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SOME PROPERTIES FOR A CLASS OF SYMMETRIC FUNCTIONS AND APPLICATIONS

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Abstract. For $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$, the symmetric function $F_n(x,r)$ is defined by

$$F_n(x,r)=F_n(x_1,x_2,\cdots,x_n;r)=\prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n}\frac{\sum_{j=1}^{r}x_{i_j}}{\sum_{j=1}^{r}(1+x_{i_j})},$$

where $r=1,2,\cdots,n$ and $i_1,i_2,\cdots,i_n$ are positive integers. In this article, the Schur convexity, Schur harmonic convexity and Schur multiplicative convexity of $F_n(x,r)$ are discussed. As applications, some inequalities are established by use of the theory of majorization.

1. Introduction

Throughout the paper we use $\mathbb{R}^n$ denote the $n-$dimensional Euclidean space, and $\mathbb{R}^n_+=\{(x_1,x_2,\cdots,x_n):x_i>0,i=1,2,\cdots,n\}$. In particular, we use $\mathbb{R}$ to denote $\mathbb{R}^1$.

For $x=(x_1,x_2,\cdots,x_n)$, $y=(y_1,y_2,\cdots,y_n)\in\mathbb{R}^n_+$ and $\alpha>0$, let

$$\begin{aligned}
x+y&=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n),\\
xy&=(x_1y_1,x_2y_2,\cdots,x_ny_n),\\
\alpha x&=(\alpha x_1,\alpha x_2,\cdots,\alpha x_n),\\
x^\alpha&=(x_1^\alpha,x_2^\alpha,\cdots,x_n^\alpha),\\
\frac{1}{x}&=\left(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n}\right)\\
\log x&=(\log x_1,\log x_2,\cdots,\log x_n)
\end{aligned}$$

and

$$e^x=(e^{x_1},e^{x_2},\cdots,e^{x_n}).$$

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For $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$, $r\in N$ and $r\leqslant n$, the Hamy symmetric function $H_n(x,r)$ is defined by T. Hara, M. Uchiyama and S. Takahasi [1] as follows:

$$H_n(x,r)=H_n(x_1,x_2,\cdots,x_n;r)=\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\left(\prod_{j=1}^{r}x_{i_j}\right)^{\frac{1}{r}},$$

where $i_1,i_2,\cdots,i_n\in N$.

Corresponding to this is the $r-$th order Hamy mean

$$\sigma_n(x,r)=\sigma_n(x_1,x_2,\cdots,x_n;r)=\frac{(n-r)!r!}{n!}H_n(x,r).$$

T. Hara, M. Uchiyama and S. Takahasi [1] established the following refinement of the classical arithmetic and geometric means inequalities:

$$G_n(x)=\sigma_n(x,n)\leqslant\sigma_n(x,n-1)\leqslant\cdots\leqslant\sigma_n(x,2)\leqslant\sigma_n(x,1)=A_n(x),$$

where $A_n(x)=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $G_n(x)=(\prod_{i=1}^{n}x_i)^{\frac{1}{n}}$ denote the classical arithmetic and geometric means of x, respectively.

The paper [2] by H. T. Ku, M. C. Ku and X. M. Zhang contains some interesting inequalities including the fact that $(\sigma_n(x,r))^{\frac{1}{r}}$ is log-concave. More results can be found in the book [3] by P. S. Bullen.

Recently, the Schur convexity of the Hamy symmetric function $H_n(x,r)$ was discussed and some analytic inequalities were established by K. Z. Guan [4].

The main purpose of this paper is to discuss the Schur convexity, Schur harmonic convexity and Schur multiplicative convexity for the symmetric function

$$F_n(x,r)=F_n(x_1,x_2,\cdots,x_n;r)=\prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n}\frac{\sum_{j=1}^{r}x_{i_j}}{\sum_{j=1}^{r}(1+x_{i_j})}. \tag{1.1}$$

As applications, some inequalities are established by use of the theory of majorization.

For the reader convenience, we recall several definitions.

