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TWO NEW INEQUALITIES FOR GAUSSIAN AND GAMMA DISTRIBUTIONS

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Abstract. Two new inequalities regarding Q function and incomplete upper bound Gamma function are established, which are related to Gaussian and Gamma distributions respectively.

1. Introduction

Let us introduce some notations first. Assume that $f(\cdot)$ is a probability density function with an interval support $[a,b]$, and $F:[a,b]\to[0,1]$ its corresponding distribution function. The corresponding reliability function or the survival function is defined by $\bar{F}(x)=1-F(x)=\int_x^b f(t)dt$. A function $g(x)$ is logarithmically concave (or log-concave for short), if its natural logarithm $ln(g(x))$ is concave. It is found in [1] that if a continuously differentiable density function, with support $[0,+\infty)$, is log-concave, then for all $\forall x,y\geqslant 0$, we have

$$\bar{F}(x+y)\leqslant\bar{F}(x)\bar{F}(y). \tag{1}$$

Moreover, if f is log-convex, then the above inequality is reversed.

Two typical distributions possessing property as (1) are Gamma distribution $\Gamma(k,x)$ and complementary error function $\operatorname{erfc}(x)$, which is given as

$$\Gamma(k,x)=\int_x^\infty\frac{t^{k-1}e^{-t}dt}{\Gamma(k)},\quad \operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}.$$

Here $\Gamma(k,x)$ is also called upper incomplete gamma function. It is also shown in [1] that such property holds for Weibull distribution, chi-squared distribution and chi distribution as well.

On the other side, the reverse inequality of (1) would rarely be occurred, since the general distributions are nearly all log-concave. It is the purpose of this paper to consider the reverse inequality of (1) at a special angle, i.e., even if (1) holds for a distribution, it is still possible to find a suitable parameter a such that

$$\bar{F}^2(x)\leqslant\bar{F}(ax). \tag{2}$$

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Obviously, (2) holds at least for $a=1$, since $\bar{F}(x)\in[0,1]$. It seems difficult to consider (2) for general distributions. As a starting point, here we study the corresponding inequality (2) for Gaussian Q function, i.e., $Q(x)=\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$, and upper incomplete gamma function respectively.

The main results of this paper are listed as following:

THEOREM 1.1. *Suppose that* $1\leqslant a\leqslant\sqrt{2}$, *then for* $\forall x\in R$

$$Q^2(x)<Q(ax), \tag{3}$$

where $Q(x)=\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$.

THEOREM 1.2. *Suppose that* $k>1$ *and* $0\leqslant a\leqslant 2^{\frac{1}{k}}$, *then for* $\forall x>0$

$$\Gamma^2(k,x)<\Gamma(k,ax), \tag{4}$$

where $\Gamma(k,x)=\int_x^\infty \frac{1}{\Gamma(k)}t^{k-1}e^{-t}dt$. *On the other side, if* $k\in(0,1]$, *inequality* (4) *holds for* $0\leqslant a<2^{\frac{1}{k}}$ *and* $\forall x>0$.

REMARK 1.1. For $a\in[0,1)$ in Theorem 1.1, the inequality (3) still holds for $x>0$ in view of the monotonicity of $Q(x)$.

If $k=1$ in Theorem 1.2, by the fact that $\Gamma(1,x)=\int_x^\infty e^{-t}dt=e^{-x}$, we know that $\Gamma^2(1,x)=\Gamma(1,2x)$.

2. Proofs for Main results

Proof of Theorem 1.1. By the fact $0\leqslant Q(x)\leqslant 1$, (3) holds naturally for $a=1$. Notice further that $Q(x)$ decreases as x increases, it is sufficient to prove (3) for $a=\sqrt{2}$.

Define $\psi(x)=Q^2(x)-Q(\sqrt{2}x)$. Clearly,

$$\lim_{x\to-\infty}\psi(x)=0,\qquad \lim_{x\to+\infty}\psi(x)=0, \tag{5}$$

and

$$\begin{aligned}
\psi'(x)&=-2Q(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}+\frac{\sqrt{2}}{\sqrt{2\pi}}e^{-x^2}\\
&=\frac{1}{\sqrt{2\pi}}e^{-x^2}\left(\sqrt{2}-2Q(x)e^{\frac{x^2}{2}}\right)\\
&\triangleq\frac{1}{\sqrt{2\pi}}e^{-x^2}\psi_1(x).
\end{aligned} \tag{6}$$

Let us study $\psi_1(x)$ first. Obviously, $\psi_1(0)=\sqrt{2}-1>0$. By the facts that $Q(-\infty)=1$ and $e^{\frac{x^2}{2}}\xrightarrow[x\to-\infty]{}+\infty$, we have $\lim_{x\to-\infty}\psi_1(x)=-\infty$. Notice further the monotonicity of $Q(x)$ and $e^{\frac{x^2}{2}}$ as $x \to -\infty$, the sign of function $\psi_1(x)$ changes once from negative to positive as x moves from $-\infty$ to 0.

