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A NOTE ON MEROMORPHIC m–VALENT STARLIKE FUNCTIONS

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Abstract. In the present investigation, we will derive sufficient conditions for starlikeness of meromorphic m-valent functions in the punctured disc.

1. Introduction and preliminaries

Let Σ_m denote the class of functions of the form

$$f(z) = \frac{1}{z^m} + \sum_{n=1}^{\infty} a_{m+n-1} z^{m+n-1}, \quad m \in \mathbb{N}^* \tag{1.1}$$

which are analytic and m-valent in the punctured disc

$$\dot{U} = \{z \in C : \ 0 < |z| < 1\} = U \setminus \{0\}.$$

A function $f \in \Sigma_m$ is said [1] to be in the class Ω_m of meromorphic m-valently starlike functions in $\dot{U}$ if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \dot{U},\ m \in \mathbb{N}^*. \tag{1.2}$$

The following definitions and lemmas will be used in the next section.

Let $\mathscr{H}(U)$ denote the space of analytic functions in U. For n a positive integer and $a \in \mathbb{C}$ let

$$\mathscr{H}_n = \{f \in \mathscr{H}(U) : \ f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots\} \tag{1.3}$$

and

$$\mathscr{H}[a,n] = \{f \in \mathscr{H}(U) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\} \tag{1.4}$$

For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in U$$

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if there exists a Schwarz function $w(z)$, analytic in U with

$$w(0)=0 \quad \text{and} \quad |w(z)|<1, \quad z\in U,$$

such that

$$f(z)=g(w(z)), \quad z\in U. \tag{1.5}$$

In particular, if the function g is univalent in U, the above subordination is equivalent to

$$f(0)=g(0) \quad \text{and} \quad f(U)\subset g(U).$$

LEMMA 1.1. [2] *Let m be a positive integer and let α be real, with $0\leqslant\alpha<m$. Let $q\in\mathscr{H}(U)$, with $q(0)=0$, $q'(0)\neq 0$ and*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\frac{\alpha}{m}. \tag{1.6}$$

Define the function h as

$$h(z)=mzq'(z)-\alpha q(z) \tag{1.7}$$

If $p\in\mathscr{H}_m$ and

$$zp'(z)-\alpha p(z)\prec h(z) \tag{1.8}$$

then $p(z)\prec q(z)$ and this result is sharp.

2. Main results

THEOREM 2.1. *If $f\in\Sigma_m$, $m\in\mathbb{N}^*$, on the form*

$$f(z)=\frac{1}{z^m}+\sum_{k=m}^{\infty}a_kz^k$$

and satisfies the condition

$$|(1-\alpha)mz^mf(z)+z^{m+1}f'(z)+\alpha m|<M, \quad \alpha\in[0,2) \tag{2.1}$$

then

$$|z^mf(z)-1|<\frac{M}{m(2-\alpha)} \tag{2.2}$$

and this result is sharp.

Proof. If we let

$$p(z)=z^mf(z)-1 \tag{2.3}$$

then $p\in\mathscr{H}_{2m}$ and (2.1) can be rewritten as

$$|zp'(z)-\alpha mp(z)|<M \tag{2.4}$$

or

$$zp'(z) - \alpha m p(z) \prec Mz. \tag{2.5}$$

If we take in Lemma 1.1

$$q(z) = \frac{Mz}{m(2-\alpha)}, \quad q \in \mathscr{H}(U),$$

with $q(0) = 0$, $q'(0) \neq 0$ and

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \frac{\alpha}{2}$$

then from (1.7), $h(z) = Mz$ and the result follows from Lemma 1.1, that is $p(z) \prec q(z)$

$$z^m f(z) - 1 \prec \frac{Mz}{m(2-\alpha)}$$

or

$$|z^m f(z) - 1| < \frac{M}{m(2-\alpha)}. \quad \square$$

By applying our previous result we can obtain a simple criterion for the starlikeness of meromorphic m-valent function.