DEFINITION 1.1. Let $E\subseteq\mathbb{R}^n$ be a set, a real-valued function F on E is said to be Schur convex if

$$F(x_1,x_1,\cdots,x_n)\leqslant F(y_1,y_2,\cdots,y_n)$$

for each pair of n-tuples $x=(x_1,\cdots,x_n)$ and $y=(y_1,\cdots,y_n)$ in E, such that $x\prec y$, that is

$$\sum_{i=1}^{k}x_{[i]}\leqslant\sum_{i=1}^{k}y_{[i]},\quad k=1,2,\cdots,n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x. F is called Schur concave if $-F$ is Schur convex .

DEFINITION 1.2. Let $E \subseteq \mathbb{R}_+^n$ be a set, a real-valued function F on E is said to be Schur multiplicatively convex if

$$F(x_1, x_2, \cdots, x_n) \leqslant F(y_1, y_2, \cdots, y_n)$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\log x \prec \log y$. F is called Schur multiplicatively concave if $\frac{1}{F}$ is Schur multiplicatively convex.

DEFINITION 1.3. Let $E \subseteq \mathbb{R}_+^n$ be a set, a real-valued function F on E is said to be Schur harmonic convex if

$$F(x_1, x_2, \cdots, x_n) \leqslant F(y_1, y_2, \cdots, y_n) \tag{1.2}$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$. F is called a Schur harmonic concave on E if inequality (1.2) is reversed.

The Schur convexity was introduced by I. Schur [5] in 1923, G. H. Hardy, J. E. Littlewood and G. Pólya were also interested in some inequalities that are related to the Schur convexity [6]. Recently, the Schur multiplicative convexity was introduced and investigated in paper [7, 8].

Very recently, the Schur harmonic convexity was introduced by Y. M. Chu and Y. P. Lv [9], and the Schur harmonic convexity for the Hamy Symmetric function $H_n(x, r)$ was discussed. To investigate the Schur harmonic convexity for the symmetric function $F_n(x, r)$ is one of the main purpose in this article.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

LEMMA 2.1. ([5]) *Suppose that $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a continuous symmetric function. If f is differentiable in $\mathbb{R}_+^n$, then f is Schur convex in $\mathbb{R}_+^n$ if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0 \tag{2.1}$$

for all $i, j = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$. And f is Schur concave in $\mathbb{R}_+^n$ if and only if inequality (2.1) is reversed. Here f is a symmetric function in $\mathbb{R}_+^n$ which means that $f(Px) = f(x)$ for any $x \in \mathbb{R}_+^n$ and any $n \times n$ permutation matrix P.

REMARK 2.1. Since f is symmetric, the Schur's condition in Lemma 2.1, that is (2.1) can be reduced to

$$(x_1-x_2)\left(\frac{\partial f}{\partial x_1}-\frac{\partial f}{\partial x_2}\right)\geqslant 0.$$

LEMMA 2.2. ([7, 8]) *Suppose that* $f:\mathbb{R}_+^n\to\mathbb{R}_+$ *is a continuous symmtric function. If* f *is differentiable in* $\mathbb{R}_+^n$*, then* f *is Schur multiplicatively convex in* $\mathbb{R}_+^n$ *if and only if*

$$(\log x_1-\log x_2)\left(x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}\right)\geqslant 0 \tag{2.2}$$

for all $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$*. And* f *is Schur multiplicatively concave in* $\mathbb{R}_+^n$ *if and only if inequality (2.2) is reversed.*

LEMMA 2.3. ([9]) *Suppose that* $f:\mathbb{R}_+^n\to\mathbb{R}_+$ *is a continuous symmetric function. If* f *is differentiable in* $\mathbb{R}_+^n$*, then* f *is Schur harmonic convex in* $\mathbb{R}_+^n$ *if and only if*

$$(x_1-x_2)\left(x_1^2\frac{\partial f}{\partial x_1}-x_2^2\frac{\partial f}{\partial x_2}\right)\geqslant 0 \tag{2.3}$$

for all $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$*. And* f *is Schur harmonic concave in* $\mathbb{R}_+^n$ *if and only if inequality (2.3) is reversed.*

