - \rho\cos\lambda\right], \end{aligned} \tag{3.2}$$

where η is given by (3.1).

Integrating (3.2) and then using (3.1), we have

$$\begin{aligned} &\int_0^{2\pi} \left|\mathrm{Re}\left[e^{i\lambda}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) - \eta\cos\lambda\right]\right| d\theta \\ &\qquad \leqslant \frac{1-\eta}{1-\rho}\int_0^{2\pi} \left|\mathrm{Re}\left[e^{i\lambda}\left(\frac{z(D^n f_l(z))'}{D^n f_l(z)}\right) - \rho\cos\lambda\right]\right| d\theta. \end{aligned} \tag{3.3}$$

Since $f_l(z) \in R_k^{\lambda}(\rho)$ for $1 \leqslant l \leqslant n$, we have

$$\int_0^{2\pi} \left| \operatorname{Re}\left[e^{i\lambda}\left(\frac{z(D^n f_l(z))'}{D^n f_l(z)} \right) - \rho\cos\lambda \right] \right| d\theta \leqslant (1-\rho)k\pi\cos\lambda. \tag{3.4}$$

Using (3.4) in (3.3), we obtain

$$\int_0^{2\pi} \left| \operatorname{Re}\left[e^{i\lambda}\left(1 + \frac{zF_n''(z)}{F_n'(z)} \right) - \eta\cos\lambda \right] \right| d\theta \leqslant (1-\eta)k\pi\cos\lambda.$$

Hence $F_n(z) \in V_k^{\lambda}(\eta)$ with η is given by (3.1). □

If $k = 2$ and $n = 0$ in Theorem 3.1, we obtain the following new result.

COROLLORY 3.2. *Let $f_l(z) \in R_2^{\lambda}(\rho)$ for $1 \leqslant l \leqslant m$ with $0 \leqslant \rho < 1$. Also let λ is real with $|\lambda| < \frac{\pi}{2}$, $\alpha_l > 0$, $1 \leqslant l \leqslant m$. If*

$$0 \leqslant (\rho - 1)\sum_{l=1}^{m} \alpha_l + 1 < 1,$$

then $F_0(z) \in V_2^{\lambda}(\eta)$ with $\eta = (\rho-1)\sum_{l=1}^{m} \alpha_l + 1$.

If $n = 0$ and $\lambda = 0$ in Theorem 3.1, we obtain the following result proved in [15].

COROLLORY 3.3. *Let $f_l(z) \in R_k^{0}(\rho)$ for $1 \leqslant l \leqslant m$ with $0 \leqslant \rho < 1$. Also let $\alpha_l > 0$, $1 \leqslant l \leqslant m$. If*

$$0 \leqslant (\rho - 1)\sum_{l=1}^{m} \alpha_l + 1 < 1,$$

then $F_0(z) \in V_k^{0}(\eta)$ with $\eta = (\rho-1)\sum_{l=1}^{m} \alpha_l + 1$.

THEOREM 3.4. *Let $f_l(z) \in V_k^{\lambda}(\rho)$ for $1 \leqslant l \leqslant n$ with $0 \leqslant \rho < 1$. Also let λ is real with $|\lambda| < \frac{\pi}{2}$, $\alpha_l > 0$, $1 \leqslant l \leqslant n$. If*

$$0 \leqslant (\beta - 1)\sum_{l=1}^{n} \alpha_l + 1 < 1,$$

then $F_n(z) \in V_k^{\lambda}(\eta)$ with $\eta = (\beta-1)\sum_{l=1}^{n} \alpha_l + 1$ and β is a root of (2.1).

Proof. From (3.2), we have

$$\int_0^{2\pi} \left| \operatorname{Re}\left[e^{i\lambda}\left(1 + \frac{zF_n''(z)}{F_n'(z)} \right) - \eta\cos\lambda \right] \right| d\theta$$

$$= \sum_{l=1}^{m} \alpha_l \int_0^{2\pi} \left| \operatorname{Re}\left[e^{i\lambda} \frac{z(D^n f_l(z))'}{D^n f_l(z)} - \rho\cos\lambda \right] \right| d\theta, \tag{3.5}$$

where η is given by (3.1).

Since $f_l(z) \in V_k^{\lambda}(\rho)$ for $1 \leqslant l \leqslant n$, then by using Lemma 2.1, we have

$$\int_0^{2\pi} \left| \operatorname{Re} \left[e^{i\lambda} \frac{z(D^n f_l(z))'}{D^n f_l(z)} - \beta \cos\lambda \right] \right| d\theta \leqslant (1-\beta) k\pi \cos\lambda, \tag{3.6}$$

where β is a root of the equation (2.1). Using (3.6) in (3.5), we obtain

$$\int_0^{2\pi} \left| \operatorname{Re} \left[e^{i\lambda} \left(1 + \frac{zF_n''(z)}{F_n'(z)} \right) - \eta \cos\lambda \right] \right| d\theta \leqslant (1-\eta) k\pi \cos\lambda.$$

Hence $F_n(z) \in V_k^{\lambda}(\eta)$ with $\eta = (\beta - 1)\sum_{l=1}^{n} \alpha_l + 1$ and β is a root of (2.1).

Set $n = 1$, $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_n = 0$ and $\lambda = 0$ in Theorem 3.4, we obtain the following result. □

COROLLORY 3.5. *Let $f(z) \in V_k(\rho)$. Then the Alexandar operator $F_0(z)$, defined in* [1]*, belongs to the class $V_k(\beta)$, where β is given by*

$$\beta = \frac{1}{4}\left[(2\rho - 1) + \sqrt{4\rho^2 - 4\rho + 9} \right].$$

For $\rho = 0$ and $k = 2$ in Corollary 3.5, we have the well known result, that is,

$$f(z) \in C(0) \text{ implies } F_0(z) \in C\left(\frac{1}{2}\right).$$

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