THEOREM 2.2. *Let $m \in \mathbb{N}^*$, $\alpha \in [0,2)$ and let*

$$M(m,\alpha) = \frac{m(2-\alpha)}{1-\alpha+\sqrt{\alpha^2+(2-\alpha)^2}} \tag{2.6}$$

If $f \in \Sigma_m$ satisfies the condition

$$|(1-\alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M(m,\alpha)$$

then $f \in \Omega_m$.

Proof. Let

$$0 < M \leqslant M(m,\alpha), \tag{2.7}$$

where $M(m,\alpha)$ is given by (2.6), and suppose that $f \in \Sigma_m$ satisfies the condition

$$|(1-\alpha)mz^m f(z) + z^{m+1} f'(z) + \alpha m| < M. \tag{2.8}$$

If we set

$$P(z) = z^m f(z), \tag{2.9}$$

then by Theorem 2.1 we obtain

$$|P(z) - 1| < \frac{M}{m(2-\alpha)} \equiv R, \quad z \in U. \tag{2.10}$$

From (2.6) we easily deduce $R < 1$, which implies $P(z) \neq 0$, $z \in U$. Hence if we let

$$p(z) = -\frac{zf'(z)}{f(z)}, \tag{2.11}$$

then $p(z) \in \mathscr{H}[m, 2m]$ and (2.8) can be written in the form

$$|-P(z)p(z) + (1-\alpha)mP(z) + \alpha m| < M. \tag{2.12}$$

We claim that this inequality implies $\mathrm{Re}\, p(z) > 0$, $z \in U$. If this is false, then there exists a point $z_0 \in U$, such that $p(z_0) = i\rho$, where ρ is real. We will show that at such a point the negation of condition (2.12) holds, that is

$$|-i\rho P(z_0) + (1-\alpha)mP(z_0) + \alpha m| \geqslant M, \tag{2.13}$$

for all real ρ.

If we let $P_0 = P(z_0)$, we have

$$\begin{aligned} |-i\rho P_0 + (1-\alpha)mP_0 + \alpha m|^2 = \rho^2|P_0|^2 + m^2(1-\alpha)^2|P_0|^2 + \alpha^2 m^2 \\ + 2\mathrm{Re}\, P_0\alpha(1-\alpha)m^2 + 2\alpha\rho m \mathrm{Im}\, P_0. \end{aligned}$$

The inequality (2.13) is equivalent to

$$\begin{aligned} E \equiv \rho^2|P_0|^2 + m^2(1-\alpha)^2|P_0|^2 + \alpha^2 m^2 \\ + 2\alpha(1-\alpha)\mathrm{Re}\, P_0 m^2 + 2\alpha\rho m \mathrm{Im}\, P_0 - M^2 \geqslant 0. \end{aligned} \tag{2.14}$$

Since from (2.10) we have

$$|P_0| > 1 - R \quad \text{and} \quad \mathrm{Re}\, P_0 > 1 - R,$$

from (2.10) and (2.14) one obtains

$$\begin{aligned} E \geqslant \rho^2|P_0|^2 + 2\alpha m \mathrm{Im}\, P_0\rho + m^2(1-\alpha)^2(1-R)^2 \\ + \alpha^2 m^2 + 2\alpha(1-\alpha)m^2(1-R) - R^2m^2(2-\alpha)^2. \end{aligned}$$

Hence $E \geqslant 0$ if

$$\alpha^2 m^2(\mathrm{Im}\, P_0)^2 \leqslant |P_0|^2\{[m\alpha + m(1-\alpha)(1-R)]^2 - R^2m^2(2-\alpha)^2 \tag{2.15}$$

or

$$\alpha^2(\mathrm{Im}\, P_0)^2 \leqslant |P_0|^2\{[1-(1-\alpha)R]^2 - R^2(2-\alpha)^2\}. \tag{2.16}$$

A simple geometric argument shows that the inequality (2.10) implies

$$(\mathrm{Im}\, P_0)^2 \leqslant R^2|P_0|^2. \tag{2.17}$$

By comparing (2.16) and (2.17) we deduce that (2.13) holds if

$$\alpha^2 R^2 \leqslant [1-(1-\alpha)R]^2 - R^2(2-\alpha)^2 \tag{2.18}$$

or

$$[(1-\alpha)^2-\alpha^2-(2-\alpha)^2]R^2-2(1-\alpha)R+1\geqslant 0. \tag{2.19}$$

This last inequality holds if $R\leqslant R_0$, where

$$R_0=\frac{1-\alpha-\sqrt{\alpha^2+(2-\alpha)^2}}{(1-\alpha)^2-\alpha^2-(2-\alpha)^2}=\frac{1}{1-\alpha+\sqrt{\alpha^2+(2-\alpha)^2}}, \tag{2.20}$$

that is $M\leqslant M(m,\alpha)$.