It is left to consider the sign of $\psi_1(x)$ when $x > 0$. By the following inequality

$$Q(x) < \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for $x > 0$, we derive

$$Q(x) e^{\frac{x^2}{2}} < \frac{1}{x\sqrt{2\pi}}.$$

Hence, for $\psi_1(x) > 0$, it is sufficient to require

$$\sqrt{2} - \frac{2}{x\sqrt{2\pi}} > 0,$$

which is equivalent to $x > \frac{1}{\sqrt{\pi}}$. This means $\psi_1(x) > 0$ for $x > \frac{1}{\sqrt{\pi}}$. Now only the case for $0 < x \leqslant \frac{1}{\sqrt{\pi}}$ is left. This can be analyzed directly as following: for $0 < x \leqslant \frac{1}{\sqrt{\pi}}$,

$$\psi_1(x) > \sqrt{2} - e^{\frac{x^2}{2}} \geqslant \sqrt{2} - e^{\frac{1}{2\pi}} = 1.4142\cdots - 1.1725\cdots = 0.2417\cdots > 0.$$

Here the approximating calculation in the last step is carried out by Matlab.

In conclusion, $\psi_1(x)$ changes its sign once from negative to positive as x moves from $-\infty$ to ∞. Thus, by (6), $\psi'(x)$ change from negative to positive as x moves from $-\infty$ to ∞, and the sign changes only once. This means $\psi(x)$ has only one local minimum. Together with (5), the assertion follows directly. $\square$

We need an upper bound for incomplete Gamma function $\Gamma(k,x)$ to prove Theorem 1.2. We refer to [2] for more details about this topic.

LEMMA 2.1. *For $k > 0$ and $x > k+1$,*

$$\Gamma(k,x) < \frac{1}{\Gamma(k)} x^k e^{-x}. \tag{7}$$

Proof. Define $\varphi(x) = \Gamma(k,x) - \frac{1}{\Gamma(k)} x^k e^{-x}$. Thus,

$$\varphi'(x) = \frac{1}{\Gamma(k)}(-x^{k-1}e^{-x} - kx^{k-1}e^{-x} + x^k e^{-x}) = \frac{1}{\Gamma(k)}(x^{k-1}e^{-x}(x-k-1)) > 0$$

for $x > k+1$. Together with the fact that $\lim_{x\to\infty}\varphi(x) = 0$, the inequality (7) follows directly. $\square$

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We divide the whole proof into two cases: (i) $k > 1$, and (ii) $k \in (0,1]$.

(i). The case $k>1$. By the facts that $x>0$ and $\Gamma(k,x)\in[0,1]$, it is sufficient to prove (4) for $a=2^{\frac{1}{k}}$. Clearly, $2^{\frac{1}{k}}\in(1,2)$, and we use a instead of $2^{\frac{1}{k}}$ below for brief.

Define $\phi(x)=\Gamma^2(k,x)-\Gamma(k,ax)$. Clearly,

$$\phi(0)=0,\quad \lim_{x\to+\infty}\phi(x)=0, \tag{8}$$

and

$$\begin{aligned}\phi'(x)&=-2\Gamma(k,x)\frac{x^{k-1}e^{-x}}{\Gamma(k)}+\frac{a(ax)^{k-1}e^{-ax}}{\Gamma(k)}\\&=\frac{2x^{k-1}e^{-x}}{\Gamma(k)}\left(e^{(1-a)x}-\Gamma(k,x)\right)\\&\triangleq\frac{2x^{k-1}e^{-x}}{\Gamma(k)}\phi_1(x).\end{aligned} \tag{9}$$

We use the fact $a^k=2$ in the above second step. Clearly, $\phi_1(0)=0$ and $\lim_{x\to\infty}\phi_1(x)=0$. By Lemma 2.1, for $x>k+1$, we have

$$\phi_1(x)>e^{(1-a)x}-\frac{1}{\Gamma(k)}x^ke^{-x}=e^{-x}\left(e^{(2-a)x}-\frac{1}{\Gamma(k)}x^k\right), \tag{10}$$

which means $\phi_1(x)>0$ when $x>x_0$ with a sufficiently large point x_0. Now let us consider the derivative of ϕ_1 below.

