

A NOTE ON MEROMORPHIC m -VALENT STARLIKE FUNCTIONS

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Abstract. In the present investigation, we will derive sufficient conditions for starlikeness of meromorphic m -valent functions in the punctured disc.

1. Introduction and preliminaries

Let Σ_m denote the class of functions of the form

$$f(z) = \frac{1}{z^m} + \sum_{n=1}^{\infty} a_{m+n-1} z^{m+n-1}, \quad m \in \mathbb{N}^* \quad (1.1)$$

which are analytic and m -valent in the punctured disc

$$\dot{U} = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}.$$

A function $f \in \Sigma_m$ is said [1] to be in the class Ω_m of meromorphic m -valently starlike functions in \dot{U} if and only if

$$\operatorname{Re} \left\{ -\frac{z f'(z)}{f(z)} \right\} > 0, \quad z \in \dot{U}, \quad m \in \mathbb{N}^*. \quad (1.2)$$

The following definitions and lemmas will be used in the next section.

Let $\mathcal{H}(U)$ denote the space of analytic functions in U . For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}_n = \{f \in \mathcal{H}(U) : f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots\} \quad (1.3)$$

and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\} \quad (1.4)$$

For two functions f and g analytic in U , we say that the function $f(z)$ is subordinate to $g(z)$ in U and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in U$$

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if there exists a Schwarz function $w(z)$, analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in U,$$

such that

$$f(z) = g(w(z)), \quad z \in U. \quad (1.5)$$

In particular, if the function g is univalent in U , the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

LEMMA 1.1. [2] *Let m be a positive integer and let α be real, with $0 \leq \alpha < m$. Let $q \in \mathcal{H}(U)$, with $q(0) = 0$, $q'(0) \neq 0$ and*

$$\operatorname{Re} \left(1 + \frac{zq''(z)}{q'(z)} \right) > \frac{\alpha}{m}. \quad (1.6)$$

Define the function h as

$$h(z) = mzq'(z) - \alpha q(z) \quad (1.7)$$

If $p \in \mathcal{H}_m$ and

$$zp'(z) - \alpha p(z) \prec h(z) \quad (1.8)$$

then $p(z) \prec q(z)$ and this result is sharp.

2. Main results

THEOREM 2.1. *If $f \in \Sigma_m$, $m \in \mathbb{N}^*$, on the form*

$$f(z) = \frac{1}{z^m} + \sum_{k=m}^{\infty} a_k z^k$$

and satisfies the condition

$$|(1 - \alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M, \quad \alpha \in [0, 2) \quad (2.1)$$

then

$$|z^m f(z) - 1| < \frac{M}{m(2 - \alpha)} \quad (2.2)$$

and this result is sharp.

Proof. If we let

$$p(z) = z^m f(z) - 1 \quad (2.3)$$

then $p \in \mathcal{H}_{2m}$ and (2.1) can be rewritten as

$$|zp'(z) - \alpha mp(z)| < M \quad (2.4)$$

or

$$zp'(z) - \alpha mp(z) \prec Mz. \tag{2.5}$$

If we take in Lemma 1.1

$$q(z) = \frac{Mz}{m(2-\alpha)}, \quad q \in \mathcal{H}(U),$$

with $q(0) = 0, q'(0) \neq 0$ and

$$\operatorname{Re} \left(1 + \frac{zq''(z)}{q'(z)} \right) > \frac{\alpha}{2}$$

then from (1.7), $h(z) = Mz$ and the result follows from Lemma 1.1, that is $p(z) \prec q(z)$

$$z^m f(z) - 1 \prec \frac{Mz}{m(2-\alpha)}$$

or

$$|z^m f(z) - 1| < \frac{M}{m(2-\alpha)}. \quad \square$$

By applying our previous result we can obtain a simple criterion for the starlikeness of meromorphic m -valent function.

THEOREM 2.2. *Let $m \in \mathbb{N}^*, \alpha \in [0, 2)$ and let*

$$M(m, \alpha) = \frac{m(2-\alpha)}{1-\alpha + \sqrt{\alpha^2 + (2-\alpha)^2}} \tag{2.6}$$

If $f \in \Sigma_m$ satisfies the condition

$$|(1-\alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M(m, \alpha)$$

then $f \in \Omega_m$.