LEMMA 2.4. ([4, 8, 10]) *Suppose that* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$ *and* $\sum_{i=1}^n x_i=s$*. If* $c\geqslant s$*, then*

$$\frac{c-x}{\frac{nc}{s}-1}=\left(\frac{c-x_1}{\frac{nc}{s}-1},\frac{c-x_2}{\frac{nc}{s}-1},\cdots,\frac{c-x_n}{\frac{nc}{s}-1}\right)\prec(x_1,x_2,\cdots,x_n)=x.$$

LEMMA 2.5. ([10]) *Suppose that* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$ *and* $\sum_{i=1}^n x_i=s$*. If* $c\geqslant 0$*, then*

$$\frac{c+x}{\frac{nc}{s}+1}=\left(\frac{c+x_1}{\frac{nc}{s}+1},\frac{c+x_2}{\frac{nc}{s}+1},\cdots,\frac{c+x_n}{\frac{nc}{s}+1}\right)\prec(x_1,x_2,\cdots,x_n)=x.$$

LEMMA 2.6. ([11]) *Suppose that* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$ *and* $\sum_{i=1}^n x_i=s$*. If* $0\leqslant\lambda\leqslant 1$*, then*

$$\frac{s-\lambda x}{n-\lambda}=\left(\frac{s-\lambda x_1}{n-\lambda},\frac{s-\lambda x_2}{n-\lambda},\cdots,\frac{s-\lambda x_n}{n-\lambda}\right)\prec(x_1,x_2,\cdots,x_n)=x.$$

3. Main Results

THEOREM 3.1. *$F_n(x,r)$ is Schur concave in $\mathbb{R}^n_+$.*

Proof. The proof is divided into four cases.
Case 1. If $r=1$, then (1.1) leads to

$$F_n(x,1)=F_n(x_1,x_2,\cdots,x_n;1)=\prod_{i=1}^{n}\frac{x_i}{1+x_i} \tag{3.1}$$

and

$$(x_1-x_2)\left(\frac{\partial F_n(x,1)}{\partial x_1}-\frac{\partial F_n(x,1)}{\partial x_2}\right)=-\frac{(x_1-x_2)^2(1+x_1+x_2)}{x_1x_2(1+x_1)(1+x_2)}F_n(x,1)\leqslant 0.$$

Case 2. If $r=n$, then

$$F_n(x,n)=F_n(x_1,x_2,\cdots,x_n;n)=\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}(1+x_i)} \tag{3.2}$$

and

$$(x_1-x_2)\left(\frac{\partial F_n(x,n)}{\partial x_1}-\frac{\partial F_n(x,n)}{\partial x_2}\right)=0.$$

Case 3. If $n\geqslant 3$ and $r=2$, then

$$\begin{aligned}F_n(x,2)&=F_n(x_1,x_2,\cdots,x_n;2)\\&=F_{n-1}(x_2,x_3,\cdots,x_n;2)\frac{(x_1+x_2)\prod_{j=3}^{n}(x_1+x_j)}{[(1+x_1)+(1+x_2)]\prod_{j=3}^{n}[(1+x_1)+(1+x_j)]}\\&=F_{n-1}(x_1,x_3,\cdots,x_n;2)\frac{(x_2+x_1)\prod_{j=3}^{n}(x_2+x_j)}{[(1+x_1)+(1+x_2)]\prod_{j=3}^{n}[(1+x_2)+(1+x_j)]}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&(x_1-x_2)\left(\frac{\partial F_n(x,2)}{\partial x_1}-\frac{\partial F_n(x,2)}{\partial x_2}\right)\\&=-2(x_1-x_2)^2F_n(x,2)\sum_{j=3}^{n}\frac{x_1+x_2+2x_j+2}{(x_1+x_j)(x_2+x_j)(2+x_1+x_j)(2+x_2+x_j)}\leqslant 0.\end{aligned}$$