Thus we have contradiction of (2.12), therefore $\operatorname{Re} p(z)>0$, $z\in U$, and $f\in\Omega_m$. $\square$

REMARK 2.1. Note that for the special case $m=1$, Theorem 2.2 reduces to a result previously obtained in [3].

COROLLARY 2.1. *Let $m\in\mathbb{N}^*$, and let $f\in\Sigma_m$ satisfies the condition*

$$|mz^m f(z)+z^{m+1}f'(z)|<\frac{2m}{3} \tag{2.21}$$

then $f\in\Omega_m$.

Since a function $f\in\Sigma_m$ can be written as

$$f(z)=\frac{1}{z^m}+g(z),\quad 0<|z|<1 \tag{2.22}$$

where $g\in\mathscr{H}_m$, Theorem 2.2 can be rewritten in the following equivalent form, that is useful for the other results.

COROLLARY 2.2. *Let $m\in\mathbb{N}^*$, $\alpha\in[0,2)$ and $f\in\Sigma_m$ have the form*

$$f(z)=\frac{1}{z^m}+g(z),$$

where $g\in\mathscr{H}_m$. If

$$|(1-\alpha)mz^m g(z)+z^{m+1}g'(z)|<M(m,\alpha),\quad z\in U \tag{2.23}$$

where $M(m,\alpha)$ is given by (2.6), then $f\in\Omega_m$.

This form has an interesting interpretation in terms of integral operators. If we let

$$h(z)=(1-\alpha)mz^m g(z)+z^{m+1}g'(z), \tag{2.24}$$

then

$$g(z)=\frac{1}{z^{(1-\alpha)m}}\int_0^z h(t)t^{-(1+\alpha m)}dt \tag{2.25}$$

which leads to the following result.

COROLLARY 2.3. *Let* $h \in \mathscr{H}_{2m}$ *and* $M(m,\alpha)$ *is given by (2.6) with* $\alpha \in [0,2)$. *If* h *satisfies the condition*

$$|h(z)| \leqslant M(m,\alpha), \quad z \in U \tag{2.26}$$

then

$$f(z) = \frac{1}{z^m} + \frac{1}{z^{(1-\alpha)m}} \int_0^z h(t)t^{-(1+\alpha m)}dt \in \Omega_m. \tag{2.27}$$

EXAMPLE 2.1. Let consider for the Corollary 2.3 the special case

$$h(z) = az^2(1+\cos z), \quad h \in \mathscr{H}_2 \tag{2.28}$$

where $m=1$ and we take $\alpha = 1$. Then

$$f(z) = \frac{1}{z} + a\int_0^z (1+\cos t)dt = \frac{1}{z} + a(z+\sin z) \tag{2.29}$$

and the assumption

$$|h(z)| \leqslant M(1,1), \quad z \in U, \tag{2.30}$$

simplifies to

$$|h(z)| = |az^2||1+\cos z| \leqslant \frac{\sqrt{2}}{2}, \quad z \in U. \tag{2.31}$$

From the fact that

$$|az^2||1+\cos z| \leqslant |a|(1+\mathrm{ch}1), \tag{2.32}$$

the condition (2.31) will be satisfied if

$$|a|(1+\mathrm{ch}1) \leqslant \frac{\sqrt{2}}{2},$$

and we obtain

$$|a| \leqslant \frac{e\sqrt{2}}{(e+1)^2} \tag{2.33}$$

or

$$|a| \leqslant 0.278\ldots \tag{2.34}$$

Hence, if we take $a = \dfrac{1}{4}$ we conclude that

$$f(z) = \frac{1}{z} + \frac{1}{4}(z+\sin z) \in \Omega_1.$$

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