Proof. Let

$$0 < M \leq M(m, \alpha), \tag{2.7}$$

where $M(m, \alpha)$ is given by (2.6), and suppose that $f \in \Sigma_m$ satisfies the condition

$$|(1-\alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M. \tag{2.8}$$

If we set

$$P(z) = z^m f(z), \tag{2.9}$$

then by Theorem 2.1 we obtain

$$|P(z) - 1| < \frac{M}{m(2-\alpha)} \equiv R, \quad z \in U. \tag{2.10}$$

From (2.6) we easily deduce $R < 1$, which implies $P(z) \neq 0$, $z \in U$. Hence if we let

$$p(z) = -\frac{zf'(z)}{f(z)}, \quad (2.11)$$

then $p(z) \in \mathcal{H}[m, 2m]$ and (2.8) can be written in the form

$$|-P(z)p(z) + (1 - \alpha)mP(z) + \alpha m| < M. \quad (2.12)$$

We claim that this inequality implies $\operatorname{Re} p(z) > 0$, $z \in U$. If this is false, then there exists a point $z_0 \in U$, such that $p(z_0) = i\rho$, where ρ is real. We will show that at such a point the negation of condition (2.12) holds, that is

$$|-i\rho P(z_0) + (1 - \alpha)mP(z_0) + \alpha m| \geq M, \quad (2.13)$$

for all real ρ .

If we let $P_0 = P(z_0)$, we have

$$\begin{aligned} |-i\rho P_0 + (1 - \alpha)mP_0 + \alpha m|^2 &= \rho^2|P_0|^2 + m^2(1 - \alpha)^2|P_0|^2 + \alpha^2 m^2 \\ &\quad + 2\operatorname{Re} P_0 \alpha(1 - \alpha)m^2 + 2\alpha\rho m \operatorname{Im} P_0. \end{aligned}$$

The inequality (2.13) is equivalent to

$$\begin{aligned} E \equiv \rho^2|P_0|^2 + m^2(1 - \alpha)^2|P_0|^2 + \alpha^2 m^2 \\ + 2\alpha(1 - \alpha)\operatorname{Re} P_0 m^2 + 2\alpha\rho m \operatorname{Im} P_0 - M^2 \geq 0. \end{aligned} \quad (2.14)$$

Since from (2.10) we have

$$|P_0| > 1 - R \quad \text{and} \quad \operatorname{Re} P_0 > 1 - R,$$

from (2.10) and (2.14) one obtains

$$\begin{aligned} E \geq \rho^2|P_0|^2 + 2\alpha m \operatorname{Im} P_0 \rho + m^2(1 - \alpha)^2(1 - R)^2 \\ + \alpha^2 m^2 + 2\alpha(1 - \alpha)m^2(1 - R) - R^2 m^2(2 - \alpha)^2. \end{aligned}$$

Hence $E \geq 0$ if

$$\alpha^2 m^2 (\operatorname{Im} P_0)^2 \leq |P_0|^2 \{ [m\alpha + m(1 - \alpha)(1 - R)]^2 - R^2 m^2 (2 - \alpha)^2 \} \quad (2.15)$$

or

$$\alpha^2 (\operatorname{Im} P_0)^2 \leq |P_0|^2 \{ [1 - (1 - \alpha)R]^2 - R^2 (2 - \alpha)^2 \}. \quad (2.16)$$

A simple geometric argument shows that the inequality (2.10) implies

$$(\operatorname{Im} P_0)^2 \leq R^2 |P_0|^2. \quad (2.17)$$

By comparing (2.16) and (2.17) we deduce that (2.13) holds if

$$\alpha^2 R^2 \leq [1 - (1 - \alpha)R]^2 - R^2 (2 - \alpha)^2 \quad (2.18)$$

or

$$[(1 - \alpha)^2 - \alpha^2 - (2 - \alpha)^2]R^2 - 2(1 - \alpha)R + 1 \geq 0. \tag{2.19}$$

This last inequality holds if $R \leq R_0$, where

$$R_0 = \frac{1 - \alpha - \sqrt{\alpha^2 + (2 - \alpha)^2}}{(1 - \alpha)^2 - \alpha^2 - (2 - \alpha)^2} = \frac{1}{1 - \alpha + \sqrt{\alpha^2 + (2 - \alpha)^2}}, \tag{2.20}$$

that is $M \leq M(m, \alpha)$.