Case 4. If $n\geqslant 4$ and $3\leqslant r\leqslant n-1$, then

$$\begin{aligned}F_n(x,r)&=F_n(x_1,x_2,\cdots,x_n;r)\\&=F_{n-1}(x_2,x_3,\cdots,x_n;r)\frac{\prod\limits_{3\leqslant i_1<i_2<\cdots<i_{r-1}\leqslant n}(x_1+\sum_{j=1}^{r-1}x_{i_j})}{\prod\limits_{3\leqslant i_1<i_2<\cdots<i_{r-1}\leqslant n}[(1+x_1)+\sum_{j=1}^{r-1}(1+x_{i_j})]}\\&\quad\times\frac{\prod\limits_{3\leqslant i_1<i_2<\cdots<i_{r-2}\leqslant n}(x_1+x_2+\sum_{j=1}^{r-2}x_{i_j})}{\prod\limits_{3\leqslant i_1<i_2<\cdots<i_{r-2}\leqslant n}[(1+x_1)+(1+x_2)+\sum_{j=1}^{r-2}(1+x_{i_j})]}\end{aligned} \tag{3.4}$$

$$= F_{n-1}(x_2, x_3, \cdots, x_n; r) \frac{\prod\limits_{3 \leqslant i_1 < i_2 < \cdots < i_{r-1} \leqslant n} (x_2 + \sum_{j=1}^{r-1} x_{i_j})}{\prod\limits_{3 \leqslant i_1 < i_2 < \cdots < i_{r-1} \leqslant n} [(1+x_2) + \sum_{j=1}^{r-1} (1+x_{i_j})]}$$
$$\times \frac{\prod\limits_{3 \leqslant i_1 < i_2 < \cdots < i_{r-2} \leqslant n} (x_2 + x_1 + \sum_{j=1}^{r-2} x_{i_j})}{\prod\limits_{3 \leqslant i_1 < i_2 < \cdots < i_{r-2} \leqslant n} [(1+x_1) + (1+x_2) + \sum_{j=1}^{r-2} (1+x_{i_j})]}.$$

and

$$(x_1 - x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1} - \frac{\partial F_n(x,r)}{\partial x_2}\right)$$
$$= -r(x_1-x_2)^2 F_n(x,r) \sum_{3 \leqslant i_1 < i_2 < \cdots < i_{r-1} \leqslant n} \frac{r + x_1 + x_2 + 2\sum_{j=1}^{r-1} x_{i_j}}{\Delta} \leqslant 0,$$

where

$$\Delta = \left(x_1 + \sum_{j=1}^{r-1} x_{i_j}\right)\left(x_2 + \sum_{j=1}^{r-1} x_{i_j}\right)\left(r + x_1 + \sum_{j=1}^{r-1} x_{i_j}\right)\left(r + x_2 + \sum_{j=1}^{r-1} x_{i_j}\right). \tag{3.5}$$

□

Therefore, Theorem 3.1 follows from Lemma 2.1 and Remark 2.1 together with Cases 1-4.

THEOREM 3.2. *$F_n(x,r)$ is Schur harmonic convex in $\mathbb{R}^n_+$.*

Proof. We divided the proof into four cases.
Case I. If $r = 1$, then (3.1) leads to

$$(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x,1)}{\partial x_1} - x_2^2 \frac{\partial F_n(x,1)}{\partial x_2}\right) = \frac{(x_1-x_2)^2}{(1+x_1)(1+x_2)} F_n(x,1) \geqslant 0.$$

Case II. If $r = n$, then (3.2) yields

$$(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x,n)}{\partial x_1} - x_2^2 \frac{\partial F_n(x,n)}{\partial x_2}\right) = \frac{n(x_1-x_2)^2(x_1+x_2)}{[\sum_{i=1}^n (1+x_i)]^2} \geqslant 0.$$

Case III. If $n \geqslant 3$ and $r = 2$, then (3.3) implies

$$(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x,2)}{\partial x_1} - x_2^2 \frac{\partial F_n(x,2)}{\partial x_2}\right)$$
$$= 2(x_1-x_2)^2 F_n(x,2) \left[\frac{1}{x_1+x_2+2}\right.$$
$$\left. + \sum_{j=3}^{n} \frac{2x_1x_2 + 2x_1x_j + 2x_2x_j + 2x_1x_2x_j + x_1x_j^2 + x_2x_j^2}{(x_1+x_j)(x_2+x_j)(2+x_1+x_j)(2+x_2+x_j)}\right] \geqslant 0.$$