$$\begin{aligned}\phi_1'(x)&=(1-a)e^{(1-a)x}+\frac{x^{k-1}e^{-x}}{\Gamma(k)}\\&=e^{-x}\left(\frac{x^{k-1}}{\Gamma(k)}-(a-1)e^{(2-a)x}\right)\triangleq e^{-x}\phi_2(x).\end{aligned} \tag{11}$$

We find that $\phi_2(0)=-(a-1)<0$ and $\lim_{x\to\infty}\phi_2(x)=-\infty$. Due to the facts that $\phi_1(x)$ has positive value for $x>x_0$, starting at $\phi_1(0)=0$, we know that its derivative $\phi_1'(x)$ must be positive somewhere between 0 and x_0, and thus for $\phi_2(x)$.

If k is a positive integer, then the $(k-1)$-th and k-th derivatives are

$$\begin{aligned}&\phi_2^{(k-1)}(x)=1-(a-1)(2-a)^{k-1}e^{(2-a)x},\\&\phi_2^{(k)}(x)=-(a-1)(2-a)^ke^{(2-a)x}<0.\end{aligned}$$

Notice further that $\phi_2^{(k-1)}(0)=1-(a-1)(2-a)^{k-1}>0$ and $\lim_{x\to\infty}\phi_2^{(k-1)}(x)=-\infty$, we know that $\phi_2^{(k-1)}(x)$ starts at a positive value and then decreases monotonically to $-\infty$. This further means that $\phi_2^{(k-2)}(x)$ starts from a negative value to a positive local maximum and then decreases monotonically to $-\infty$, and so on till $\phi_2'(x)$. Thus, $\phi_2(x)$ increases piecewise monotonically from $\phi_2(0)<0$ to a positive maximum and then decreases to $-\infty$. And then $\phi_1'(x)$ changes its sign twice, i.e., from negative to positive and then negative. Hence, $\phi_1(x)$ decreases from 0 to a negative minimum and then increase to positive maximum and then decreases to 0. And this further holds for $\phi'(x)$, which means the sign of $\phi'(x)$ changes from negative to positive once. Finally, we know that $\phi(x)$ decreases from $\phi(0)=0$ to a negative minimum and then increases to 0. This means $\phi(x)<0$ as desired.

When k is not an integer, the $[k]$-th and $([k]+1)$-th derivatives are

$$\phi_2^{([k])}(x) = \frac{(k-1)\cdots(k-[k])x^{k-[k]-1}}{\Gamma(k)} - (a-1)(2-a)^{[k]}e^{(2-a)x},$$
$$\phi_2^{([k]+1)}(x) = \frac{(k-1)\cdots(k-[k]-1)x^{k-[k]-2}}{\Gamma(k)} - (a-1)(2-a)^{[k]}e^{(2-a)x} < 0.$$

Notice further that $\phi_2^{([k])}(0)=+\infty$ and $\lim_{x\to\infty}\phi_2^{([k])}(x)=-\infty$, we know that $\phi_2^{([k])}(x)$ decreases monotonically from $+\infty$ to $-\infty$ as x moves from 0 to ∞. The rest reasoning is similar to the above case.

(ii). The proof for the case $k\in(0,1]$ is similar. It is sufficient to consider $a\in(1,2^{\frac{1}{k}})$ by the monotonicity of $\Gamma(k,x)$. By the same definition of $\phi(x)$, the same assertions (8) follow. Observe that $1<a<2^{\frac{1}{k}}$ this time, we have different derivative of ϕ as:

$$\phi'(x) = \frac{x^{k-1}e^{-x}}{\Gamma(k)}\left(a^k e^{(1-a)x} - 2\Gamma(k,x)\right) \triangleq \frac{x^{k-1}e^{-x}}{\Gamma(k)}\varphi(x),$$

and thus, $\phi'(0)<0$ since $\varphi(0)=a^k-2<0$, and $\lim_{x\to\infty}\phi'(x)=0$. By Lemma 2.1 and similar to (10), we also know $\phi'(x)$ is positive for sufficiently large x. The rest proof is nearly the same to the counterpart of case (i). $\square$

REFERENCES

[1] A. BARICZ, *A functional inequality for the survival function of the Gamma distribution*, J. Inequal. Pure and Appl. Math., **9**, 1 (2008), Article 13.

[2] P. NATALINI AND B. PALUMBO, *Inequalities for the incomplete Gamma Function*, Mathematical Inequalities & Applications, **3**, 1 (2000), 69–77.

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