Thus we have contradiction of (2.12), therefore $\operatorname{Re} p(z) > 0$, $z \in U$, and $f \in \Omega_m$. \square

REMARK 2.1. Note that for the special case $m = 1$, Theorem 2.2 reduces to a result previously obtained in [3].

COROLLARY 2.1. Let $m \in \mathbb{N}^*$, and let $f \in \Sigma_m$ satisfies the condition

$$|mz^m f(z) + z^{m+1} f'(z)| < \frac{2m}{3} \tag{2.21}$$

then $f \in \Omega_m$.

Since a function $f \in \Sigma_m$ can be written as

$$f(z) = \frac{1}{z^m} + g(z), \quad 0 < |z| < 1 \tag{2.22}$$

where $g \in \mathcal{H}_m$, Theorem 2.2 can be rewritten in the following equivalent form, that is useful for the other results.

COROLLARY 2.2. Let $m \in \mathbb{N}^*$, $\alpha \in [0, 2)$ and $f \in \Sigma_m$ have the form

$$f(z) = \frac{1}{z^m} + g(z),$$

where $g \in \mathcal{H}_m$. If

$$|(1 - \alpha)mz^m g(z) + z^{m+1} g'(z)| < M(m, \alpha), \quad z \in U \tag{2.23}$$

where $M(m, \alpha)$ is given by (2.6), then $f \in \Omega_m$.

This form has an interesting interpretation in terms of integral operators. If we let

$$h(z) = (1 - \alpha)mz^m g(z) + z^{m+1} g'(z), \tag{2.24}$$

then

$$g(z) = \frac{1}{z^{(1-\alpha)m}} \int_0^z h(t)t^{-(1+\alpha m)} dt \tag{2.25}$$

which leads to the following result.

COROLLARY 2.3. Let $h \in \mathcal{H}_{2m}$ and $M(m, \alpha)$ is given by (2.6) with $\alpha \in [0, 2)$. If h satisfies the condition

$$|h(z)| \leq M(m, \alpha), \quad z \in U \quad (2.26)$$

then

$$f(z) = \frac{1}{z^m} + \frac{1}{z^{(1-\alpha)m}} \int_0^z h(t)t^{-(1+\alpha m)} dt \in \Omega_m. \quad (2.27)$$

EXAMPLE 2.1. Let consider for the Corollary 2.3 the special case

$$h(z) = az^2(1 + \cos z), \quad h \in \mathcal{H}_2 \quad (2.28)$$

where $m = 1$ and we take $\alpha = 1$. Then

$$f(z) = \frac{1}{z} + a \int_0^z (1 + \cos t) dt = \frac{1}{z} + a(z + \sin z) \quad (2.29)$$

and the assumption

$$|h(z)| \leq M(1, 1), \quad z \in U, \quad (2.30)$$

simplifies to

$$|h(z)| = |az^2||1 + \cos z| \leq \frac{\sqrt{2}}{2}, \quad z \in U. \quad (2.31)$$

From the fact that

$$|az^2||1 + \cos z| \leq |a|(1 + \operatorname{ch}1), \quad (2.32)$$

the condition (2.31) will be satisfied if

$$|a|(1 + \operatorname{ch}1) \leq \frac{\sqrt{2}}{2},$$

and we obtain

$$|a| \leq \frac{e\sqrt{2}}{(e+1)^2} \quad (2.33)$$

or

$$|a| \leq 0.278\dots \quad (2.34)$$

Hence, if we take $a = \frac{1}{4}$ we conclude that

$$f(z) = \frac{1}{z} + \frac{1}{4}(z + \sin z) \in \Omega_1.$$

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