Case IV. If $n \geqslant 4$ and $3 \leqslant r \leqslant n-1$, then from (3.4) we get

$$\begin{aligned}
&(x_1-x_2)\left(x_1^2\frac{\partial F_n(x,r)}{\partial x_1}-x_2^2\frac{\partial F_n(x,r)}{\partial x_2}\right)\\
=\ &r(x_1-x_2)^2F_n(x,r)\\
&\times\Bigg[\sum_{3\leqslant i_1<i_2<\cdots<i_{r-2}\leqslant n}\frac{x_1+x_2}{(x_1+x_2+\sum_{j=1}^{r-2}x_{i_j})(r+x_1+x_2+\sum_{j=1}^{r-2}x_{i_j})}\\
&+\sum_{3\leqslant i_1<i_2<\cdots<i_{r-1}\leqslant n}\frac{\Theta}{\Delta}\Bigg]\geqslant 0,
\end{aligned}$$

where $\Theta=(x_1+x_2)\left(\sum_{j=1}^{r-1}x_{i_j}\right)^2+2x_1x_2\sum_{j=1}^{r-1}x_{i_j}+r(x_1+x_2)\sum_{j=1}^{r-1}x_{i_j}+rx_1x_2$ and Δ is defined as in (3.5).

Therefore, Theorem 3.2 follows from Lemma 2.3 and Cases I-IV. $\square$

Next, we denote by

$$\Omega_n(t,r)=\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n : t\leqslant x_i\leqslant\sqrt{(r-1)^2t^2+r(r-1)t}\}$$

for $t>0$ and $2\leqslant r\leqslant n-1$.

For the Schur multiplicative convexity or concavity of the symmetric function $F_n(x,r)$, we have the following Theorem 3.3.

THEOREM 3.3. (i) *$F_n(x,1)$ is Schur multiplicatively concave in $\mathbb{R}^n_+$;*
(ii) *$F_n(x,n)$ is Schur multiplicatively convex in $\mathbb{R}^n_+$;*
(iii) *If $n\geqslant 3$ and $2\leqslant r\leqslant n-1$, then $F_n(x,r)$ is Schur multiplicatively convex in $\Omega_n(t,r)$ for any $t>0$.*

Proof. (i) From (3.1) we clearly see that

$$\begin{aligned}
&(\log x_1-\log x_2)\left(x_1\frac{\partial F_n(x,1)}{\partial x_1}-x_2\frac{\partial F_n(x,1)}{\partial x_2}\right)\\
&=-\frac{(x_1-x_2)(\log x_1-\log x_2)}{(1+x_1)(1+x_2)}F_n(x,1)\leqslant 0.
\end{aligned}\tag{3.6}$$

Therefore, Theorem 3.3 (i) follows from (3.6) and Lemma 2.2.

(ii) Equation (3.2) leads to

$$\begin{aligned}
&(\log x_1-\log x_2)\left(x_1\frac{\partial F_n(x,n)}{\partial x_1}-x_2\frac{\partial F_n(x,n)}{\partial x_2}\right)\\
&=\frac{n(x_1-x_2)(\log x_1-\log x_2)}{[\sum_{i=1}^n(1+x_i)]^2}\geqslant 0.
\end{aligned}\tag{3.7}$$

Therefore, Theorem 3.3 (ii) follows from (3.7) and Lemma 2.2.

(iii) We divided the proof into two cases.

Case A. If $n \geqslant 3$ and $r=2$, then (3.3) implies

$$
\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1\frac{\partial F_n(x,2)}{\partial x_1} - x_2\frac{\partial F_n(x,2)}{\partial x_2}\right) \qquad (3.8)\\
&= 2(x_1-x_2)(\log x_1 - \log x_2)F_n(x,2)\left[\frac{1}{(x_1+x_2)(x_1+x_2+2)}\right.\\
&\left. + \sum_{j=3}^{n}\frac{x_j^2+2x_j-x_1x_2}{(x_1+x_j)(x_2+x_j)(x_1+x_j+2)(x_2+x_j+2)}\right] \geqslant 0
\end{aligned}
$$

for any $t>0$ and $x=(x_1,x_2,\cdots,x_n)\in\Omega_n(t,2)$.

Case B. If $n \geqslant 4$ and $3 \leqslant r \leqslant n-1$, then (3.4) yields

$$
\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1\frac{\partial F_n(x,r)}{\partial x_1} - x_2\frac{\partial F_n(x,r)}{\partial x_2}\right)\\
&= r(x_1-x_2)(\log x_1 - \log x_2)F_n(x,r)\\
&\times\left[\sum_{3\leqslant<i_1<i_2<\cdots<i_{r-2}\leqslant n}\frac{1}{(x_1+x_2+\sum_{j=1}^{r-2}x_{i_j})(r+x_1+x_2+\sum_{j=1}^{r-2}x_{i_j})}\right.\\
&\left. + \sum_{3\leqslant<i_1<i_2<\cdots<i_{r-1}\leqslant n}\frac{r\sum_{j=1}^{r-1}x_{i_j}+(\sum_{j=1}^{r-1}x_{i_j})^2-x_1x_2}{\Delta}\right] \geqslant 0
\end{aligned}
$$

for any $t>0$ and $x=(x_1,x_2,\cdots,x_n)\in\Omega_n(t,r)$, where Δ is defined as in (3.5).

Therefore, Theorem 3.3 (iii) follows from Cases A and B together with Lemma 2.2. □

4. Applications

In this section, we establish some inequalities by use of Theorems 3.1–3.3 and the theory of majorization.

The following Theorem 4.1 easily follows from Theorems 3.1 and 3.2 together with Lemmas 2.4–2.6.

THEOREM 4.1. *Suppose that* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$ *and* $\sum_{i=1}^{n}x_i=s$. *If* $c_1\geqslant s, c_2\geqslant 0$, $0\leqslant\lambda\leqslant 1$ *and* $r\in\{1,2,\cdots,n\}$, *then*

(1) $F_n(x,r)\leqslant F_n\left(\dfrac{c_1-x}{\frac{nc_1}{s}-1},r\right)$;

(2) $F_n\left(\dfrac{1}{x},r\right)\geqslant F_n\left(\dfrac{\frac{nc_1}{s}-1}{c_1-x},r\right)$;

(3) $F_n(x,r)\leqslant F_n\left(\dfrac{c_2+x}{\frac{nc_2}{s}+1},r\right)$;

(4) $F_n\left(\frac{1}{x},r\right) \geqslant F_n\left(\frac{\frac{nc_2}{s}+1}{c_2+x},r\right)$;

(5) $F_n(x,r) \leqslant F_n\left(\frac{s-\lambda x}{n-\lambda},r\right)$;

(6) $F_n(\frac{1}{x},r) \geqslant F_n\left(\frac{n-\lambda}{s-\lambda x},r\right)$.

For $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$, if we denote $H_n(x)=\frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$, then we clearly see that

$$(A_n(x),A_n(x),\cdots,A_n(x)) \prec (x_1,x_2,\cdots,x_n) \tag{4.1}$$

and

$$\left(\frac{1}{H_n(x)},\frac{1}{H_n(x)},\cdots,\frac{1}{H_n(x)}\right) \prec \left(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n}\right). \tag{4.2}$$

It follows from (1.1) and (4.1)–(4.2) together with Theorems 3.1–3.2 that the following Theorem 4.2 is obvious.

THEOREM 4.2. *Suppose that* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$. *If* $r\in\{1,2,\cdots,n\}$, *then*

(1) $$\prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n}\frac{\sum_{j=1}^r x_{i_j}}{r+\sum_{j=1}^r x_{i_j}} \leqslant \left[\frac{A_n(x)}{A_n(1+x)}\right]^{\frac{n!}{r!(n-r)!}};$$

(2) $$\prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n}\frac{\sum_{j=1}^r \frac{1}{x_{i_j}}}{r+\sum_{j=1}^r \frac{1}{x_{i_j}}} \geqslant \left[\frac{1}{A_n(1+x)}\right]^{\frac{n!}{r!(n-r)!}};$$

(3) $$\prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n}\frac{\sum_{j=1}^r \frac{1}{x_{i_j}}}{r+\sum_{j=1}^r \frac{1}{x_{i_j}}} \leqslant \left[\frac{1}{1+H_n(x)}\right]^{\frac{n!}{r!(n-r)!}};$$

(4) $$\prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n}\frac{\sum_{j=1}^r x_{i_j}}{r+\sum_{j=1}^r x_{i_j}} \geqslant \left[\frac{H_n(x)}{1+H_n(x)}\right]^{\frac{n!}{r!(n-r)!}}.$$

If we take $r=1$ in Theorem 4.2(1), (3) and (4), respectively, then we get

COROLLARY 4.1. *If* $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$, *then*

(1) $\frac{G_n(x)}{G_n(1+x)} \leqslant \frac{A_n(x)}{A_n(1+x)}$;

(2) $G_n(1+x) \geqslant 1+H_n(x)$;

(3) $\frac{G_n(x)}{G_n(1+x)} \geqslant \frac{H_n(x)}{1+H_n(x)}$.

REMARK 4.1. Inequality in Corollary 4.1(1) was proved by V. Govedarica and M. V. Jovanović in [12].

REMARK 4.2. If we take $\sum_{i=1}^{n} x_i = 1$ in Corollary 4.1(1), then we get the Weierstrass inequality [13, p. 260]

$$\prod_{i=1}^{n}(x_i^{-1}+1) \geqslant (n+1)^n.$$

THEOREM 4.3. *Let $\mathscr{A} = A_1A_2\cdots A_{n+1}$ be a $n-$dimensional simplex in $\mathbb{R}^n$ and P be an arbitrary point in the interior of $\mathscr{A}$. If B_i is the intersection point of straight line A_iP and hyperplane $\sum_i = A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}, i = 1,2,\cdots,n+1$. Then for $r \in \{1,2,\cdots,n+1\}$ we have*

$$(1) \quad \prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n+1} \frac{\sum_{j=1}^{r}\frac{PB_{i_j}}{A_{i_j}B_{i_j}}}{r+\sum_{j=1}^{r}\frac{PB_{i_j}}{A_{i_j}B_{i_j}}} \leqslant \left(\frac{1}{n+2}\right)^{\frac{(n+1)!}{r!(n-r+1)!}};$$

$$(2) \quad \prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n+1} \frac{\sum_{j=1}^{r}\frac{PA_{i_j}}{A_{i_j}B_{i_j}}}{r+\sum_{j=1}^{r}\frac{PA_{i_j}}{A_{i_j}B_{i_j}}} \leqslant \left(\frac{n}{2n+1}\right)^{\frac{(n+1)!}{r!(n-r+1)!}};$$

$$(3) \quad \prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n+1} \frac{\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{PB_{i_j}}}{r+\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{PB_{i_j}}} \geqslant \left(\frac{n+1}{n+2}\right)^{\frac{(n+1)!}{r!(n-r+1)!}};$$

$$(4) \quad \prod_{1\leqslant i_1<i_2\cdots<i_r\leqslant n+1} \frac{\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{PA_{i_j}}}{r+\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{PA_{i_j}}} \geqslant \left(\frac{n+1}{2n+1}\right)^{\frac{(n+1)!}{r!(n-r+1)!}}.$$

Proof. It is easy to see that $\sum_{i=1}^{n+1}\frac{PB_i}{A_iB_i} = 1$ and $\sum_{i=1}^{n+1}\frac{PA_i}{A_iB_i} = n$, these identities imply that

$$\left(\frac{1}{n+1},\frac{1}{n+1},\cdots,\frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2},\cdots,\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right) \tag{4.3}$$

and

$$\left(\frac{n}{n+1},\frac{n}{n+1},\cdots,\frac{n}{n+1}\right) \prec \left(\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2},\cdots,\frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right). \tag{4.4}$$

Therefore, Theorem 4.3 follows from (4.3), (4.4), Theorem 3.1, Theorem 3.2 and (1.1). □

REMARK 4.3. D. S. Mitrinović, J. E. Pečarić and V. Volenec [14, p. 473–479] established a series of inequalities for $\frac{PA_i}{A_iB_i}$ and $\frac{PB_i}{A_iB_i}$, $i = 1,2,\cdots,n+1$. Obviously, our inequalities in Theorem 4.3 are different from theirs.

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